Homework 1; Solutions

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Assignment Applied Analysis pg 30-32 problems 4, 5, 6, 12, 15, 16, 17

1 Exercise 1.4

Problem statement: Let $(X, d_X), (Y, d_Y)$ be metric spaces. Let $Z = X \times Y$ and define a metric $d : Z \times Z \to Z$ by $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ where $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$. We want to show that Z is a metric space with respect to this metric.

Solution: We need to check that the function d defined above indeed defines a metric. We want to check
i. $d(z_1, z_2) \geq 0, \forall z_1, z_2 \in Z$
ii. $d(z_1, z_2) = d(z_2, z_1), \forall z_1, z_2 \in Z$
iii. $d(z_1, z_2) \leq d(z_1, z) + d(z, z_2), \forall z_1, z_2, z \in Z$

Part i: Let $z_1, z_2 \in Z$. By definition above $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. Since d_X and d_Y are both metrics, we have that $d_X(x_1, x_2) \geq 0$ and $d_Y(y_1, y_2) \geq 0$. By the axioms of total ordering on real numbers, the sum of two nonnegative numbers is a nonnegative number. Thus $d(z_1, z_2) \geq 0$. (If you do not remember the order axioms or how to prove the last assertion, do it for youself. It is a good exercise).

Part ii: Let $z_1, z_2 \in Z$. By definition above $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2) = d_X(x_2, x_1) + d_Y(y_2, y_1) = d(z_2, z_1)$. (Since both d_X and d_Y are metrics, we have that they are symmetric in their arguments).

Part iii: Let $z_1, z_2, z \in Z$ where $z = (\bar{x}, \bar{y})$. Consider $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2) \leq d_X(x_1, x) + d_X(x, x_2) + d_Y(y_1, y) + d_Y(y, y_2), \forall x \in X, \forall y \in Y$. Then $d(z_1, z_2) \leq d_X(x_1, \bar{x}) + d_X(\bar{x}, x_2) + d_Y(y_1, \bar{y}) + d_Y(\bar{y}, y_2) = d(z_1, z) + d(z, z_2)$.

2 Exercise 1.5

Problem statement: Let $(X, \|\cdot\|)$ be a normed linear space. We want to show that the following two functions define a metric:
i. $d(x, y) = \|x - y\|$
ii. $d(x, y) = \|x - y\| 1 + \|x - y\|$

Solution: See collueges work below.

3 Execise 1.6

Problem statement: Assume that $\mathbb{R}$ equipped with its usual distance function is complete. We want to show that $\mathbb{R}^n$ equipped with the following norms is a complete normed linear space (also known as a Banach Space):
i. the sum norm
ii. the maximum norm
iii. the Euclidean norm is complete

Solution: Define the following functions:
1. $f_1(x) = \|x\|_{sum} = |x_1| + \cdots + |x_n|$
2. $f_2(x) = \|x\|_{Euclidean} = \sqrt{x_1^2 + \cdots + x_n^2}$
3. $f_3(x) = \|x\|_{max} = max\{|x_1|, \cdots, |x_n|\}$
Notice that we can relate these functions to each other in the following way. $\forall \bar{x} \in \mathbb{R}^n$ we have that $f_3(x) \leq f_2(x) \leq f_1(x)$ (see the lemma below). Then let $\bar{x}_n$ be a sequence in $\mathbb{R}^n$ that is Cauchy with respect to the metric defined by f_1. Let $\epsilon > 0$. Since $\bar{x}_n$ Cauchy in $\mathbb{R}^n \Rightarrow \{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R} \Rightarrow \exists y_i \in \mathbb{R}$ such that $\lim n \to \infty x_i = y_i$ (since we assumed that the space $\mathbb{R}$ was complete with respect to the euclidean metric (ie the absolute value function).
Then we claim that $\bar{y} = (y_1, ..., y_n$ is our desired limit. Consider $\lim n \to \infty f(\bar{x}_n - \bar{y}) = f(\lim n \to \infty(\bar{x}_n - \bar{y})) = f(0) = 0$. Thus we have our desired result. (The reason we can bring the limit through the argument of the function lies in the fact that the norm is a continuous function (which we have not yet proved). However, this is a good fact to get used to manipulating. If the suspense of waiting until it is covered in class is too great, I urge you to prove it for yourself or find a reference that shows the inherent beauty behind this result).

