# Mat 201A: Homework 2 Solutions
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**Excercise 1.19** Following the construction of the Cantor set $C$ by the removal of middle thirds, we dfine a function $F$ on the complement of the Cantor set $[0,1] \setminus C$ as follows. First, we define $F(x) = 1/2$ for $1/3 < x < 2/3$. Then $F(x) = 1/4$ for $1/9 < x < 2/9$ and $F(x) = 3/4$ for $7/9 < x < 8/9$, and so on. Prove that $F$ extends to a unique continuous function $F : [0,1] \to \mathbb{R}$. Prove that $F$ is differentiable at every $x \in \mathbb{R} \setminus C$ and $F'(x) = 0$. This is called the *Cantor function*. Its graph is sometimes called the *devil's staircase*.

**Solution.** For brevity, let $S = [0,1] \setminus C$. First, note some properties of $S$.

$$\begin{aligned}
\mu(S) &= \frac{1}{3} + 2 \cdot \left(\frac{1}{9}\right) + 4 \cdot \left(\frac{1}{27}\right) + \dots \\
&= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} \\
&= \frac{1}{3}\left(\frac{1}{1-2/3}\right) = 1.
\end{aligned}$$

Now, this simple calculation gives us motivation for extending $F$ to a continuous function on $[0,1]$. Let $\widetilde{F}$ be the extension of $F : S \to [0,1]$, to $\widetilde{F} : [0,1] \to [0,1]$. For points $x \notin S$, define the function

$$\widetilde{F}(x) = \sup\{F(y) : y < x \text{ and } y \in S\}.$$

With this construction, we have that $\widetilde{F}$ is a nondecreasing function on the entire interval. In particular, at each open interval it's defined on, we can extend the function to endpoints of their interval, for example

$$F(x) = 1/2 \text{ for } 1/3 < x < 2/3 \Rightarrow \widetilde{F}(x) = 1/2 \text{ for } 1/3 \le x \le 2/3.$$

Notice that $F$ maps each interval given by some $\left(\frac{3^a+1}{3^k}, \frac{3^a+2}{3^k}\right)$ to its image by replacing all the $2's$ in the base 3 expansion of the left end point's expansion with $1's$. For instance,

$$(1/3, 2/3)_{10} = (0.1, 0.2)_3 \mapsto 1/2_{10} = .1_2$$

where we've assigned the function on the entire interval to be the left end point, but in base 2. Here are a few more examples

$$\begin{aligned}
(1/9, 2/9)_{10} &= [0.01, 0.02]_3 \mapsto 1/4_{10} = 0.01_2 \\
(7/9, 8/9)_{10} &= [0.\mathbf{2}1, 0.22]_3 \mapsto 3/4_{10} = 0.\mathbf{1}1_2 \\
(1/27, 2/27)_{10} &= [0.001, 0.002]_3 \mapsto 1/8_{10} = 0.001_2 \\
(7/27, 8/27)_{10} &= [0.0\mathbf{2}1, 0.022]_3 \mapsto 3/8_{10} = 0.0\mathbf{1}1_2 \\
(19/27, 20/27)_{10} &= [0.\mathbf{2}01, 0.202]_3 \mapsto 5/8_{10} = 0.\mathbf{1}01_2 \\
(25/27, 26/28)_{10} &= [0.\mathbf{22}1, 0.222]_3 \mapsto 7/8_{10} = 0.\mathbf{11}1.
\end{aligned}$$

This motivates me to think that as long as my error in the image is less than some $2^{-N}$, then I can always find an interval the preimage sits in to be no less than $3^{-N}$ away.

Formally, let $\epsilon > 0$. Let $N \in \mathbb{N}$ satisfy $2^{-N} < \epsilon$. By the denseness of $S$ (recall its measure is 1 in $[0,1]$), we can always find $y \in S$ such that

$$|y - x_0| < 3^{-N} = \delta < (2/3)^N \epsilon.$$

Graphically, this is saying we can interate far enough in $\bigcup[(3^a+1)3^k,(3^a+2)/3^k]$ so that for each $x_0 \in [0,1]$, $y$ lies on an interval within $3^{-N}$ away from the $x_0$. Let $S_y := [(3^c+1)/3^N,(3^c+2)/3^N]$ satisfiy $y \in S_y$ for some $c \in \mathbb{N}$.

Now, for this same $N$, $\widetilde{F}$ maps the left endpoint of $S_y$ to its base 2 expansion. Since we replace all the $2's$ appearing in the base 3 expansion, with $1's$ in the base 2 expansion, the difference must be less than $2^{-N}$

$$|F(y) - \widetilde{F}(x_0)| \leq 2^{-N} < \epsilon.$$

Therefore, $\widetilde{F}$ is the continuous extension of $F$.

