Analysis Preliminary Exam Workshop: Distributions and Sobolev Spaces

1. Distributions

A distribution is a linear functional on a space of test functions. Distributions include all locally integrable functions and have derivatives of all orders (great for linear problems) but cannot be multiplied in any natural way (not so great for nonlinear problems). One can use many different spaces of test functions. We will consider distributions on the space $\mathcal{D}(\Omega)$ of smooth compactly supported test functions where $\Omega \subset \mathbb{R}^n$ is an open set (which we call simply "distributions"), distributions on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of smooth, rapidly decreasing functions ("tempered distributions"), and distributions on the space $\mathcal{D}(\mathbb{T}^n)$ of smooth periodic functions ("periodic distributions").

2. Test functions

If $\Omega \subset \mathbb{R}^n$ is an open set (possibly equal to $\mathbb{R}^n$), then the space $\mathcal{D}(\Omega)$ consists of all smooth functions ϕ whose support $\operatorname{supp}\phi$ is a compact subset of Ω (i.e. $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ as a set).

The topology on $\mathcal{D}(\Omega)$ corresponds to the following notion of convergence of test functions: $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp}\phi_n \subset K$ for every $n \in \mathbb{N}$ and $\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly on Ω for every multi-index $\alpha \in \mathbb{N}_0^n$.

The space $\mathcal{D}(\Omega)$ is a topological vector space, but its topology is not metrizable. Nevertheless, somewhat remarkably, sequential continuity of a functional is equivalent to continuity.

3. Distributions in $\mathcal{D}'(\Omega)$

A distribution $T \in \mathcal{D}'(\Omega)$ is a continuous linear functional

$$T : \mathcal{D}(\Omega) \to \mathbb{C}, \qquad \phi \mapsto \langle T, \phi \rangle$$

Here, continuity means that

$$\langle T, \phi_n \rangle \to \langle T, \phi \rangle \qquad \text{if } \phi_n \to \phi \text{ in } \mathcal{D}(\Omega).$$

Examples: (a) If $f \in L^1_{\text{loc}}(\Omega)$, then T_f defined by

$$\langle T_f, \phi \rangle = \int_\Omega f \phi \, dx$$

is a distribution, called a regular distribution. The function f is determined by the distribution T_f up to pointwise a.e. equivalence. We typically identify T_f with the function f and write $T_f = f$.
(b) If $c \in \Omega$ then δ_c defined by

$$\langle \delta_c, \phi \rangle = \phi(c)$$

is a distribution, called the δ-distribution supported at c. We usually write $\delta_0 = \delta$. This is not a regular distribution; nevertheless it is often referred to as the δ-function.
If $T \in \mathcal{D}'(\Omega)$ and $f \in C^\infty(\Omega)$, we define $fT \in \mathcal{D}'(\Omega)$ by

$$\langle fT, \phi \rangle = \langle T, f\phi \rangle.$$

If $\alpha \in \mathbb{N}_0^n$ is any multi-index, we define the derivative $\partial^\alpha T \in \mathcal{D}'(\Omega)$ by

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle.$$

Example: The α-derivative of the δ-distribution is given by

$$\langle \partial^\alpha \delta_c, \phi \rangle = (-1)^{|\alpha|} \partial^\alpha \phi(c).$$

We say that a sequence of distributions (T_n) converges to a distribution T in $\mathcal{D}'(\Omega)$, written $T_n \rightharpoonup T$, if

$$\langle T_n, \phi \rangle \to \langle T, \phi \rangle \qquad \text{for every } \phi \in \mathcal{D}(\Omega).$$

(This is weak-$*$ convergence in $\mathcal{D}'$.)

4. Tempered distributions

Tempered distributions are distributions on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. If (ϕ_k) is a sequence in $\mathcal{S}(\mathbb{R}^n)$, we say that $\phi_k \to \phi$ in $\mathcal{S}$ if

$$x^\alpha \partial^\beta \phi_k \to x^\alpha \partial^\beta \phi$$

uniformly as $k \to \infty$ for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. (This topology is metrizable, but does not come from a norm.)

A tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ is a continuous linear functional

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}.$$

If $T \in \mathcal{S}'(\mathbb{R}^n)$, we define the Fourier transform $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$$

and the inverse Fourier transform by $\langle \check{T}, \phi \rangle = \langle T, \check{\phi} \rangle$, where $\check{\phi} = \mathcal{F}^{-1}\phi$.
Example: For $\delta \in \mathcal{S}'(\mathbb{R}^n)$, we have

$$\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^{n/2}} \int \phi(x)\, dx = \frac{1}{(2\pi)^{n/2}} \langle 1, \phi \rangle,$$

so $\hat{\delta} = (2\pi)^{-n/2}$.

THEOREM 1. *The Fourier transform*

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \qquad \mathcal{F} : T \mapsto \hat{T}$$

is a one-to-one, onto continuous map with inverse $\mathcal{F}^{-1}$.