Lemma 3.1. $\sqrt{x_1^2 + \cdots + x_n^2} \leq |x_1| + \cdots + |x_n|$

Proof (Outline): Consider the absolute value as $|x_i| = \sqrt{x_i^2}$ Compare both $\sqrt{x_1^2 + \cdots + x_n^2}$ and $|x_1| + \cdots + |x_n|$ using this definition. Square both of these expressions and then compare. Using distributivity of multiplication over addition and commutativity of addition simplify. Show that the expression . Consider the relationship $(|x_1| + \cdots + |x_n|)^2$ has all the summands included in the expression $x_1^2 + \cdots + x_n^2$ and additional (nonnegative) summands. This gives us our desired result. (Hidden in this process is the result that shows that $f_3(x) \leq f_2(x)$.

Remark 3.2. *This problem gives insight to a very classical process that we will repeat many times throughout this course. Specifically, to show that a space is complete, begin with a Cauchy sequence, find a candidate for the limit point and show that the sequence indeed converges to that limit point with respect to the metric with whcih we are working).*

4 Exercise 1.12

Problem statement: Let $(X, d_X), (Y, d_Y), and (Z, d_Z)$ be metric spaces. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. We want to show that $h = g \circ f : X \to Z$ is continuous.

Solution: Let $\zeta \subset Z$ be an open set. Then the inverse image of ζ in Y, call it $g^{-1}(\zeta) \subset$ is an open set by proposition 1.46 (specifically, since g is continuous, we have that the inverse image of an open set is open). Since $g^{-1}(\zeta) = U$ is an open set in Y and f is continuous, we have that $f^{-1}(U)$ is an open set in X by proposition 1.46 (f is continuous $\Rightarrow$ inverse image of an open set is open). Then we have that $f^{-1}(g^{-1}(\zeta)) \subset X$ is an open set. Notice that $h^{-1} = (g \circ f)^{-1} = f^{-1} \circ g^{-1}$. By the above argument, we see that the inverse image of an open set in Z is open in X (under the map h). Then by the reverse dirction of proposition 1.46, we see that h is continuous.

Remark 4.1. *In the above proof, the phrase inverse image of an open set is open connotes that we have some map between the domain and the codomain. It is important to keep in mind that this statement is always made with reference to a particular map. This hints at a larger theme in mathematics: We can study a particular space and its attributes using maps to and from our space of interest. Using the resulting structures, we can infer much about the structure of the space we are investigating. We will see this much more as the days roll by.*

5 Exercise 1.15

Problem statement: Let $K \subset X$ be a compact subset of metric space W. We want to show:
Part i: K is closed and bounded.
Partii: Let $Y \subset K$ be closed. Then Y is compact.

Part i:
First we will show closed. Let K be compact $\Rightarrow$ K is sequentially compact by Thm 1.62 $\Rightarrow$ K is complete $\Rightarrow \forall \{y_n\}_{n=1}^{\infty}$ Cauchy in K, $\lim_{n \to \infty} y_n \in K \Rightarrow$ K is closed by Proposition 1.41. Now we will show bounded. Since K compact, by Thm 1.62 K sequentially compact. Then by Thm 1.59 K is totally bounded. By definition of totally bounded, we have $\forall \epsilon > 0$, $\exists \{k_1, k_2, ..., k_n\} \subset K$ such that $K \subset \bigcup_{i=1}^{n} B(\epsilon, k_i)$. Then if we take $M = 2 \cdot n \cdot \epsilon + 1$, we have that diam K = sup{ d(x, y) $|x, y \in K\} < M$. Then we see that K is bounded (by the definition of bounded.

Part ii:
Let $Y \subset X$ be closed. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in Y $\Rightarrow$ $\{y_n\}_{n=1}^{\infty}$ is a sequence in X $\Rightarrow$ there exists a convergence subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ that converges to some $y \in K \Rightarrow y \in Y$ since Y is closed $\Rightarrow$ Y is sequentially compact $\Rightarrow$ Y is compact.

6 Exercise 1.16: *Urysohn's lemma*

Problem statement: Let $F \subset \mathbb{R}^n$ be closed. Let $G \subset \mathbb{R}^n$ be open. Let $F \subset G$. We want to show that the function $f(x) = \frac{d(x,G^c)}{d(x,G^c)+d(x,F)}$ satisfies the following properties:
i. $0 \leq f(x) \leq 1$
ii. $f(x) = 0, \forall x \in G^c$, $f(y) = 1, \forall y \in F$
iii. It is continuous.

