Chapter 3

Contraction mapping theorem

★ Latex meeting tomorrow
office hrs rescheduled
tomorrow 4-6pm ?
毎週すること?

pg 61 def contraction

(i) Let (X,d) be a metric space

(ii) Mapping $T: X \to X$ is a contraction mapping if there is some ~~$0<c$~~ $0 \leq c < 1$ s.t.

$$d(T(x), T(y)) \leq c\, d(x,y)$$

pg 62 If $T: X \to X \Rightarrow x \in X$ s.t.

$$T(x) = x$$

is a fixed point

does not depend on the metric $d: X \times X \to \mathbb{R}$

Theorem 3.2 pg 62

this is demonstrative mathematics... the study of this type of math is much different than solving problems?

Antecedents

(I) $T: X \to X$ a contraction mapping

(II) X, d a complete metric space

conclusion

(a) solution x to $T(x) = x$ exists

(b) it is unique

Proof

Constructive

$x_{n+1} = T(x_n)$

- define a sequence $x_0 \in X$ $\quad T_1^n(x_0) = x_n$
- show it is Cauchy (use $\Delta \leq$ & geometric series)
-

Formal proof

Let $x_0 \in X$.

Let $T^n(x_0) = x_n$ or $x_{n+1} = T(x_n)$

d

WLOG assume $n \geq m \geq 1$

$$d(x_n, x_m) = d(T^n x_0, T^m x_0)$$

$$= d(T^{n-m} T^m x_0, T^m x_0)$$

少

$$\leq c^m d(T^{n-m} x_0, x_0)$$

def of contraction

uh oh! Now what

what did we do in step 少 that made our problem more easily bounded

$$= c^m d(T^{n-m} x_0 + T^{n-m-1} x_0 + T^{n-m-1} x_0 - \cdots + T x_0, x_0)$$

(秋) $$\leq c^m \Big[d(T^{n-m} x_0, T^{n-m-1} x_0) + d(T^{n-m-1} x_0, T^{n-m-2} x_0) + \cdots + d(T x_0, x_0) \Big]$$

What tools do we have on our space

side thoughts

$$x_{n+1} = T(x_n) = T\,T\,x_{n-1} = T^2 x_{n-1} = T^3 x_{n-2} = T^{n+1}$$

- counting argument take cases 4, 8, 10 as examples

$d(T(x), T(y)) \leq c\, d(x, y)$

def of T

$$d(T^2(x), T^2(y)) \leq c\, d(T(x), T(y)) \leq c^2 d(x, y)$$

side thought

say I foreget geometric sequence

$$(1+x+x^2+\cdots+x^n)(1-x)$$

$$\begin{array}{l} 1+x+\cdots+x^n \\ -x-\cdots-x^n-x^{n-1} \end{array} = 1-x^{n-1}$$

$$\Rightarrow \quad 1+x+x^2+\cdots+x^n = \frac{1-x^{n-1}}{1-x}$$

how is this applicable

(夏) $\quad \leq \quad c^m \left[\frac{1}{1-c}\right] d(x_1, x_0)$

$\Rightarrow$ Cauchy $\qquad$ why is this true

$\Rightarrow \exists x \in X$ s.t. $\lim_{n\to\infty} x_n = x \;/\; \lim_{n\to\infty} x_{n+1} = x$

$$\Rightarrow Tx = T \lim_{n\to\infty} x_n$$

$$\overset{?}{=} \lim_{n\to\infty} T x_n$$

$$= \lim_{n\to\infty} \cancel{T}\, x_{n+1}$$

$$= x \quad \checkmark$$

side board

Lets focus in on one of these summands

how many summands exist?

$n-m$

why?

$$d(Tx_0, x_0) = d(x_1, x_0)$$

$$\begin{aligned} d(T^k x_0, T^{k-1} x_0) &= d(T\,T^{k-1} x_0, T^{k-1} x_0) \\ &\le c^{k-1} d(Tx_0, x_0) \\ &= c^{k-1} d(Tx_0, x_0) = c^{k-1} d(x_1, x_0) \end{aligned}$$

Returning to 春 秋 冬 夏

$$\le c^m \left[c^{n-m-1} d(x_1, x_0) + \cdots + c^k d(x_1, x_0) + \cdots + d(x_1, x_0) \right]$$

$$= c^m \left[c^{n-m-1} + \cdots + c^k + \cdots + 1 \right] d(x_1, x_0)$$

$$= c^m \left[\sum_{k=0}^{n-m-1} c^k \right] d(x_1, x_0)$$

how can I recognize

Recall: this is the importance of ~~say~~ speak about past problems

what is true about c?

夏) $$\le c^m \left[\sum_{k=0}^{\infty} c^k \right] d(x_1, x_0)$$

Geometric sequence?

Direct uniqueness proof:

Suppose two such points x & y exist

$$0 \le d(x,y) \le d(Tx, Ty)$$
$$\le c\, d(x,y)$$
$$\Rightarrow d(x,y) = 0$$

Why do we care?