Solutions 5
Analysis Prelim Workshop
Fall 2013

Problem 5. (Fall, 2011) Let $u(x) = (1 + |\log x|)^{-1}$. Prove that $u \in W^{1,1}(0,1)$ and $u(0) = 0$ but $(u/x) \notin L^1(0,1)$.

Solution. Since $u \in C^\infty(0,1)$ is smooth, its pointwise derivative $v = u'$,

$$v(x) = \frac{1}{x(1 + |\log x|)^2},$$

is also its weak derivative (i.e., $\int_0^1 u\phi'\,dx = -\int_0^1 v\phi\,dx$ for every $\phi \in C_c^\infty(0,1)$). The substitution $t = 1 + |\log x|$ gives

$$\int_0^1 \frac{1}{x(1 + |\log x|)^\alpha}\,dx = \int_1^\infty \frac{1}{t^\alpha}\,dt,$$

which is finite if $\alpha > 1$ and infinite if $\alpha \le 1$. It follows that $v \in L^1(0,1)$ and $u \in W^{1,1}(0,1)$. Moreover, u extends to an absolutely continuous function on $[0,1]$ with $u(0) = \lim_{x\to 0^+}(1 + |\log x|)^{-1} = 0$. The previous calculation (with $\alpha = 1$) shows that $(u/x) \notin L^1(0,1)$.

Problem 6. (Spring, 2011) Let $C^{0,\alpha}([0,1])$ be the Banach space of Hölder continuous functions on $[0,1]$ with exponent $0 < \alpha \le 1$ and norm

$$\|u\|_{C^{0,\alpha}} = \sup_{x\in[0,1]} |u(x)| + \sup_{x\ne y\in[0,1]} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

Prove that the closed unit ball $B = \left\{u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}} \le 1\right\}$ in $C^{0,\alpha}([0,1])$ is a compact subset of $C([0,1])$ with the sup-norm topology.

Solution. By the Arzelà-Ascoli theorem, B is a compact subset of $C([0,1])$ if and only if it is closed, bounded, and equicontinuous. If $u \in B$, then $\|u\|_\infty \le \|u\|_{C^{0,\alpha}} \le 1$, where $\|\cdot\|_\infty$ denotes the sup-norm, so B is bounded, and $|u(x) - u(y)| \le |x-y|^\alpha < \epsilon$ if $|x-y| < \epsilon^{1/\alpha}$, so B is equicontinuous. Finally, if $u_n \in B$ and $u_n \to u$ in $C([0,1])$, then $u_n \to u$ pointwise and

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} = \lim_{n\to\infty} \frac{|u_n(x) - u_n(y)|}{|x-y|^\alpha} \le 1 \qquad \text{for all } x \ne y \in [0,1]$$

so $u \in B$, and B is closed.

Problem 2. (Spring, 2012) Let $X \subset L^2(0, 2\pi)$ be the set of functions u such that

$$u(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx}, \qquad |a_k| \le \frac{1}{1+|k|}.$$

Prove that X is a compact subset of $L^2(0, 2\pi)$.

Solution. The H^s-Sobolev norm of $u \in X$ with Fourier coefficients a_k satisfies

$$\|u\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |a_k|^2 \le \sum_{k\in\mathbb{Z}} \frac{(1+|k|^2)^s}{(1+|k|)^2}.$$

The series on the right converges if $2 - 2s > 1$ or $s < 1/2$. It follows that X is a bounded subset of $H^s(0, 2\pi)$ for $0 < s < 1/2$, and the Rellich theorem implies that X is a precompact subset of $L^2(0, 2\pi)$. Furthermore, if $u_n \to u$ as $n \to \infty$ in $L^2(0, 2\pi)$ and $u_n \in X$, then by the continuity of the inner product,

$$|a_k| = \frac{1}{2\pi}\left|\int_0^{2\pi} u(x)e^{-ikx}\,dx\right| = \lim_{n\to\infty} \frac{1}{2\pi}\left|\int_0^{2\pi} u_n(x)e^{-ikx}\,dx\right| \le \frac{1}{1+|k|},$$

so $u \in X$, and X is closed, which proves that X is compact.

Remark. For completeness, we prove the version of Rellich's theorem used here. (It wouldn't be necessary to do this in an exam!)

If $s > 1/2$, then H^s-functions are Hölder continuous, and the result follows directly from Sobolev embedding and the Arzelà-Ascoli theorem: bounded sets in H^s are bounded in $C^{0,\alpha}$ with $\alpha = s - 1/2 > 0$; so they are bounded and equicontinuous and therefore precompact in $C([0, 2\pi])$; which implies that they are precompact in L^2, since uniform convergence is stronger than L^2-convergence.

This argument doesn't work directly if $0 < s \le 1/2$, when H^s-functions needn't even be continuous, but we can fix it up. The idea is to approximate a bounded sequence of H^s-functions uniformly in L^2 by sequences of smooth functions (we simply truncate their Fourier series), apply the Arzelà-Ascoli theorem and a diagonal argument to show that there is a subsequence of the original sequence all of whose approximate subsequences converge uniformly, and conclude that the subsequence converges in L^2.

