Solutions 2
Analysis Prelim Workshop
Fall 2013

Problem 1. (Spring, 2013) Consider the Hilbert space $L^2(0,1)$ with inner product $\langle f, h\rangle = \int_0^1 fh\,dx$. Let

$$V = \left\{ f \in L^2(0,1) : \int_0^1 x f(x)\,dx = 0 \right\}$$

and $g(x) = 1$. Find the closest element to g in V.

Solution. By the projection theorem, the closest element to g in V is $h = Pg$ where $P : L^2(0,1) \to L^2(0,1)$ is the orthogonal projection onto V. Moreover, the error $g - h \in V^\perp$ is orthogonal to V.
Since $V = \langle x\rangle^\perp$, we have $V^\perp = \langle x\rangle^{\perp\perp} = \langle x\rangle$. It follows that $g - h = kx$ for some constant k, so $h(x) = 1 - kx$. Then $h \in V$ if

$$\int_0^1 x h(x)\,dx = \int_0^1 (x - kx^2)\,dx = \frac{1}{2} - \frac{1}{3}k = 0,$$

so $k = 3/2$ and the closest element is

$$h(x) = 1 - \frac{3}{2}x.$$

Problem 1. (Fall, 2012) Show that the space of continuous functions on $[0,1]$ with the sup-norm $\|f\| = \max |f(x)|$ is not a Hilbert space.

Solution. A norm $\|\cdot\|$ is derived from an inner product if and only if it satisfies the parallelogram law

$$\|f - g\|^2 + \|f + g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right),$$

but this fails for the sup-norm.
For example, consider $f(x) = 1$, $g(x) = x$. Then

$$\|f\| = 1, \quad \|g\| = 1, \quad \|f - g\| = 1, \quad \|f + g\| = 2,$$

and

$$\|f - g\|^2 + \|f + g\|^2 \neq 2\left(\|f\|^2 + \|g\|^2\right).$$

Problem 4. (Fall, 2012) A bounded operator on a Hilbert space is normal if it commutes with its adjoint. Define $V : L^2(0,1) \to L^2(0,1)$ by

$$(Vf)(x) = \int_0^x f(t)\,dt$$

and $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$. Are either of V, S normal?

Solution. Neither V nor S is normal.
To compute V^*, we consider for $f, g \in L^2(0,1)$

$$\langle Vf, g\rangle = \int_0^1 \left(\int_0^x f(t)\,dt\right)\overline{g(x)}\,dx = \int_0^1 f(t)\overline{\left(\int_t^1 g(x)\,dx\right)}\,dt = \langle f, V^*g\rangle,$$

where

$$(V^*g)(x) = \int_x^1 g(t)\,dt.$$

The exchange in the order of integration is justified by Fubini's theorem, since

$$\int_0^1 \left(\int_0^x |f(t)|\,dt\right)|g(x)|\,dx \le \left(\int_0^1 |f(t)|\,dt\right)\left(\int_0^1 |g(x)|\,dx\right) \le \|f\|_{L^2}\|g\|_{L^2}$$

is finite. If $f = 1$, then

$$(Vf)(x) = \int_0^x 1\,dt = x, \qquad (V^*Vf)(x) = \int_x^1 t\,dt = \frac{1}{2}(1 - x^2),$$
$$(V^*f)(x) = \int_x^1 1\,dt = 1 - x, \qquad (VV^*f)(x) = \int_0^x (1-t)\,dt = x - \frac{1}{2}x^2,$$

so $V^*V \ne VV^*$.
To compute S^*, we consider for $x = (x_i)$ and $y = (y_i)$ in $\ell^2(\mathbb{N})$

$$\langle Sx, y\rangle = \sum_{i=2}^{\infty} x_{i-1}\bar{y}_i = \sum_{i=1}^{\infty} x_i\bar{y}_{i+1} = \langle x, S^*y\rangle,$$

where $S^*y = (y_2, y_3, y_4, \dots)$. If $x = (1, 0, 0, \dots)$, then

$$Sx = (0, 1, 0, 0, \dots), \qquad S^*Sx = (1, 0, 0, \dots),$$
$$S^*x = (0, 0, 0, \dots), \qquad SS^*x = (0, 0, 0, \dots),$$

so $S^*S \ne SS^*$.