Analysis Preliminary Exam
Spring 2014: Solutions

1. Let (g_n) be a sequence of absolutely continuous functions on $[0, 1]$ with $|g_n(0)| \leq 1$. Suppose also that for each n, $|g_n'(x)| \leq 1$ for Lebesgue almost everywhere $x \in [0, 1]$. Show that there is a subsequence of (g_n) that converges uniformly to a Lipschitz function on $[0, 1]$.

Solution.

- Since g_n is absolutely continuous, it is differentiable pointwise a.e. and, by the fundamental theorem of calculus for the Lebesgue integral,

$$g_n(x) = g_n(0) + \int_0^x g_n'(t)\,dt, \qquad 0 \leq x \leq 1. \tag{1}$$

- It follows that

$$\|g_n\|_\infty = \sup_{x\in[0,1]} |g_n(x)| \leq |g_n(0)| + \int_0^1 |g_n'(t)|\,dt \leq 2,$$

  so $\{g_n\}$ is a bounded subset of $C([0, 1])$.

- It also follows from (1), and the assumption $|g_n'| \leq 1$, that

$$|g_n(x) - g_n(y)| = \left|\int_y^x g_n'(t)\,dt\right| \leq |x - y|, \tag{2}$$

  which shows that $\{g_n\}$ is equicontinuous (i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|g_n(x) - g_n(y)| < \epsilon$ for every $n \in \mathbb{N}$; take $\delta = \epsilon$).

- Since the set $\{g_n\}$ is bounded and equicontinuous, the Arzelà-Ascoli theorem implies that it is precompact in $C([0, 1])$, so (g_n) has a uniformly convergent subsequence (g_{n_k}).

- If $g_{n_k} \to g$ uniformly, then it converges pointwise and

$$|g(x) - g(y)| = \lim_{k\to\infty} |g_{n_k}(x) - g_{n_k}(y)| \leq |x - y|$$

  from (2), so the limit g is Lipschitz.

2. Let T be a linear operator from a Banach space X to a Hilbert space H. Show that T is bounded if and only if $x_n \rightharpoonup x$ implies that $T(x_n) \rightharpoonup T(x)$ for every weakly convergent sequence (x_n) in X.

Solution.

- Suppose that $T : H \to H$ is bounded. If $y \in H$, then $\phi : x \mapsto \langle Tx, y\rangle$ is a bounded linear functional on X, since it is obviously linear and

$$|\langle Tx, y\rangle| \leq \|T\|\|y\|\|x\|,$$

  so $\|\phi\| \leq \|T\| \, \|y\|$. If $x_n \rightharpoonup x$ in X, then $\phi(x_n) \to \phi(x)$ by the definition of weak convergence. It follows that

$$\langle Tx_n, y\rangle \to \langle Tx, y\rangle$$

  for every $y \in H$, meaning that $Tx_n \rightharpoonup Tx$ in H.

- To prove the converse, suppose that T is not bounded. Then for every $n \in \mathbb{N}$, there exists $y_n \in X$ with $\|y_n\| = 1$ and $\|Ty_n\| \geq n^2$. Let $x_n = y_n/n$. Then $x_n \to 0$ strongly and therefore weakly in X, but Tx_n does not converge weakly in H since $\|Tx_n\| \geq n$ and every weakly convergent sequence is bounded.

3. Let $f, f_k : E \to [0, +\infty)$ be non-negative Lebesgue integrable functions on a measurable set $E \subset \mathbb{R}^n$. If (f_k) converges to f pointwise almost everywhere

$$\int_E f_k\, dx \to \int_E f\, dx,$$

show that

$$\int_E |f - f_k|\, dx \to 0.$$

Solution.