5. Periodic distributions

Let $\mathcal{D}(\mathbb{T}^n)$ denote the space of smooth periodic functions $\phi : \mathbb{T}^n \to \mathbb{C}$, where $\phi_k \to \phi$ in $\mathcal{D}(\mathbb{T}^n)$ if $\partial^\alpha \phi_k \to \partial^\alpha \phi$ uniformly for every multi-index $\alpha \in \mathbb{N}_0^n$.
A periodic distribution $T \in \mathcal{D}'(\mathbb{T}^n)$ is a continuous linear functional

$$T : \mathcal{D}(\mathbb{T}^n) \to \mathbb{C}.$$

For $k \in \mathbb{Z}^n$, we define the kth Fourier coefficient of T by

$$\hat{T}_k = \frac{1}{(2\pi)^n} \langle T, e^{-ik\cdot x} \rangle.$$

THEOREM 2. *If* $T \in \mathcal{D}'(\mathbb{T}^n)$*, the Fourier series*

$$T = \sum_{k \in \mathbb{Z}^n} \hat{T}_k e^{-ik\cdot x}$$

converges to T *in the sense of distributions.*

6. Weak derivatives

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in L^1_{\mathrm{loc}}(\Omega)$. If, for some multi-index $\alpha \in \mathbb{N}_0^n$, there exists a function $g \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega f \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega g\phi \, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega)$$

then f is weakly differentiable of order α and $g = \partial^\alpha f$ is the weak derivative of f of order α. Equivalently, f is weakly differentiable of order α if the distributional derivative $\partial^\alpha T_f$ is a regular distribution T_g, in which case $g = \partial^\alpha f$.

An integration by parts shows that if f is smooth, then the weak derivative is equal a.e. to the usual pointwise derivative.

Example: If $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is given by $f(x) = |x|$, then f is weakly differentiable with $f'(x) = \operatorname{sgn} x$, but f' is not weakly differentiable (its distributional derivative is 2δ, which is not regular).

7. Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of the functions $f \in L^p(\Omega)$ that have weak derivatives $\partial^\alpha f \in L^p(\Omega)$ of all orders $|\alpha| \le k$. For $1 \le p < \infty$, we use the norm

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^p} \right)^{1/p},$$

and for $p = \infty$

$$\|f\|_{W^{k,\infty}} = \max_{|\alpha| \le k} \|\partial^\alpha f\|_{L^\infty}.$$

For $p = 2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$, and use the inner product

$$(f, g)_{H^k} = \sum_{|\alpha| \le k} (\partial^\alpha f, \partial^\alpha g)_{L^2}.$$

THEOREM 3. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $k \in \mathbb{N}$, and $1 \le p \le \infty$. Then $W^{k,p}(\Omega)$ is a Banach space and $H^k(\Omega)$ is a Hilbert space. The space $W^{k,p}(\Omega)$ is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$. In particular, $H^k(\Omega)$ is a separable Hilbert space.*

THEOREM 4 (Meyers-Serrin). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $k \in \mathbb{N}$, and $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Note that $C_c^\infty(\Omega)$ is not, in general, dense in $W^{k,p}(\Omega)$ (see 10 below) and $u \in C^\infty(\Omega)$ need not be in $W^{k,p}(\Omega)$, or even bounded e.g. $1/x \in C^\infty(0,1)$, but $1/x \notin L^1(0,1)$. In the case $\Omega = \mathbb{R}^n$, however, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for $1 \leq p < \infty$.

8. The Sobolev spaces $H^k(\mathbb{R}^n)$ and $H^k(\mathbb{T}^n)$

We can use the Fourier transform to give an alternative description of the L^2-Sobolev spaces on $\mathbb{R}^n$ and $\mathbb{T}^n$. The space $H^k(\mathbb{R}^n)$ consists of the functions $f \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^k |\hat{f}(\xi)|^2 \, d\xi < \infty$$

with the (equivalent) inner product

$$(f, g)_{H^k(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^k \overline{\hat{f}(\xi)} \hat{g}(\xi) \, d\xi.$$

The Sobolev space $H^k(\mathbb{T}^n)$ consists of the periodic functions $f \in L^2(\mathbb{T}^n)$ such that

$$\sum_{m \in \mathbb{Z}^n} \left(1 + |m|^2\right)^k |\hat{f}_m|^2 < \infty$$

with inner product

$$(f, g)_{H^k(\mathbb{T}^n)} = \sum_{m \in \mathbb{Z}^n} \left(1 + |m|^2\right)^k \overline{\hat{f}_m} \hat{g}_m.$$

9. Embedding theorems

There are two crucial embedding properties of Sobolev spaces: (a) sufficiently many L^p-derivatives imply that a function is continuous (or has improved integrability); (b) for functions defined on bounded sets, uniform estimates of L^p-derivatives imply compactness in suitable spaces of functions with lower-order derivatives.

If $1 \leq p < n$, the Sobolev conjugate $p < p^* < \infty$ of p is defined by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

First consider Sobolev spaces on $\mathbb{R}^n$.

