Banach Spaces Problem Set

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November 23, 2009

1 Introduction and Motivation

2 Applied Analysis Problem 5.3 pg 121

Part i:

1. Let $\delta : C([0,1]) \to \mathbb{R}$ be the linear functional that evaluates a function at the origin: $\delta(f) = f(0)$.
2. Let $C([0,1])$ be equipped with the sup-norm.
   - $\|f\|_\infty = \sup\limits_{0 \le x \le 1} |f(x)|$

We want to show that δ is bounded. Let $f \in C([0,1])$. Then since δ is a linear functional $\delta(f) \in \mathbb{R}$ and

$$\begin{aligned} |\delta(f)| &= |f(0)| && \text{by definition of } \delta \\ &\le \|f\|_\infty && \text{by definition of sup and an upper bound} \end{aligned}$$

We see by definition 5.2 and exercise 5.1 that

$$\begin{aligned} \|\delta\|_{op} &= \sup_{\|f\|_\infty = 1} \|\delta(f)\|_\infty \\ &= \sup_{\|f\|_\infty = 1} \|f(0)\|_\infty && \text{by definition of } delta \\ &\le \sup_{\|f\|_\infty = 1} \|f\|_\infty \\ &= 1 \end{aligned}$$

Thus we see that the operator norm δ is bounded above by 1. We will show that the operator norm of δ is actually equal to one by finding a $f \in C([0,1])$ with $\|f\|_\infty = 1$. Specifically, let $f(x) = 1, \forall x \in [0,1]$. Then we see that

$$\begin{aligned} |\delta(f)| &= |f(0)| && \text{by definition of } delta \\ &= |1| = 1 \\ &\le \sup_{\|f\|_\infty = 1} \|f\|_\infty && \text{by definition of sup} \end{aligned}$$

By the above we see that $\|\delta\|_{op} = 1$

Part ii:

1. Let $\delta : C([0,1]) \to \mathbb{R}$ be the linear functional that evaluates a function at the origin: $\delta(f) = f(0)$.
2. Let $C([0,1])$ be equipped with the sup-norm.

- $\|f\|_1 = \int\limits_0^1 |f(x)|dx$

We want to show that δ is unbounded. Let $M \in \mathbb{R}$. Let us consider the ratio:

$$\begin{aligned} \frac{\|\delta(f)\|_1}{\|f\|_1} &= \frac{\int\limits_0^1 |\delta(f)|dx}{\int\limits_0^1 |f(x)|dx} && \text{by definition of the 1-norm} \\ &= \frac{\int\limits_0^1 |f(0)|dx}{\int\limits_0^1 |f(x)|dx} && \text{by definition of } \delta \end{aligned}$$

Then we have translated this problem into finding a function that has a constant value at 1 and a very small integral. We can now apply the concept of sequences of functions to present a classic analysis proof to the desired. Two such sequences are:

$$\begin{aligned} &\{f_n\}_{n=1}^{\infty} && f_n = (1-x)^n \\ &\{g_n\}_{n=1}^{\infty} && g_n = e^{-nx} \end{aligned}$$

To be explicit... DETAILS!

3 Applied Analysis Problem 5.5 pg 121

Part i: version 1

1. Let $K : C([0,1]) \to C([0,1])$ be an operator that takes in a function and puts out a function
2. Let $K(f) = \int\limits_{y=0}^{y=1} k(x,y)f(y)dy$
3. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function

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We want to show that K is bounded. First we will use a theorem in this section to translate this problem. Notice that from theorem we know that K is bounded $\Leftrightarrow$ K is continuous. We also know from chapter 1, theorem 1.34 that K is continuous $\Leftrightarrow$ K is sequentially continuous (we see this holds since the statement of 1.34 requires only that the domain and codomain are metric spaces. Also any operator is by definition a function). Thus it is sufficient to show that K is continuous. We see using the below lemma, that $g(x,y) = k(x,y)f(y) \Rightarrow K$ is continuous. Thus our result holds

Lemma 3.1. *[Similar to Corollary 12.36 and it's proof]*
Antecedents:

1. *Let $g(x,\cdot)$ be a measurable function $\forall x \in X$*

2. *Let $g(\cdot, y)$ be a continuous function $\forall y \in Y$*
3. *Let there be some $\bar{g}$ such that $|g(x,y)| \leq \bar{g}(y), \forall x \in X, \forall y \in Y$*

Consequents:

- $G(x) = \int\limits_Y g(x,y) d\mu(y)$ *is continuous*

Proof: Apply Dominated Convergence theorem!

