SOLUTIONS 4
ANALYSIS PRELIM WORKSHOP
Fall 2013

Problem 5. (Spring, 2013) Prove that the image of the space $C^k(\mathbb{T})$ of k-times continuously differentiable functions on the unit circle under the Fourier transform is contained in the set of sequences satisfying $c_n = o(|n|^{-k})$ and contains the set of sequences satisfying $c_n = o(|n|^{-k-1-\epsilon})$ for some $\epsilon > 0$.

Solution. First, suppose that $f \in C^k(\mathbb{T})$ has the Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-inx}\,dx.$$

Integrating by parts k-times, we get that

$$c_n = \frac{1}{2\pi}\frac{1}{(in)^k}\int_{\mathbb{T}} f^{(k)}(x)e^{-inx}\,dx.$$

The Riemann-Lebesgue lemma implies that

$$\frac{1}{2\pi}\int_{\mathbb{T}} f^{(k)}(x)e^{-inx}\,dx \to 0 \qquad \text{as } |n| \to \infty,$$

which proves that $c_n = o(|n|^{-k})$.

Second, suppose that $c_n = o(|n|^{-k-1-\epsilon})$ as $|n| \to \infty$ for some $\epsilon > 0$. Then there is a constant M such that $|n|^{k+1+\epsilon}|c_n| \le M$ for all $n \in \mathbb{Z}$. If $1 \le j \le k$, then

$$\sum_{n\in\mathbb{Z}} |n|^j |c_n| \le M \sum_{n\neq 0} \frac{1}{|n|^{k-j+1+\epsilon}} < \infty,$$

since $k - j + 1 + \epsilon > 1$, so the Fourier series

$$\sum_{n\in\mathbb{Z}} (in)^j c_n e^{inx}$$

converges uniformly to a continuous function f_j (by the M-test). Since the convergence is uniform, a standard theorem from calculus implies that we can differentiate the Fourier series of f term by term and $f^{(j)} = f_j$, which proves that $f \in C^k(\mathbb{T})$.

Problem 4. (Fall, 2012) The heat kernel on $\mathbb{R}^3$ is given by

$$H_t(x) = \frac{1}{(4\pi t)^{3/2}} e^{-|x|^2/(4t)}.$$

Prove that if $u \in L^3(\mathbb{R}^3)$, then $t^{1/2}\|H_t * u\|_{L^\infty(\mathbb{R}^3)} \to 0$ as $t \to 0^+$.

Solution. By Young's inequality, with $r = \infty$, $p = 3/2$ and $q = 3$ (so that $1 + 1/r = 1/p + 1/q$), we have

$$\|H_t * u\|_{L^\infty} \leq \|H_t\|_{L^{3/2}} \|u\|_{L^3}.$$

Using the substitution $x = t^{1/2}y$, $dx = t^{3/2}dy$, we compute that

$$\|H_t\|_{L^{3/2}} = \frac{1}{(4\pi t)^{3/2}} \left(\int_{\mathbb{R}^3} e^{-3|x|^2/(8t)}\, dx \right)^{2/3} = \frac{c}{t^{1/2}},$$

where

$$c = \frac{1}{(4\pi)^{3/2}} \left(\int_{\mathbb{R}^3} e^{-3|y|^2/8}\, dy \right)^{2/3}.$$

It follows that

$$t^{1/2}\|H_t * u\|_{L^\infty} \leq c\|u\|_{L^3}.$$

To prove that the limit is zero, we approximate u by $\phi \in C_c(\mathbb{R}^3)$. Since

$$\|H_t\|_{L^1} = a, \qquad a = \frac{1}{(4\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-|y|^2/4}\, dy,$$

(in fact, $a = 1$) we have

$$\begin{aligned} t^{1/2}\|H_t * u\|_{L^\infty} &\leq t^{1/2}\|H_t * (u - \phi)\|_{L^\infty} + t^{1/2}\|H_t * \phi\|_{L^\infty} \\ &\leq c\|u - \phi\|_{L^3} + a t^{1/2}\|\phi\|_{L^\infty}. \end{aligned}$$

Given any $\epsilon > 0$, pick $\phi \in C_c(\mathbb{R}^3)$ such that $c\|u - \phi\|_{L^3} < \epsilon/2$, which is possible since $C_c(\mathbb{R}^3)$ is dense in $L^3(\mathbb{R}^3)$, and then choose $\delta > 0$ such that $a\delta^{1/2}\|\phi\|_{L^\infty} < \epsilon/2$. Then $0 < t < \delta$ implies that $t^{1/2}\|H_t * u\|_{L^\infty} < \epsilon$, which proves the result.