- Since $f, f_k \geq 0$, we have

  $$f + f_k - |f - f_k| \geq 0,$$

  so Fatou's lemma implies that

  $$\int_E \liminf_{k\to\infty} (f + f_k - |f - f_k|)\, dx \leq \liminf_{k\to\infty} \int_E (f + f_k - |f - f_k|)\, dx.$$

- Since $f_k \to f$ pointwise a.e. and $\int_E f_k\, dx \to \int_E f\, dx$, this gives

  $$\int_E 2f\, dx \leq 2\int_E f\, dx - \limsup_{k\to\infty} \int_E |f - f_k|\, dx,$$

  so

  $$\limsup_{k\to\infty} \int_E |f - f_k|\, dx \leq 0.$$

  Since $|f - f_k| \geq 0$, it follows that $\lim \int_E |f - f_k|\, dx = 0$.

Remark. It's essential to use the positivity of f_k here. For example, if $f_k : (0, 1) \to \mathbb{R}$ is defined (for $k \geq 2$) by

$$f_k(x) = \begin{cases} -k & \text{if } 0 < x < 1/k, \\ 0 & \text{if } 1/k \leq x \leq 1 - 1/k, \\ k & \text{if } 1 - 1/k < x < 1, \end{cases}$$

then $f_k \to 0$ pointwise and $\int_0^1 f_k\, dx \to 0$, but $\int_0^1 |f_k|\, dx \to 2$.

4. Let P_1 and P_2 be a pair of orthogonal projections onto H_1 and H_2, respectively, where H_1 and H_2 are closed subspaces of a Hilbert space H. Prove that P_1P_2 is an orthogonal projection if and only if P_1 and P_2 commute. In that case, prove that P_1P_2 is the orthogonal projection onto $H_1 \cap H_2$.

Solution.

- A linear operator $P : H \to H$ is an orthogonal projection if and only if $P^2 = P$ (projection) and $P^* = P$ (orthogonal).

- If P_1P_2 is an orthogonal projection, then

$$P_1P_2 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1,$$

  where we use the identity $(AB)^* = B^*A^*$ and the orthogonality of the P_i, so P_1 and P_2 commute.

- If P_1, P_2 commute and $P = P_1P_2$, then

$$\begin{aligned} P^2 &= P_1P_2P_1P_2 = P_1^2P_2^2 = P_1P_2 = P, \\ P^* &= (P_1P_2)^* = P_2^*P_1^* = P_2P_1 = P_1P_2 = P, \end{aligned}$$

  so P_1P_2 is an orthogonal projection.

- If P_1, P_2 commute and $P = P_1P_2$, then $\operatorname{ran} P \subset \operatorname{ran} P_1 = H_1$ and $\operatorname{ran} P = \operatorname{ran}(P_2P_1) \subset \operatorname{ran} P_2 = H_2$, so $\operatorname{ran} P \subset H_1 \cap H_2$. To get the reverse inclusion, suppose that $x \in H_1 \cap H_2$. Then $P_1P_2x = P_1x = x$, since P_i is the identity operator on its range H_i, meaning that $x \in \operatorname{ran} P$ and $\operatorname{ran} P \supset H_1 \cap H_2$. It follows that P_1P_2 is the orthogonal projection onto $H_1 \cap H_2$.

5. Let H be a (separable) Hilbert space with orthonormal basis $\{f_k\}_{k=1}^{\infty}$. Prove that the linear operator $T : H \to H$ defined by

$$T(f_k) = \frac{1}{k} f_{k+1}, \qquad k \geq 1$$

is compact but has no eigenvalues.

Solution.

- For $x = \sum_{k=1}^{\infty} x_k f_k \in H$, we have

  $$Tx = \sum_{k=1}^{\infty} \frac{x_k}{k} f_{k+1}.$$

  If $Tx = \lambda x$ for $\lambda \in \mathbb{C}$, then

  $$\lambda x_1 = 0, \qquad \lambda x_{k+1} = \frac{x_k}{k} \quad \text{for every } k \in \mathbb{N}.$$

- If $\lambda \neq 0$, then $x_1 = 0$ and, by induction, $x_k = 0$ for every $k \in \mathbb{N}$. On the other hand, if $\lambda = 0$, then it follows immediately that $x_k = 0$. In either case, $x = 0$ and λ is not an eigenvalue.