Part i: version 2 As an alternate to the heavy machinary and theorem recollection used to prove the above, we can use a different approach to prove this same result. Specifically, let f be any continuous function. By definition, we have that $K(f)$ is a continuous real valued function. Then for any $x \in [0,1]$ consider:

$$\begin{aligned}
|Kf(x)| &= |\int\limits_{y=0}^{y=1} k(x,y) f(y) d\mu(y)| && \text{by definition of K} \\
&\leq \int\limits_0^1 |k(x,y) f(y)| d\mu(y) && \text{by measure theory} \\
&= \int\limits_0^1 |k(x,y)||f(y)| d\mu(y) && \text{since absolute value is a homomorphism} \\
&\leq \max_{y\in[0,1]} |f(y)| \int\limits_0^1 |k(x,y)| d\mu(y) && \text{by theorem 1.62} \\
&\leq \max_{y\in[0,1]} |f(y)| \sup_{x\in[0,1]} \int\limits_0^1 |k(x,y)| d\mu(y) \\
&\leq \max_{y\in[0,1]} |f(y)| \max_{x\in[0,1]} \int\limits_0^1 |k(x,y)| d\mu(y) && \text{by remark below} \\
&= \|f\|_\infty \max_{x\in[0,1]} \int\limits_0^1 |k(x,y)| d\mu(y)
\end{aligned}$$

Thus, we see that $\|K\|_{op} = \sup\limits_{\|f\|_\infty \neq 0} \frac{\|Kf\|_\infty}{\|f\|_\infty} \leq$ HERE. $\Rightarrow K$ is bounded.

Remark 3.2. *In the above problem, we assumed that $k : [0,1] \times [0,1] \to \mathbb{R}$ was a continuous function. We know that if $[0,1] \times [0,1]$ is compact, then we can use theorem 1.68 to conclude that a continuous function on a compact set achieves in minimum and maximum. It is outside the scope of Applied Analysis to give results covered in Topology. However, see Gamelin and Greene "An Introduction to Topology" page 94 Theorem 10.8 for Tychonoff's theorem. The statement of this theorem is as follows: Let $X_1, X_2, ..., X_n$ be compact spaces. Then $X_1 \times X_2 \times \cdots \times X_n$ is compact. The proof of this theorem is by induction. Also, if one can get the first case the rest of the proof follows with less thought. The major trick of this proof is to look at intersection of coordinate slices. See Gamelin and Greene for more.*

Part ii: [Solution provided by Mihaela Ifrim and Anna Vershynina] Now we want to show that K is bounded. As we will see more in the future, usually once we found a plausible upper bound, we can create a function that shows that this upper bound is indeed equal to the norm of K. Here we want to show that $\|K\|_{op} = \max_{x\in[0,1]} \int_0^1 |k(x,y)|d\mu(y)$ It is sufficient to show

$$\max_{x\in[0,1]} \int_0^1 |k(x,y)|d\mu(y) \leq \|K\|_{op} \tag{1}$$

by order axioms of real numbers and part i above. Let $\tilde{x}$ be the specific $x \in [0,1]$ such that $\int_0^1 |k(\tilde{x},y)|d\mu(y) = \max_{x\in[0,1]} \int_0^1 |k(x,y)|d\mu(y)$. We know this exists following reasoning from remark above.
To prove that (1), we will introduce some auxillary problem solving elements. This is a classy presentation (not my own idea) that illustrates beautiful analysis style thinking.

Let

$$A_n = \{y \in [0,1] | k(x,y) \geq \frac{1}{n}\}$$
$$B_n = \{y \in [0,1] | k(x,y) \leq \frac{-1}{n}\}$$

. By definition, these sets are disjoint. We also have that they are closed since the inverse of a closed (JUSTIFICATION).

By Urysohn's lemma, there is a continuous function

$$\bar{f}_n(y) = \begin{cases} 1 & \text{if } y \in A_n \\ 0 & \text{if } y \in B_n \end{cases}$$

Define a new function

$$f_n(x) = 2\bar{f}_n(y) - 1 = \begin{cases} 1 & \text{if } y \in A_n \\ -1 & \text{if } y \in B_n \end{cases}$$

Since f_n is a linear combination of continuous functions, it is continuous. Moreover, we can calculate that $\|f_n\|_\infty = 1$ by definition. Then we have that

$$\|K\|_{op} = \sup_{\|f\|_\infty = 1} \|Kf\|_\infty$$

$$\geq \sup_{\|f_n\|_\infty = 1} \|Kf_n\|_\infty$$

Remark 3.3. *[Exercise 1.16: Urysohn's lemma]*
Let $F \subset \mathbb{R}^n$ be closed. Let $G \subset \mathbb{R}^n$ be open. Let $F \subset G$. We saw in homework one that the function $f(x) = \frac{d(x,G^c)}{d(x,G^c)+d(x,F)}$ satisfies the following properties: i. $0 \leq f(x) \leq 1$, ii. $f(x) = 0, \forall x \in G^c$, $f(y) = 1, \forall y \in F$, iii. It is continuous.

4 Applied Analysis Problem 5.7 pg 121

Part i: The Kernal

1. Let $K : C([0,1]) \to C([0,1])$ be an operator like that define above
2. Let $K(f) = \int\limits_{y=0}^{y=1} sin(\pi(x-y))f(y)dy$
   - Here is an explicit $k(x,y) = sin(\pi(x-y))$ illustrating above idea.