It seems trivial, but at every point in $S$, the function is defined to be constant. Since the derivative of a constant is 0, then $F'(s) = 0 \; \forall s \in S$.

□

**Excercise 1.20** Let $X$ be a normed linear space. A series $\sum x_n$ in $X$ is *absolutely convergent* if $\sum ||x_n||$ converges to a finite value in $\mathbb{R}$. Prove that $X$ is a Banach space if and only if every absolutely convergent series converges.

**Solution** So the idea is to prove the following statement.

$$X \text{ is Banach} \Leftrightarrow \left(\sum ||x_n|| < \infty \Rightarrow \sum x_n \text{ converges in } X\right)$$

" $\Leftarrow$ ". Let $r_k$ and $s_k$ be the partial sums of the series $\sum x_n$ and $\sum ||x_n||$, respectively, that is,

$$r_k = \sum_{n=1}^{k} x_n, \text{ and } s_k = \sum_{n=1}^{k} ||x_n||.$$

If $s_k$ converges, then

$$\lim_{k\to\infty} s_k = \sum_{n=1}^{k} ||x_n|| = s, \text{ where } s \in \mathbb{R}.$$

We can use the fact that if a sequence is convergent, then the sequence is also Cauchy. For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall\, j,l > N$,

$$||s_l - s_j|| = \sum_{n=j+1}^{l} ||x_n|| < \epsilon.$$

Now since $s_k \longrightarrow s$ implies $r_n \longrightarrow r$ by assumption

$$\lim_{k\to\infty} r_k = \sum_{n=1}^{k} x_n = r, \text{ where } r \in X,$$

we can construct a Cauchy sequence from $r_n$. By triangle inequality, we have

$$||r_l - r_k|| = \left|\left|\sum_{n=k+1}^{l} x_n\right|\right| \leq \sum_{n=k+1}^{l} ||x_n|| < \epsilon.$$

Therefore $r_n$ is a Cauchy sequence and converges to $r$. Since this is true for all Cauchy sequences, $X$ is a Banach space.

" $\Rightarrow$ ". Let $X$ be a Banach space. This implies every Cauchy sequence $(x_k) \in X$ will converge to a limit $y \in X$. Now, let's assume there is a series derived from this sequence that absolutely converges in $\mathbb{R}$, that is, $s = \sum ||x_n|| < \infty$. We can construct a Cauchy sequence for $r_k$ with this bound

$$||r_l - r_j|| = \left|\left|\sum_{n=j+1}^{l} x_n\right|\right| \leq \sum_{n=j+1}^{l} ||x_n|| = ||s_l - s_j|| < \epsilon.$$

Therefore, $r_k$ is a Cauchy sequence, and by completeness, $\exists\, r \in X$ such that

$$\lim_{k\to\infty} r_k = \sum_{n=1}^{\infty} x_n = r.$$

Therefore, absolute convergence implies convergence. □

**Excercise 1.21** Suppose $X$ is a Banach space, and $(x_{mn})$ is a doubly indexed sequence in $X$ such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} ||x_{mn}|| < \infty.$$

Prove that

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} x_{mn}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} x_{mn}\right).$$

**Solution.** The following proof is thanks to William R. Wade's textbook: "Introduction to Analysis". First let's prove a well known theorem about uniform convergence for $\mathbb{R}$.

**Weierstrass $M$-Test.** Let $f_k$ be defined on a set $E$ and suppose $M_k \geq 0$ satisfies

$$\sum_{k=1}^{\infty} M_k < \infty.$$

If $|f_k(y)| \leq M_k$ for $k \in \mathbb{N}$ and $y \in E$, then $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on $E$.

**Proof.** Let $\epsilon > 0$ and we use the Cauchy Criterion to choose $N \in \mathbb{N}$ such that $m > n \geq N$ implies $\sum_{k=n}^{m} M_k < \epsilon$. Thus,

$$\left|\left|\sum_{k=n}^{\infty} f_k(y)\right|\right| \leq \sum_{k=n}^{m} ||f_k(y)|| \leq \sum_{k=n}^{m} M_k < \epsilon$$

for $m > n \geq N$ and $y \in E$. Hence, the partial sums of $\sum_{k=n}^{m} f_k$ are uniformly Cauchy and the partial sums of $\sum_{k=1}^{\infty} ||f_k(y)||$ are Cauchy for each $y \in E$. □

