Convolution Problem
Analysis Prelim Workshop
Fall 2012

Problem 3. (Spring, 2012) For $\epsilon > 0$, we set

$$\eta_\epsilon(x) = \frac{1}{\pi} \sin\left(\frac{\epsilon \pi x}{x^2 + \epsilon^2}\right) \frac{\epsilon}{x^2 + \epsilon^2},$$

and define the convolution for $u \in L^2(\mathbb{R})$:

$$\eta_\epsilon * u(x) = \int_{\mathbb{R}} \eta_\epsilon(x - y) u(y)\, dy.$$

For $\epsilon > 0$, prove that $\sqrt{\epsilon}(\eta_\epsilon * u)(x)$ is bounded as a function of x and ϵ and that $\eta_\epsilon * u$ converges strongly in $L^2(\mathbb{R})$ as $\epsilon \to 0^+$. What is the limit?

Solution. We have

$$\eta_\epsilon(x) = \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right), \qquad \eta(x) = \frac{1}{\pi}\sin\left(\frac{\pi x}{x^2+1}\right)\frac{1}{x^2+1}.$$

Since

$$|\eta(x)| \leq \frac{|x|}{(x^2+1)^2},$$

we see that $\eta \in L^1(\mathbb{R})$, and since η is an odd function,

$$\int_{\mathbb{R}} \eta(x)\, dx = 0.$$

Moreover, for all $\epsilon > 0$,

$$\|\eta_\epsilon\|_{L^1(\mathbb{R})} = \|\eta\|_{L^1(\mathbb{R})}, \qquad \|\eta_\epsilon\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{\epsilon}}\|\eta\|_{L^2(\mathbb{R})}.$$

By Young's inequality, $\|\eta_\epsilon * u\|_{L^\infty(\mathbb{R})} \leq \|\eta_\epsilon\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}$, so

$$\sqrt{\epsilon}\, \|\eta_\epsilon * u\|_{L^\infty(\mathbb{R})} \leq \|\eta\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}$$

is a bounded function of (x, ϵ).

We claim that $\eta_\epsilon * u \to 0$ in $L^2(\mathbb{R})$ as $\epsilon \to 0^+$. This is a consequence of the mollification theorem (see Theorem 8.14 in *Real Analysis* by Folland, which includes the case $\int \eta\, dx = 0$). The proof in Folland uses the L^p-continuity of translations of L^p-functions and the Minkowski integral inequality. We'll give an alternative proof that doesn't rely on the Minkowski integral inequality.

Theorem 1. *Suppose that* $\eta \in L^1(\mathbb{R}^n)$*, with* $\int \eta\, dx = 0$*, and* $u \in L^p(\mathbb{R}^n)$*, where* $1 \le p < \infty$*. For* $\epsilon > 0$ *let* $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$*. Then* $\eta_\epsilon * u \to 0$ *in* $L^p(\mathbb{R}^n)$ *as* $\epsilon \to 0^+$*.*

Proof. First we show, by density, that it is sufficient to prove the result for continuous functions with compact support.

If $\eta, \rho \in L^1$ and $u, v \in L^p$, then by Young's inequality and the fact that $\|\eta_\epsilon\|_{L^1} = \|\eta\|_{L^1}$, we get

$$\|\eta_\epsilon * u - \rho_\epsilon * v\|_{L^p} \le \|\eta - \rho\|_{L^1}\|u\|_{L^p} + \|\rho\|_{L^1}\|u - v\|_{L^p}.$$

If $\eta \in L^1$, $u \in L^p$, and $\delta > 0$ is given, choose $\rho, v \in C_c(\mathbb{R}^n)$ such that

$$\|\eta - \rho\|_{L^1} < \frac{\delta}{4\|u\|_{L^p}}, \qquad \|u - v\|_{L^p} < \frac{\delta}{4\|\rho\|_{L^1}},$$

which is possible since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Then

$$\|\eta_\epsilon * u\|_{L^p} \le \|\eta_\epsilon * u - \rho_\epsilon * v\|_{L^p} + \|\rho_\epsilon * v\|_{L^p} < \|\rho_\epsilon * v\|_{L^p} + \frac{\delta}{2}.$$

Assuming the result holds for continuous functions with compact support, we choose $\epsilon' > 0$ such that $\|\rho_\epsilon * v\|_{L^p} < \delta/2$ for $0 < \epsilon < \epsilon'$. Then

$$\|\eta_\epsilon * u\|_{L^p} < \delta \qquad \text{for } 0 < \epsilon < \epsilon',$$

which proves the result for $\eta \in L^1$, $u \in L^p$.