We want to find $\{f \in C([0,1]) | K(f) = 0\} = \{f \in C([0,1]) | \int\limits_0^1 sin(\pi(x-y))f(y)dy$

Consider

$$\begin{aligned}
\int_{y=0}^{y=1} sin(\pi(x-y))f(y)dy &= \int_0^1 (sin(\pi x)cos(\pi y) - cos(\pi x)sin(\pi y))f(y)dy && \text{by remark below} \\
&= \int_0^1 sin(\pi x)cos(\pi y)f(y)dy - \int_0^1 cos(\pi x)sin(\pi y)f(y)dy && \text{by measure theory} \\
&= sin(\pi x)\int_0^1 cos(\pi y)f(y)dy - cos(\pi x)\int_0^1 sin(\pi y)f(y)dy \\
&= \alpha sin(\pi x) + \beta cos(\pi x)
\end{aligned}$$

where $\alpha = \int\limits_0^1 cos(\pi y)f(y)dy \in \mathbb{R}$ and $\beta = -\int\limits_0^1 sin(\pi y)f(y)dy \in \mathbb{R}$.
Now, interpreting C([0,1]) as a linear space, we can look at a linear combination of the vectors sin(x) and cos(x). These vectors can be shown to be linearly independent. Thus, we see that $f \in ker(K)$ if and only if $\alpha = 0$ and $\beta = 0$.

Remark 4.1.

$$\begin{aligned}
sin(x+y) = sin(x)cos(y) + cos(x)sin(y) &\Rightarrow sin(x+-y) = sin(x)cos(-y) + cos(x)sin(-y) \\
&\Rightarrow sin(x-y) = sin(x)cos(y) - cos(x)sin(y)
\end{aligned}$$

The proof of this identity comes from combining the following two theorems:

1. *Euler's identity:* $e^{ix} = cos(x) = isin(x)$ *(the proof of this follows from Taylor's series expansion of* $e^ix, sin(x), cos(x)$ *and identity* $i^4 = 1$*).*
2. *exponentiation is a homomorphism (ie* $e^{x+y} = e^x \cdot e^y$*).*

Part ii: The Range Now we would like to find the range of our operator K. We showed above that K(f) non-zero if and only if α and β nonzero. Thus we have $Ran(K) \subset span\{sin(\pi x), cos(\pi x)\}$.

Claim: span$\{sin(\pi x), cos(\pi x)\} \subset Ran(K)$.

5 Applied Analysis Problem 5.8 pg 121

6 Applied Analysis Problem 5.11 pg 122

1. Let $\{T_n\}_{n=1}^{\infty} \to T$ uniformly.
   - $\lim\limits_{n\to\infty} \|T_n - T\|_{op} = 0$ (by definition 5.39 page 109)

We want to show that $\|T_n\|_{op} \to \|T\|_{op}$ as real numbers. Before we begin, recall that B(X,Y) is a linear space whose norm is the operator norm discussed in the problems of this section. Consider

$$\begin{aligned}
|\|T_n\|_{op} - \|T\|_{op}| &= |\|T_n - 0\|_{op} - \|T - 0\|_{op}| \\
&= |d_B(T_n, 0) - d_B(T, 0)| && \text{where } d_b \text{ is metric induced by operator norm} \\
&\leq |d_B(T_n, T)| && \text{by exercise 1.3 on pg 30 of Applied Analysis} \\
&= \|T_n - T\|_{op}
\end{aligned}$$

Letting $\epsilon > 0$ we have by assumption that we can find an N such that $n \geq N \Rightarrow \|T_n - T\|_{op} < \epsilon$. Thus we have our desired convergence.

7 Applied Analysis Problem 5.12 pg 122

1. Let $\{T_n\}_{n=1}^{\infty} \to T$ uniformly.
   - $\lim\limits_{n\to\infty} \|T_n - T\|_{op} = 0$ (by definition 5.39 page 109)

We want to show that $T_n \to T$ strongly. We know that $T_n \to T$ converges strongly if

- $\lim\limits_{n\to\infty} T_n(x) = T(x)$ where this equality is with respect to the norm in Y (by definition 5.44 page 111)
- $\lim\limits_{n\to\infty} \|T_n(x) - T(x)\|_Y = 0$ by reinterpreting above definition.

Then consider

$$\begin{aligned}
\|T_n(x) - T(x)\|_Y &= \|(T_n - T)(x)\|_Y && \text{since B(X, Y) a linear space} \\
&\leq \|(T_n - T)\|_{op} \|x\|_X && \text{by 5.1 on page 95}
\end{aligned}$$

Then, by our hypothesis and experience with taking limits, we see that $T_n \to T$ strongly.
(SPECIFICS: Let $\epsilon > 0$. Use uniform convergence to find an $N \in \mathbb{N}$ such that $\|(T_n - T)\|_{op} < \frac{\epsilon}{\|x\|_X}$)