THEOREM 5 (Gagliardo, Nirenberg, Sobolev). *If $1 \le p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

and there exists a constant C, depending only on p and n, such that

$$\|u\|_{L^{p^*}} \le C\,\|Du\|_{L^p} \qquad \text{for all } u \in W^{1,p}(\mathbb{R}^n).$$

THEOREM 6 (Morrey). *If $n < p \le \infty$, then $W^{1,p}(\mathbb{R}^n) \subset C^{0,\alpha}(\mathbb{R}^n)$ of Hölder continuous functions with exponent α where $\alpha = 1 - n/p$, and if $u \in W^{1,p}(\mathbb{R}^n)$,*

$$|u(x) - u(y)| \le C(p,n)\|Du\|_{L^p}|x-y|^\alpha$$

a.e. in $x, y \in \mathbb{R}^n$

Next consider open subsets of $\mathbb{R}^n$ with smooth (e.g. C^1) boundary $\partial\Omega$, and closure $\overline{\Omega}$.

THEOREM 7. *If $1 \le p \le \infty$ and $\Omega \subset \mathbb{R}^n$ is an open set with smooth boundary, then*

$$\begin{aligned}
&W^{1,p}(\Omega) \subset L^{p^*}(\Omega) && \text{if } p < n,\\
&W^{1,p}(\Omega) \subset L^q(\Omega) && \text{for all } p \le q < \infty \text{ if } p = n\\
&W^{1,p}(\Omega) \subset L^\infty(\Omega) \cap C^{0,\alpha}(\overline{\Omega}) && \text{if } p > n, \text{ where } \alpha = 1 - n/p
\end{aligned}$$

and these embeddings are continuous.

For smooth, *bounded* open sets we get compact embeddings (a Sobolev version of the Arzelà-Ascoli theorem).

THEOREM 8 (Rellich, Kondrachov). *If $1 \le p \le \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary, then*

$$\begin{aligned}
&W^{1,p}(\Omega) \subset L^q(\Omega) && \text{for all } 1 \le q < p^* \text{ if } p < n,\\
&W^{1,p}(\Omega) \subset L^q(\Omega) && \text{for all } p \le q < \infty \text{ if } p = n\\
&W^{1,p}(\Omega) \subset C(\overline{\Omega}) && \text{if } p > n
\end{aligned}$$

and these embedding are compact.

Examples: Suppose $n = 1$. The space $W^{1,1}(0,1)$ consists of the absolutely continuous functions on $[0,1]$. If f is absolutely continuous, then its pointwise derivative exists a.e. and is equal to its weak derivative, and f satisfies the fundamental theorem of calculus ($f(x) = f(0) + \int_0^x f'(t)\,dt$). (Since $p = n = 1$, the continuity of $f \in W^{1,1}(0,1)$ does not follow from the general Sobolev embedding theorem, but the one-dimensional case is particularly simple.)

The space $H^1(0,1)$ consists of functions that are Hölder continuous with exponent $\alpha = 1 - 1/2 = 1/2$, as can be seen directly from the absolute continuity of $f \in H^1(0,1)$ and the Cauchy-Schwartz inequality:

$$|f(x) - f(y)| = \left| \int_y^x f'(t)\,dt \right| \leq \|f'\|_{L^2} |x-y|^{1/2}.$$

Moreover, bounded sets in $H^1(0,1)$ are precompact in $C([0,1])$, as can be seen directly from the previous Hölder estimate and the Arzelà-Ascoli theorem.

10. The Poincaré inequality

In general, the smooth functions with compact support are not dense in $W^{k,p}(\Omega)$. We define

$$W_0^{k,p}(\Omega) = \overline{C_c^\infty(\Omega)}$$

where the closure is taken with respect to the $W^{k,p}$-norm. Intuitively, $W_0^{k,p}(\Omega)$ is the space of Sobolev functions that vanish on the boundary. For $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, we have $u \in W_0^{1,p}(\Omega)$ if and only if $u = 0$ on $\partial\Omega$.

Example: Suppose $n = 1$ and $\Omega = (0,1)$. The space $W_0^{1,1}(0,1)$ consists of all absolutely continuous functions $u : [0,1] \to \mathbb{C}$ such that $u(0) = 0$, $u(1) = 0$. The space $W_0^{2,1}(0,1)$ consists of all $C^1([0,1])$-functions such that u' is absolutely continuous and $u(0) = u'(0) = 0$, $u(1), u'(1) = 0$.

$H_0^1(0,1)$ consists of all functions $u \in H^1(0,1)$ such that $u(0) = 0$, $u(1) = 0$. The space $H^2(0,1) \cap H_0^1(0,1)$ consists of all functions $u \in H^2(0,1)$ such that $u(0) = 0$, $u(1) = 0$.

THEOREM 9 (Poincaré). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 \leq p < \infty$. Then there exists $C(p,\Omega)$ such that for all $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \leq C \|Du\|_{L^p(\Omega)}$$

This result means that we can use $\|Du\|_{L^p}$ as an equivalent norm on $W_0^{1,p}(\Omega)$ (but not on $W^{1,p}(\Omega)$ since $\|Du\|_{L^p}$ does not control the L^p-norm of constants).