Now, suppose we have the description above. Set

$$s_m = \sum_{n=1}^{\infty} x_{mn}$$

for each $m \in \mathbb{N}$. Let $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Then, for each $m \in \mathbb{N}$ let

$$f_m(0) = \sum_{n=1}^{\infty} x_{mn}, \quad f_m(1/k) = \sum_{n=1}^{k} x_{mn}.$$

By completeness, $f_m(0)$ exists and by definition of series convergence,

$$\lim_{k\to\infty} f_m(1/k) = f_m(0).$$

So $f_m$ is continuous at $0 \in E$ for each $m \in \mathbb{N}$. Moreover, we have that $||f_m(y)|| \leq s_m$ for all $y \in E$ and $m \in \mathbb{N}$, the Weierstrass $M$-Test implies that

$$f(y) = \sum_{m=1}^{\infty} f_m(y)$$

converges uniformly on $E$. Hence, $f$ is continuous at $0 \in E$. By sequential continuity, we have $1/n \to 0$ implies $f(1/n) \to f(0)$. Therefore,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn} = \lim_{k\to\infty}\sum_{m=1}^{k}\sum_{n=1}^{\infty} x_{mn} = \lim_{k\to\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{k} x_{mn}$$
$$= \lim_{k\to\infty}\sum_{n=1}^{\infty} f_m(1/k) = \lim_{k\to\infty} f(1/k) = f(0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{mn}.$$

**Excercise 1.24** Let $f : X \to \mathbb{R}$ be a real-valued function on a set [Note, it should be a metric space, not just a set.] $X$. The *epigraph* epi $f$ of $f$ is the subset of $X \times \mathbb{R}$ consisting of points that lie above the graph of $f$ :

$$\text{epi } f = \{(x,t) \in X \times \mathbb{R}\,|t \geq f(x)\}.$$

Prove that a function is lower semicontinuous if and only if its epigraph is a closed set.

**Solution.** "$\Rightarrow$ ." Suppose a function is lower semicontinuous. Then for all sequences $x_n \to x$, we have

$$\liminf(f(x_n)) \geq f(x).$$

Consider a sequence $(x_n, t_n) \in$ epi $f$ such that $(x_n, t_n) \to (x_0, t_0) \in X \times \mathbb{R}$ as $n \to \infty$. By definition of epigraph, for each $n$, we have

$$f(x_n) \leq t_n \Rightarrow f(x_0) = \liminf_{n\to\infty} f(x_n) \leq \liminf_{n\to\infty} t_n = t_0.$$

This shows that the pair $(x_0, t_0)$ live in the epigraph. Therefore, every convergent sequence converges to a point in the epigraph, therefore epi $f$ is closed.

"$\Leftarrow$ ." Let the epigraph be closed. Then for all sequences $(x_n, t_n)$ converging to $(x_0, t_0) \in X \times \mathbb{R}$, we have that $(x_0, t_0) \in$ epi $f$. Consider the sequence $(x_n, f(x_n)) \in X \times \mathbb{R}$. This sequence is in the epigraph for all $n$. Because the epigraph is closed, we can always find a sequence $(x_n, f(x_n)) \to (x_0, t_0) \in$ epi $f$. Now we use the definition of the epigraph to arrive at the desired result

$$f(x_0) \leq t_0 \leq \liminf_{n\to\infty} f(x_n).$$

Therefore, the function is lower semi-continuous. □

**Excercise 1.26** Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial of two real variables. Suppose that $p(x,y) \geq 0$ for all $x, y \in \mathbb{R}$. Does every such function attain its infimum? Prove or disprove.

**Solution.** Let$p(x,y,) = (xy-1)^2 + y^2$. Then notice that $p(x,y) \geq 0 \forall (x,y) \in \mathbb{R}^2$. In fact, $p(x,y) = 0$ if and only $y = 0$ and $x = -1/y$. However, these two conditions cannot be simultaneously satisfied. Now, let's construct a sequence with infimum equal to 0. Then, we will arrive at a disproof.

$$p(x_n, y_n) = p(n, 1/n) = (n/n-1)^2 + 1/n^2 = 1/n^2.$$

We know the $\inf(p(x,y)) \leq \inf(p(n,1/n)) = 0$ However, $p(x,y) \neq 0 \,\forall\, (x,y) \in \mathbb{R}^2$. Therefore,

$$\boxed{p(x,y) = y^2 + (xy-1)^2}$$

proves that not all two-variable polynomials attain their infimum.

