Solutions 3
Analysis Prelim Workshop
**Fall 2013**

**Problem 2.** (Spring, 2013) Let $(A_n)$ be a sequence of bounded linear operators on a Banach space that converges in norm to an operator $A$, such that the $A_n$ have the same spectrum $\sigma_0 = \sigma(A_n)$. Show that $\sigma_0 \subset \sigma(A)$.

**Solution.** We prove the equivalent statement that if $\lambda \in \rho(A)$, then $\lambda \in \rho_0$, where $\rho(A)$ is the resolvent set of $A$ and $\rho_0 = \mathbb{C} \setminus \sigma_0$ is the resolvent set of the $A_n$.
If $\lambda \in \rho(A)$, then $\lambda I - A$ is invertible and

$$\lambda I - A_n = (\lambda I - A)(I - T_n), \qquad T_n = (\lambda I - A)^{-1}(A_n - A).$$

Since

$$\|T_n\| \leq \left\|(\lambda I - A)^{-1}\right\| \|A_n - A\| \to 0 \qquad \text{as } n \to \infty,$$

we have $\|T_n\| < 1$ for sufficiently large $n$, and then $I - T_n$ is invertible by the Neumann series $(I - T_n)^{-1} = I + T_n + T_n^2 + \dots$. It follows that $\lambda I - A_n$ is invertible and $\lambda \in \rho_0$.

**Problem 2.** (Fall, 2012) Suppose that $\phi : [0,1] \to \mathbb{R}$ is a continuous function, and the linear operator $T : L^2(0,1) \to L^2(0,1)$ is given by

$$(Tf)(x) = \phi(x) \int_0^1 \phi(t) f(t)\, dt.$$

(a) Show that $T$ is self-adjoint. (b) Show that there exists a number $\lambda \geq 0$ such that $T^2 = \lambda T$; (c) Find the spectral radius $r(T)$ of $T$.

**Solution.** (a) For $f, g \in L^2(0,1)$, we compute that

$$\begin{aligned}
\langle Tf, g\rangle &= \int_0^1 (Tf)(x)\overline{g(x)}\, dx \\
&= \int_0^1 \phi(x) \left(\int_0^1 \phi(t) f(t)\, dt\right) \overline{g(x)}\, dx \\
&= \int_0^1 f(t) \overline{\left(\phi(t) \int_0^1 \phi(x) g(x)\, dx\right)}\, dt \\
&= \langle f, Tg\rangle,
\end{aligned}$$

which proves that $T$ is self-adjoint.

(b) For $f \in L^2(0,1)$, we compute that

$$
\begin{aligned}
(T^2 f)(x) &= \phi(x) \int_0^1 \phi(t)(Tf)(t)\, dt \\
&= \phi(x) \int_0^1 \phi(t) \left( \phi(t) \int_0^1 \phi(s) f(s)\, ds \right) dt \\
&= \left( \int_0^1 \phi^2(t)\, dt \right) \left( \phi(x) \int_0^1 \phi(s) f(s)\, ds \right) \\
&= \lambda (Tf)(x),
\end{aligned}
$$

where

$$\lambda = \int_0^1 \phi^2(t)\, dt \geq 0.$$

(c) Since $T$ is self-adjoint, $\|T^2\| = \|T\|^2$ and $r(T) = \|T\|$. From (b), $\|T^2\| = \lambda \|T\|$, so $\|T\|^2 = \lambda \|T\|$ and $\|T\| = \lambda$ (since $\|T\| \neq 0$ unless $\lambda = 0$). It follows that $r(T) = \lambda$.

**Problem 3.** (Fall, 2012) Let $T$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. (a) Show that if $\|T\| \leq 1$, then $T$ and its adjoint $T^*$ have the same fixed points i.e., $Tx = x$ if and only if $T^* x = x$. (b) Let $\lambda$ be an eigenvalue of $T$. Is it true that $\bar{\lambda}$ must be eigenvalue of $T^*$? Is it true that $\bar{\lambda}$ must be in the spectrum of $T^*$?

**Solution.** (a) Suppose that $Tx = x$. Then, since $\|T^*\| = \|T\| \leq 1$,

$$
\begin{aligned}
\|T^* x - x\|^2 &= \langle T^* x - x, T^* x - x \rangle \\
&= \langle T^* x, T^* x \rangle - \langle T^* x, x \rangle - \langle x, T^* x \rangle + \langle x, x \rangle \\
&= \|T^* x\|^2 - \langle x, Tx \rangle - \langle Tx, x \rangle + \|x\|^2 \\
&= \|T^* x\|^2 - \|x\|^2 \\
&\leq \|T^*\|^2 \|x\|^2 - \|x\|^2 \\
&\leq 0.
\end{aligned}
$$

It follows that $\|T^* x - x\| = 0$, so $T^* x = x$. Then same argument applied to $T^*$ instead of $T$ shows that $T^* x = x$ implies that $Tx = x$, so $T$ and $T^*$ have the same fixed points.