Part iii. Outline. It is sufficient to prove that f is sequentially continuituous by proposition 1.34 Let x_n be a sequence in $\mathbb{R}^n$ that converges to some $x \in \mathbb{R}^n$. We want to show that $limlimitsn \to \infty f(x_n) = f(x)$. Use the below lemma to prove that the function $d(x,S) = inf\{d(x,\sigma)|\sigma \in S\}$ is continuous. Then pull the limit into the argument of the function and we have our desired property. For a detailed solution, see one of your colluge's work included below.

Lemma 6.1. *[Lemma part ii] The metric function is continuous.*

Remark 6.2. *This lemma is generalized by the following statements: i. The inner product is continuous ii. The norm is continuous All of these statements come in very useful when taking the limits of sequences and passing this through from the argument of a function to the output. We will see many examples of such ideas throughout the definitions given in chapter 2 and later into the*

Remark 6.3. *This problem will be used later in your career as an analyist. For example, on the Davis preliminary exam in Analysis given in Fall 2009, this lemma was used in the proof of Problem 3 (show that the continuous functions of compact support are dense in L^P). It is very helpful in the construction of a series of functions that approximate integrable functions. Keep a special place for this in your mind and remember this theorem for later.*

7 Exercise 1.17

Problem statement: Let (X,d) be a complete metric space. Let Y $\subset$ X. We want to show:
(i)(Y, d) is complete $\Rightarrow$ Y is closed subset of X.
(ii) Y closed $\Rightarrow$ Y is complete

Part i:
Let Y be a complete metric space. We want to show that Y is closed. By the below lemmas, if we can show that for any $y \in \bar{Y}$, y is also in Y, we know that Y is closed. Let $y \in \bar{Y} \Rightarrow B(r,x) \neq \emptyset, \forall r > 0 \Rightarrow \exists \{y_n\}_{n=1}^{\infty}$ converging to y (specifically, choose each y_n to be in the ball of radius $\frac{1}{n}$ around the point y). $\Rightarrow \{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence $\Rightarrow y \in Y$ since Y assumed to be complete.

Lemma 7.1. *[Lemma part i] The following are equivalent:*
a. $y \in \bar{Y}$
b.B(r,y)$\neq \emptyset$, $\forall r > 0$
c. There is a sequence $\{y_n\}_{n=1}^{\infty}$ *in E that converges to y.*

Proof that $(a) \Leftrightarrow (b)$:
Part ii:
Suppose that Y is closed. Let $\{y_n\}_{n=1}^{\infty}$ be a Cauchy sequence in Y. Since $Y \subset X$ and X complete, we know that there is some $y \in X$ such that $y_n \rightarrow y$ with respect to the metric for X. Then $y \in \bar{Y} \Rightarrow y \in Y$ since Y is assumed to be closed.

Lemma 7.2. *[Lemma part i and part ii] A subset S of a topological space X is closed* $\Leftrightarrow S = \bar{S}$.

Proof: Let S be closed $\Rightarrow S^c$ is open. Also we have that S $\bigcap S^c = \emptyset$.

Lemma 7.3. *[Lemma part ii] If S is a subset of a topological space X and if a sequence* $x_{i}{}_{i=1}^{\infty}$ *converges to* $x \in X$*, then* $x \in \bar{S}$.

Lemma 7.4. *[Lemma part ii] A subset* S^c *of a topological space X is open* $\Leftrightarrow$ *S = int(S).*

Remark 7.5. *For the proofs of these lemmas as well as many more facinating details, see "Introduction to Topology" (Second Edition) by T.W. Gamlin and R.E. Greene.*

Exercise 1.5

Given $(X, \|\cdot\|)$ is a normed linear space

(1.2) says (X, d) is a metric space with $d(x,y) = \|x-y\|$, $x,y \in X$

Proof: (a) $d(x,y) \geq 0$ for $\forall x,y \in X$ since $\|x-y\| \geq 0$

(b) $d(x,y) = \|x-y\|$
$= \|y-x\|$
$= d(y,x)$

$\Rightarrow d(x,y)$ is a symmetric function.

(c) Let $z \in X$, then
$d(x,z) + d(z,y) = \|x-z\| + \|z-y\|$
$\geq \|x-z+z-y\|$
$= \|x-y\|$
$= d(x,y)$

$\Rightarrow$ Triangular Inequality

$\Rightarrow d(x,y) = \|x-y\|$ defines a metric on X.