# Problem 1.21

Suppose that $X$ is a Banach space, and $(x_{mn})$ is a doubly indexed sequence in $X$ such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|x_{mn}\| < \infty$$

Prove that

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}x_{mn}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}x_{mn}\right)$$

We know that

$$\|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{mn}\| \leq \sum_{m=1}^{\infty}\|\sum_{n=1}^{\infty}x_{mn}\| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|x_{mn}\| < \infty$$

By repeated application of the triangle inequality ( similar to the previous problem ). This implies that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{mn} = L$ converges to a value in $X$ (because $X$ is complete ). If we look at the sequence of rectangular partial sums

$$s_{mm} = \sum_{m=1}^{M}\sum_{n=1}^{M}x_{mn} = \sum_{m=1}^{M}x_{mm}$$

We know this sequence converges to $L$ since the series converges to $L$. This also implies that each column and each row of each rectangle in the partial sum converges to a finite value. Since each the limits of summation are finite for a particular summation, the order of summation does not matter. So

$$s_{mm} = \sum_{m=1}^{M}\sum_{n=1}^{M}x_{mn} = \sum_{n=1}^{M}\sum_{m=1}^{M}x_{mn} = s'_{mm}$$

We know that the sequence $(s_{mn})$ converges to $L$ as $m, n \to \infty$ but since each $s_{mn} = s'_{mm}$ this implies that $(s'_{mm})$ converges to $L$ as well, and we have our result.

## Problem 1.24

Let $f : X \to \mathbb{R}$ be a real-valued function on a set $X$. The epigraph epi$f$ of $f$ is the subset of $X \times \mathbb{R}$ consisting of points that lie above the graph of $f$:

$$\text{epi}f = \{(x,t) \in X \times \mathbb{R} | t \geq f(x)\}$$

Prove that a function is lower semicontinuous if and only if its epigraph is a closed set.

- Assume a function $f$ is lower semicontinuous. We need to show that epi$f$ is closed.

  We assume epi$f$ is not closed and try to reach a contradiction. Since epi$f$ is not closed, then there is a sequence in epi$f$ that converges to a point outside of epi$f$. So $\{(x_n, t_n)\} \subset$ epi$f$ such that $(x_n, t_n) \to (x,t) \notin$ epi$f$ . Since $(x_n, t_n) \in$ epi$f$ this means that $t_n \geq f(x_n)$. If we take the limit of the infimum of each side, then

  $$\lim_{n\to\infty} \inf t_n = t \geq \lim_{n\to\infty} \inf f(x_n)$$

  Since $(x,t) \notin$ epi$f \implies f(x) > t$ . Putting this together yields

  $$\lim_{n\to\infty} \inf f(x_n) \leq t < f(x)$$

  Now using the fact that $f$ is lower semicontinuous, $f$ must also satisfy

  $$\lim_{n\to\infty} \inf f(x_n) \geq f(x)$$

  This is a contradiction, and thus epi$f$ must be closed.

- We now assume that epi$f$ is closed. We need to show that $f$ is lower semicontinuous.

  We will assume that the function $f$ is not lower semicontinuous and try to reach a contradiction. If $f$ is not lower semicontinuous then

  $$\lim_{n\to\infty} \inf f(x_n) < f(x)$$

  Then there exists a sequence such that $x_n \to x$ with $f(x_n) \leq t < f(x)$ . This is true because at $x$ it is not lower semicontinuous and this sequence is within the epigraph of $f$. Since the epigraph is closed $f(x) \leq t$. This is a contradiction and thus $f$ must be lower semicontinuous.

## Problem 1.26

Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function of two real variables. Suppose that $p(x, y) \geq 0$ for all $x, y \in \mathbb{R}$. Does every such function attain its infimum? Prove or disprove.

- We are given that $p(x, y) \geq 0$ for all $x, y \in \mathbb{R}$. We are going to show that not every function attains its infimum.

If we look at the polynomial $p(x, y) = (xy - 1)^2 + y^2 \geq 0$ . Since it is bounded below then the infimum $0 \leq m < \infty$ . Now if we look at a sequence

$$(x_n, y_n) = (n^2 + n, \frac{1}{n^2})$$

Then

$$p(x_n, y_n) = ((n^2 + n)\frac{1}{n^2} - 1)^2 + \left(\frac{1}{n^2}\right)^2$$

Simplifying

$$p(x_n, y_n) = \frac{1}{n^2} + \frac{1}{n^4}$$

Taking the limit

$$\lim_{n \to \infty} p(x_n, y_n) = 0$$

So it has an infimum at 0. However the polynomial never attains it. If it did then there would be some point $(x, y)$ such that $p(x, y) = 0 = (xy - 1)^2 + y^2$ . Since both terms need to be zero for this to be true, $y = 0$, but then the first term cannot be 0. Therefore there is no point $(x, y)$ such that $p(x, y) = 0$ and thus $p(x, y)$ never attains its infimum.