(b) It is not true that if $\lambda$ is an eigenvalue of $T$, then $\bar{\lambda}$ must be eigenvalue of $T^*$. For example, let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the left shift operator, whose adjoint is the right shift operator,

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots), \quad T^*(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots).$$

Then 0 is an eigenvalue of $T$, with eigenvector $(1, 0, 0, \dots)$, but $T^*$ is one-to-one and 0 is not an eigenvalue.

It is true that if $\lambda$ is an eigenvalue of $T$, then $\bar{\lambda}$ is in the spectrum of $T^*$. Since

$$\overline{\operatorname{ran}(T^* - \bar{\lambda} I)} = \ker(T - \lambda I)^{\perp},$$

the range of $T^* - \bar{\lambda} I$ is a proper, non-dense subspace of $\mathcal{H}$. If $\bar{\lambda}$ is not in the point spectrum of $T^*$, then it is in the residual spectrum.

**Problem 2.** (Fall, 2010) Define the Fourier transform $\hat{\phi} = \mathcal{F}\phi$ of a Schwartz function $\phi \in \mathcal{S}(\mathbb{R})$ by

$$\hat{\phi}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x) e^{-ix\xi}\, dx, \qquad \phi(x) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) e^{ix\xi}\, d\xi.$$

and the Fourier transform $\hat{T} = \mathcal{F}T$ of a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$ by $\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. Compute the Fourier transform of the function

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

**Solution.** Using the definition of the Fourier transform given above, we have

$$\mathcal{F}[f'] = i\xi \mathcal{F} f, \qquad \mathcal{F}[xf] = i(\mathcal{F} f)'.$$

Moreover,

$$\mathcal{F}\delta = \frac{1}{2\pi}, \qquad \mathcal{F}[1] = \delta, \qquad \mathcal{F}[x] = i\delta'.$$

We write $f$ as

$$f(x) = \frac{1}{2}\left[x + x\sigma(x)\right] \qquad \sigma(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

It follows that

$$\hat{f}(\xi) = \frac{i}{2}\left[\delta'(\xi) + \hat{\sigma}'(\xi)\right].$$

To compute $\hat{\sigma}$, we note that the distributional derivative of $\sigma$ is

$$\sigma' = 2\delta.$$

Taking the Fourier transform of this equation, we get

$$i\xi\hat{\sigma}(\xi) = \frac{1}{\pi}$$

The general distributional solution of this equation is

$$\hat{\sigma}(\xi) = -\frac{i}{\pi}\,\mathrm{p.v.}\frac{1}{\xi} + c\delta(\xi)$$

where $c$ is an arbitrary constant. (Here, we use $\xi \cdot \mathrm{p.v.}(1/\xi) = 1$ and the fact that $\xi T(\xi) = 0$ implies $T(\xi) = c\delta(\xi)$. Give proofs!)

We determine $c$ by symmetry considerations. Define $R\phi(x) = \phi(-x)$ for test functions and $\langle Rf, \phi\rangle = \langle f, R\phi\rangle$ for distributions. We say that $f \in \mathcal{S}'$ is odd if $Rf = -f$ and even if $Rf = f$. Then $\mathrm{p.v.}(1/\xi)$ is odd and $\delta$ is even; moreover, $\widehat{Rf} = R\hat{f}$, so $\hat{f}$ is odd if $f$ is odd. (Give proofs!) Since $\sigma$ is odd, $\hat{\sigma}$ is odd, so we must have $c = 0$, and

$$\hat{\sigma}(\xi) = -\frac{i}{\pi}\,\mathrm{p.v.}\frac{1}{\xi}.$$

It follows that

$$\hat{f}(\xi) = \frac{i}{2}\delta'(\xi) + \frac{1}{2\pi}\frac{d}{d\xi}\mathrm{p.v.}\frac{1}{\xi}.$$

Finally, we note that the derivative of the principal value distribution can be expressed explicitly as a finite-part distribution

$$\frac{d}{d\xi}\mathrm{p.v.}\frac{1}{\xi} = -\mathrm{f.p.}\frac{1}{\xi^2}$$

where

$$\begin{aligned}\langle \mathrm{f.p.}\frac{1}{\xi^2}, \phi\rangle &= \lim_{\epsilon\to 0^+}\left[\int_{|\xi|>\epsilon}\frac{\phi(\xi)}{\xi^2}\,d\xi - \frac{2\phi(0)}{\epsilon}\right] \\ &= \int_0^\infty \frac{\phi(\xi) - 2\phi(0) + \phi(-\xi)}{\xi^2}\,d\xi.\end{aligned}$$

(The proof that this is a distribution and that it is the distributional derivative of $\mathrm{p.v.}(1/\xi)$ is left as another exercise.) Hence

$$\hat{f}(\xi) = \frac{i}{2}\delta'(\xi) - \frac{1}{2\pi}\mathrm{f.p.}\frac{1}{\xi^2}.$$