Now suppose that $\eta, u \in C_c(\mathbb{R}^n)$. By rescaling ϵ, we may assume without loss of generality that $\operatorname{supp}\eta \subset B_1$, where $B_r \subset \mathbb{R}^n$ is the ball of radius r and center 0. Since the integral of η is zero and $\operatorname{supp}\eta_\epsilon \subset B_\epsilon$,

$$\eta_\epsilon * u(x) = \int_{B_\epsilon} \eta_\epsilon(y)\,[u(x-y) - u(x)]\;dy.$$

If $\operatorname{supp} u \subset B_r(0)$, then $\operatorname{supp}\eta_\epsilon * u \subset B_{r+\epsilon}$. Moreover, since $u \in C_c(\mathbb{R}^n)$ is uniformly continuous, for every $\delta > 0$ there exists $0 < \epsilon' \le 1$ such that

$$\sup_{|y-z|<\epsilon'} |u(y) - u(z)| < \frac{\delta}{|B_{r+1}|^{1/p}\|\eta\|_{L^1}}$$

where $|B_r|$ denotes the measure of B_r. It follows that, for $0 < \epsilon < \epsilon'$,

$$|\eta_\epsilon * u(x)| \le \left(\sup_{|y-z|<\epsilon'} |u(y) - u(z)|\right)\int |\eta_\epsilon|\,dy < \frac{\delta}{|B_{r+1}|^{1/p}},$$

which implies that

$$\|\eta_\epsilon * u\|_{L^p} = \left(\int_{B_{r+\epsilon}} |\eta_\epsilon * u(x)|^p \, dx \right)^{1/p} < \delta.$$

This proves the result. ▮

For completeness, we give a Hölder-inequality proof of Young's theorem for convolution with an L^1-functions.

Theorem 2. *Suppose that $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, where $1 \le p \le \infty$. Then $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}$.*

Proof. The result is immediate for $p = \infty$, since

$$|f * g(x)| = \left| \int f(y) g(x-y) \, dy \right| \le \|f\|_{L^1} \|g\|_{L^\infty}.$$

For $1 \le p < \infty$, Hölder's inequality gives

$$\begin{aligned} \int |f(x-y) g(y)| \, dy &= \int |f(x-y)|^{1/p'} |f(x-y)|^{1/p} |g(y)| \, dy \\ &\le \left(\int |f(x-y)| \, dx \right)^{1/p'} \left(\int |f(x-y)| \, |g(y)|^p \, dy \right)^{1/p} \\ &\le \|f\|_{L^1}^{1/p'} \left(\int |f(x-y)| \, |g(y)|^p \, dy \right)^{1/p} \end{aligned}$$

Using Fubini's theorem, we then get that

$$\begin{aligned} \left(\int |f * g(x)|^p \, dx \right)^{1/p} &\le \|f\|_{L^1}^{1/p'} \left[\int \left(\int |f(x-y)| \, |g(y)|^p \, dy \right) dx \right]^{1/p} \\ &\le \|f\|_{L^1}^{1/p'} \left[\int \left(\int |f(x-y)| \, |g(y)|^p \, dx \right) dy \right]^{1/p} \\ &\le \|f\|_{L^1} \|g\|_{L^p}, \end{aligned}$$

which proves the result. In particular, since $f * g \in L^p(\mathbb{R}^n)$, the integral defining the convolution converges for x pointwise a.e. in $\mathbb{R}^n$. ▮