Solutions 1
Analysis Prelim Workshop
Fall 2013

Problem 4. (Spring, 2013) Prove that

$$\lim_{\epsilon \to 0^+} \int_0^\infty \frac{\epsilon}{\epsilon^2 + x} \sin\left(\frac{1}{x}\right) dx = 0.$$

Solution. We give two solutions.

(a) Split the integral into two parts and estimate them separately:

$$\left|\int_0^1 \frac{\epsilon}{\epsilon^2 + x} \sin\left(\frac{1}{x}\right) dx\right| \le \int_0^1 \frac{\epsilon}{\epsilon^2 + x} dx = \epsilon \int_0^{1/\epsilon^2} \frac{1}{1+y} dy = \epsilon \log(1 + \epsilon^{-2});$$
$$\left|\int_1^\infty \frac{\epsilon}{\epsilon^2 + x} \sin\left(\frac{1}{x}\right) dx\right| \le \int_1^\infty \frac{\epsilon}{x} \cdot \frac{1}{x} dx = \epsilon,$$

where we use $|\sin(1/x)| \le 1$ and the substitution $x = \epsilon^2 y$ in the first integral, and $|\sin(1/x)| \le 1/x$ in the second integral. It follows that

$$\left|\int_0^\infty \frac{\epsilon}{\epsilon^2 + x} \sin\left(\frac{1}{x}\right) dx\right| \le \epsilon \log(1 + \epsilon^{-2}) + \epsilon \to 0 \qquad \text{as } \epsilon \to 0^+.$$

(b) Apply Hölder's inequality with $1 < p < 2$ and $2 < p' < \infty$:

$$\left|\int_0^\infty \frac{\epsilon}{\epsilon^2 + x} \sin\left(\frac{1}{x}\right) dx\right| \le \left(\int_0^\infty \frac{\epsilon^p}{(\epsilon^2 + x)^p} dx\right)^{1/p} \left(\int_0^\infty \left|\sin\left(\frac{1}{x}\right)\right|^{p'} dx\right)^{1/p'}.$$

As in (a), we have

$$\int_0^\infty \left|\sin\left(\frac{1}{x}\right)\right|^{p'} dx = \int_0^1 \left|\sin\left(\frac{1}{x}\right)\right|^{p'} dx + \int_1^\infty \left|\sin\left(\frac{1}{x}\right)\right|^{p'} dx \le 1 + \frac{1}{p'-1},$$
$$\int_0^\infty \frac{\epsilon^p}{(\epsilon^2 + x)^p} dx = \epsilon^{2-p} \int_0^\infty \frac{1}{(1+y)^p} dy = \frac{\epsilon^{2-p}}{p-1},$$

so

$$\left|\int_0^\infty \frac{\epsilon}{\epsilon^2 + x} \sin\left(\frac{1}{x}\right) dx\right| \le C(p, p')\epsilon^{(2-p)/p} \to 0 \qquad \text{as } \epsilon \to 0^+.$$

Problem 1. (Spring, 2011) Let $f_n(x) = ne^{-nx}$. Prove that $f_n \not\rightharpoonup f$ in $L^1(0,1)$.

Solution. Suppose for contradiction that a weak limit $f \in L^1(0,1)$ exists, meaning that

$$\int_0^1 ne^{-nx} g(x)\, dx \to \int_0^1 f(x)g(x)\, dx \qquad \text{for all } g \in L^\infty(0,1).$$

For $0 < \epsilon < 1$, let $g_\epsilon = \chi_{(\epsilon,1)} \operatorname{sgn} f$, where

$$\chi_{(\epsilon,1)}(x) = \begin{cases} 1 & \text{if } \epsilon < x < 1, \\ 0 & \text{if } 0 < x < \epsilon \end{cases}, \qquad \operatorname{sgn} f(x) = \begin{cases} 1 & \text{if } f_n(x) \geq 0, \\ -1 & \text{if } f_n(x) < 0. \end{cases}$$

Then $g_\epsilon \in L^\infty(0,1)$ and

$$\left| \int_0^1 ne^{-nx} g_\epsilon(x)\, dx \right| \leq \int_\epsilon^1 ne^{-nx}\, dx = e^{-n\epsilon} - e^{-n} \to 0 \qquad \text{as } n \to \infty.$$

It follows that

$$\int_0^1 f g_\epsilon\, dx = \int_\epsilon^1 |f|\, dx = 0$$

for all $0 < \epsilon < 1$, which implies that $f = 0$ pointwise a.e. on $(0,1)$, so 0 is the only possible weak limit.

On the other hand, consider $g = 1 \in L^\infty(0,1)$. We have

$$\lim_{n\to\infty} \int_0^1 ne^{-nx} \cdot 1\, dx = \lim_{n\to\infty} \left(1 - e^{-n}\right) = 1 \neq \int_0^1 0 \cdot 1\, dx,$$

which contradicts the weak convergence of (f_n) to 0.