THEOREM 1. *If $s > 0$, then $H^s(0, 2\pi)$ is compactly embedded in $L^2(0, 2\pi)$.*

Proof. We need to show that a bounded sequence in H^s has a subsequence that converges strongly in L^2. If

$$u(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx}, \qquad a_k = \frac{1}{2\pi}\int_0^{2\pi} u(x) e^{-ikx},$$

we use as norms

$$\|u\|_{L^2} = \left(\frac{1}{2\pi}\int_0^{2\pi} |u|^2\, dx\right)^{1/2} = \left(\sum_{k\in\mathbb{Z}} |a_k|^2\right)^{1/2},$$
$$\|u\|_{H^s} = \left(\sum_{k\in\mathbb{Z}} (1+k^2)^s |a_k|^2\right)^{1/2}.$$

For $N \in \mathbb{N}$, we denote the orthogonal projection $u^N \in C^\infty([0, 2\pi])$ of $u \in L^2(0, 2\pi)$ onto the space of trigonometric polynomials of degree less than or equal to N by

$$u^N(x) = \sum_{|k|\le N} a_k e^{ikx}.$$

If $u \in H^s$, then

$$\begin{aligned} \|u - u^N\|_{L^2} &= \left(\sum_{|k|>N} |a_k|^2\right)^{1/2} \\ &\le \frac{1}{(1+N^2)^{s/2}}\left(\sum_{|k|>N} (1+k^2)^s |a_k|^2\right)^{1/2} \\ &\le \frac{\|u\|_{H^s}}{(1+N^2)^{s/2}}. \end{aligned} \tag{1}$$

Now suppose that (u_n) is a bounded sequence in H^s with $\|u_n\|_{H^s} \le R$ for all $n \in \mathbb{N}$. Denoting the Fourier coefficients of u_n by $a_{n,k}$, we have

$$\left|u_n^N(x)\right| \le \sum_{|k|\le N} |a_{n,k}| \le (1+2N)^{1/2}\left(\sum_{|k|\le N} |a_{n,k}|^2\right)^{1/2} \le C_N R,$$

where C_N is a generic constant depending on N, and

$$\begin{aligned}\left|u_n^N(x) - u_n^N(y)\right| &\le \sum_{|k|\le N} |a_{n,k}| \cdot \left|e^{ikx} - e^{iky}\right| \\ &\le \sum_{|k|\le N} |a_{n,k}| \cdot \sqrt{2}\,|kx - ky| \\ &\le \sqrt{2}\left(\sum_{|k|\le N} k^2\right)^{1/2} \left(\sum_{|k|\le N} |a_{n,k}|^2\right)^{1/2} |x-y| \\ &\le C_N R|x-y|.\end{aligned}$$

It follows that $\{u_n^N : n \in \mathbb{N}\}$ is a bounded, equicontinuous subset of $C([0,2\pi])$ for every $N \in \mathbb{N}$, so it is precompact by the Arzelà-Ascoli theorem.

Using a diagonal argument, we can extract a subsequence (u_{n_j}) of the original sequence (u_n) such that $(u_{n_j}^N)_{j=1}^\infty$ converges uniformly as $j \to \infty$ for every $N \in \mathbb{N}$. To do this, choose a subsequence $(u_{n_j^1})$ of (u_n) so that $(u_{n_j^1}^1)$ converges uniformly, then choose a subsequence $(u_{n_j^2})$ of $(u_{n_j^1})$ so that $(u_{n_j^2}^2)$ converges uniformly, and so on to get successive subsequences $(u_{n_j^N})$ such that $(u_{n_j^N}^M)$ converges uniformly as $j \to \infty$ for every $1 \le M \le N$, and define $u_{n_j} = u_{n_j^j}$.

Using (1) and the inequality $\|u\|_{L^2} \le \sqrt{2\pi}\|u\|_{L^\infty}$, we get that

$$\begin{aligned}\|u_{n_i} - u_{n_j}\|_{L^2} &\le \|u_{n_i} - u_{n_i}^N\|_{L^2} + \|u_{n_i}^N - u_{n_j}^N\|_{L^2} + \|u_{n_j}^N - u_{n_j}\|_{L^2} \\ &\le \frac{2R}{(1+N^2)^{s/2}} + \sqrt{2\pi}\|u_{n_i}^N - u_{n_j}^N\|_{L^\infty}.\end{aligned}$$

Given $\epsilon > 0$, choose N sufficiently large that

$$\frac{2R}{(1+N^2)^{s/2}} < \frac{\epsilon}{2}.$$

Since $(u_{n_j}^N)$ converges uniformly, it is uniformly Cauchy, and there exists $J \in \mathbb{N}$ such that

$$\sqrt{2\pi}\|u_{n_i}^N - u_{n_j}^N\|_{L^\infty} < \frac{\epsilon}{2} \qquad \text{for all } i, j > J.$$

It follows that $\|u_{n_i} - u_{n_j}\|_{L^2} < \epsilon$ for all $i, j > J$, so the subsequence (u_{n_j}) is Cauchy in L^2, and therefore it converges in L^2. ▮