(1.4) $d(x,y) = \dfrac{\|x-y\|}{1+\|x-y\|}$

Proof: (a) $d(x,y) \geq 0$ because $\|x-y\| \geq 0$ and $1+\|x-y\| \geq 1$

(b) $d(x,y) = \dfrac{\|x-y\|}{1+\|x-y\|}$
$= \dfrac{\|y-x\|}{1+\|y-x\|}$
$= d(y,x)$

$\Rightarrow d(x,y)$ is a symmetric function.

(c) Let $z \in X$, then
$d(x,z) + d(z,y) - d(x,y)$

$= \dfrac{\|x-z\|}{1+\|x-z\|} + \dfrac{\|z-y\|}{1+\|z-y\|} - \dfrac{\|x-y\|}{1+\|x-y\|}$

$= \dfrac{\|x-z\|+\|z-y\|-\|x-y\|+2\|x-z\|\|z-y\|+\|x-y\|\|x-z\|\|z-y\|}{(1+\|x-z\|)(1+\|z-y\|)(1+\|x-y\|)} \geq 0$

because $\|x-z\| + \|z-y\| \geq \|x-y\|$

$\Rightarrow d(x,z) + d(z,y) - d(x,y) \geq 0$

$\Rightarrow d(x,z) + d(z,y) \geq d(x,y)$ $\Rightarrow$ Triangular Inequality

$\Rightarrow d(x,y) = \dfrac{\|x-y\|}{1+\|x-y\|}$ defines a metric on X.

Excercise 1.16 Suppose that F and G are closed and open subsets of
Show that there is a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that:

(1) $0 \le f(x) \le 1$;
(2) $f(x) = 1$ for $x \in F$;
(3) $f(x) = 0$ for $x \in G^c$.

HINT. Consider the function

$$f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)}.$$

This result is called *Urysohn's lemma*.

Solution. Let $f(x)$ be defined as above. Notice that the metric

$$d(x, S) = \inf\{||x - s|| : \forall s \in S\}$$

is defined in terms of a norm, so the metric can never be negative. Sin
be nonnegative,

$$0 \le d(x, G^c), \text{ and } 0 \le d(x, F) \Rightarrow 0 \le f$$

Notice f attains a maximum when $d(x, F)$ attains its minimum. This i

$$d(x, F) = \inf\{||x - y|| : y \in F\} = 0 \Leftrightarrow x$$

The last if and only if comes from the fact that F is a closed set. An in
set itself. When $x \in F$, $d(x, F) = 0$ so $f(x) = 1$. Moreoever, the functi

$$d(x, G^c) = \inf\{||x - z|| : z \in G^c\} = 0 \Leftrightarrow x$$

We also have the last claim because G^c is closed. Therefore, since $f(x)$
1, that is, $0 \le f(x) \le 1$.

To prove continuity, we use the following lemma.

Lemma. Let H be a closed subset of $\mathbb{R}^n$. Let (x_n) be a convergent se
some norm $|| \cdot ||$, the metric

(1) $$d(x, H) = \inf\{||x - y|| : \forall y \in H\},$$

is continuous.

Proof. Let H be an arbitrary closed set, and (x_n) with limit x a con
points in H such that $d(x_n, H) = d(x_n, y)$ and $d(x, H) = d(x, z)$, res
because H is closed. Notice that

$$d(x_n, y) \le d(x_n, z)$$

by the infimum. Then by the triangle inequality, we have

$$d(x_n, H) \le d(x_n, z) \le d(x_n, x) + d(x, z) = d(x_n.$$

Rearranging

Similarly,

(2) $$d(x, H) = d(x, z) \le d(x, z) \le d(x, x_n) + d(x_n, z) = d(x, x_n) + d(x_n, H)$$

(3) $$\Rightarrow d(x, H) - d(x_n, H) \le d(x, x_n).$$

Since $|d(x_n, H) - d(x, H)|$ is bounded by $d(x_n, x)$, we can conclude if a sequence $(x_n) \to x$ as $n \to \infty$, then $d(x_n, H) \to d(x, H)$ as $n \to \infty$. Therefore, the metric given by equation (1) is continuous. □

We apply this lemma to two closed sets: F and G^c. Now, since $f(x)$ is a sum and ratio of continuous functions, $f(x)$ itself is also continuous, unless its denominator is 0. However, $F \subset G$ implies F and G^c are disjoint sets. This is true even for the boundary because F must be a proper subset of the open set G since F is closed. This implies $f(x)$ is well defined for all values x. Therefore, $f(x)$ is continuous. □