- An operator $T : H \to H$ is compact if it is the operator-norm limit of compact operators. Let $T_n = P_n T$ where P_n is the orthogonal projection onto the subspace spanned by $\{f_1, f_2, \ldots, f_n\}$. Then T_n has finite rank, so it is compact.

- Moreover, we get that

  $$\begin{aligned}\|T - T_n\| &= \sup_{\|x\|=1} \|(T - T_n)x\| = \sup_{\|x\|=1} \left\| \sum_{k=n}^{\infty} \frac{x_k}{k} f_{k+1} \right\| \\ &= \sup_{\|x\|=1} \left(\sum_{k=n}^{\infty} \frac{|x_k|^2}{k^2} \right)^{1/2} \leq \frac{1}{n} \sup_{\|x\|=1} \left(\sum_{k=n}^{\infty} |x_k|^2 \right)^{1/2} \leq \frac{1}{n}.\end{aligned}$$

  It follows that $T_n \to T$ with respect to the operator norm, so T is compact.

- Alternatively, one can show directly that the image of the unit ball under T is precompact, either by showing it's totally bounded or by use of a diagonal argument.

6. Let $H_1 = L^2([-\pi,\pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with inner product

$$(F,G) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\theta})\overline{G(e^{i\theta})}\,dx.$$

Let H_2 be the space $L_2(\mathbb{R})$. Using the mapping

$$x \mapsto \frac{i-x}{i+x},$$

of $\mathbb{R}$ to the unit circle, show that:

(a) The correspondence $U : F \mapsto f$ with

$$f(x) = \frac{1}{\pi^{1/2}(i+x)} F\left(\frac{i-x}{i+x}\right)$$

gives a unitary mapping of H_1 to H_2.

b) As a result,

$$\left\{\pi^{-1/2}\left(\frac{i-x}{i+x}\right)^n \frac{1}{i+x}\right\}_{n=-\infty}^{\infty}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

Solution.

- (a) A mapping $U : H_1 \to H_2$ is unitary if and only if it is onto and preserves inner products. Let

$$e^{i\theta} = \frac{i-x}{i+x}, \qquad x = i\left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right) = \tan\left(\frac{\theta}{2}\right).$$

  Then $\theta \mapsto x$ defines a diffeomorphism of $(-\pi,\pi)$ onto $\mathbb{R}$, and

$$ie^{i\theta}\,d\theta = -\frac{2i}{(i+x)^2}\,dx, \quad \text{or} \quad d\theta = \frac{-2}{(i-x)(i+x)}\,dx.$$

- If $F, G \in L^2(-\pi,\pi)$, then

$$\begin{aligned}\langle F,G\rangle_{H_1} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F\left(\frac{i-x}{i+x}\right)\overline{G\left(\frac{i-x}{i+x}\right)}\,\frac{-2}{(i-x)(i+x)}\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\pi^{1/2}(i+x)}F\left(\frac{i-x}{i+x}\right)\overline{\frac{1}{\pi^{1/2}(i+x)}G\left(\frac{i-x}{i+x}\right)}\,dx \\ &= \langle UF, UG\rangle_{H_2},\end{aligned}$$

  and similarly $\langle f,g\rangle_{H_2} = \langle U^{-1}f, U^{-1}g\rangle_{H_1}$, which shows that U is unitary.

- (b) A unitary transformation maps an orthonormal basis to an orthonormal basis. The functions

$$\pi^{-1/2}\left(\frac{i-x}{i+x}\right)^n \frac{1}{i+x} = U(e^{in\theta})$$

are the images of the standard orthonormal basis vectors $\{e^{in\theta} : n \in \mathbb{Z}\}$ of $L^2(-\pi,\pi)$, so they form an orthonormal basis of $L^2(\mathbb{R})$.