```
\documentclass[12point]{article}
\topmargin 0.0in
\headheight 0.0in
\headsep 0.0in
\oddsidemargin 1in
%\hoffset -1in
\textwidth 5in
\textheight 9in

\usepackage{amsfonts, amstext, amsthm, amsmath, amssymb}
\usepackage{enumerate}
%\usepackage{graphics}
\usepackage{graphicx}
\usepackage{amscd}

\newenvironment{sgitem}{
\begin{enumerate}
  \setlength{\itemsep}{1pt}
  \setlength{\parskip}{0pt}
  \setlength{\parsep}{0pt}
}{\end{enumerate}}


\begin{document}

\newcommand{\red}{\textcolor{red}}
\newcommand{\blue}{\textcolor{blue}}
\newcommand{\black}{\textcolor{black}}
\newcommand{\brown}{\textcolor{brown}}
\newcommand{\green}{\textcolor{darkgreen}}
\newcommand{\yellow}{\textcolor{yellow}}
\newcommand{\pink}{\textcolor{pink}}

\newcommand{\N}{\mathbb{N}}
\newcommand{\Z}{\mathbb{Z}}
\newcommand{\Q}{\mathbb{Q}}
\newcommand{\R}{\mathbb{R}}
\newcommand{\C}{\mathbb{C}}
\newcommand{\A}{\mathcal{A}}
\newcommand{\F}{\textbf{F}}                    %%%new command for fields
\newcommand{\G}{G_{\mathcal A}}
\newcommand{\D}{\mathfrak D}
\newcommand{\B}{\mathcal B}
\newcommand{\limit}[2]{lim_{#1\rightarrow #2}}
\newcommand{\abs}[1]{\left|#1\right|}     %|x|


\title{MAT 201A - Analysis}
\author{Kristen Freeman}
\date{Friday, October 9, 2009}
\maketitle


\begin{itemize}
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\item[\bf{1.19}] Following the construction of the Cantor set $C$ by the removal of mi
ddle thirds, we define a function $F$ on the complement of the Cantor set $[0,1] \back
slash C$ as follows.  First, we define $F(x) = 1/2$ for $1/3 < x < 2/3$.  Then $F(x) =
 1/4$ for $1/9 < x < 2/9$ and $F(x) = 3/4$ for $7/9 < x < 8/9$, and so on.  Prove that
 $F$ extends to a unique continuous function $F:[0,1] \to \R$.  Prove that $F$ is diff
erentiable at every $x \in \R \backslash C$ and $F'(x) = 0$.  This function is called
the \textit{Cantor function}.  Its graph is sometimes called the \textit{devil's stair
case}.

\begin{proof}
First we want to show that $F$ is a continuous function.   To do this let us define $F
$ as the limit of a sequence of continuous functions, $\{f_n\}$, on $[0,1]$ where $f_0
(x) = x$, $f_1(x) = 1/2$ on the interval $(1/3,2/3)$ and is just a linear extension of
f of this interval so that $f_1$ is continuous and $f_1(0) = 0$, $f_1(1)=1$, and so on
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 where for each new $f_n$ we are setting more intervals of length $1/3^n$ to some cons
tant value.  It should be clear that $\lim_{n \to \infty} f_n = F$ defined in the prob
lem statement.  Then we have a theorem (which can be found in Rudin's, ``Principles of
 Mathematical Analysis'') that states when given a sequence of continuous functions if
 they converge uniformly to some function $F$ then $F$ is continuous.  To show that $\
{f_n\}$ converges uniformly it is enough to show that for every $\epsilon > 0$ there e
xists an $N$ such that for $n >N$ $\sup_{x \in [0,1]} \abs{f_n(x) - F(x)} < \epsilon$.
  Both $f_n(x)$ and $F(x)$ are equal to horizontal line segments on certain intervals
but otherwise  they do not necessarily match up.  On $f_n(x)$ the vertical distance be
tween two successive horizontal line segments is $1/2^n$ and so we find that on the in
tervals of $x$ that does not go to a constant, $f_n(x)$ is contained in an open interv
al of length $1/2^n$.  Also let us use the fact that $\{f_n(x)\} \to F(x)$ so for any
$\epsilon > 0$ there exists $N$ such that $n > N \implies \abs{f_n(x) - F(x)} < \epsil
on$; so by definition $f_n(x)$ is getting really close to $F(x)$. From this we can sor
t of see that we want to find $N$ such that $1/2^N < \epsilon$.  For any $\epsilon > 0
$ there exists a degree $N$ such that $2^{-N} < \epsilon$.  Let our $N$ be this to sho
w that $\{f_n\}$ converges uniformly to $F$.  Therefore $F$ is a continuous function.


Suppose there was another way to define the Cantor function that satisfies the above c
onditions, $F(x) = 1/2$, etc. Let $\tilde{F}, F$ be two different extensions of this f
unction from $[0,1]\backslash C$ to $[0,1]$.  Let $p(x) = F(x) - \tilde{F}(x)$. It fol
lows that $p(x) = 0$ on $[0,1] \backslash C$.  Then using the fact that the cantor fun
ction is continuous we have that $p(x)=0$ on $C$ because $p(x)$ is also continuous (si
nce it is the difference of two continuous functions).  Therefore $F(x) = \tilde{F}(x)
$ which is a contradiction.  So the Cantor function is unique.

Now we want to show that $F$ is differentiable on $[0,1] \backslash C$, which means we
 must show that the following limit exists and is finite,
\[ \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}. \]
If $x \in [0,1] \backslash C$ then there must exist some open interval of length $1/3^
n$ which contains $x$.  Let $d$ be the minimum distance between $x$ and the end of tha
t interval; note this distance will always exist because $x$ is in an open interval.
Then for $h < d$ we have that $F(x+h) - F(x) = 0$ (since everything on that interval w
as sent to some constant).  Thus our limit is 0, and following our definition that mea
ns that $F'(x) = 0$.

So to recap, $F(x)$ is a unique continuous map on $[0,1]$ and it is differentiable on
$[0,1] \backslash C$, and $F'(x) = 0$.
\end{proof}



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\item[\bf{1.20}] Let $X$ be a normed linear space.  A series $\sum x_n$ in $X$ is \tex
tit{absolutely convergent} if $\sum \| x_n \|$ converges to a finite value in $\R$.  P
rove that $X$ is a Banach space if and only if every absolutely convergent series conv
erges.

\begin{proof}
($\Rightarrow$)  Assume $X$ is a Banach space.  Let $\{x_n\}$ be an absolutely converg
ent series, then we have that
\[ \sum_{n=1}^\infty \|x_n\| \to M < \infty.\]
We want to show that $\{x_n\}$ converges.  Let $S_N = \sum_{n=1}^N x_n$ be our sequenc
e of partial sums.  For $N > M$ we have that $S_N - S_M = \sum_{n=M+1}^N x_n$ and $\|S
_N - S_M \|=  \| \sum_{n=M+1}^N x_n \|$.  Since this sum is finite we can use the tria
ngle inequality and so $\|S_N - S_M \| \leq \sum_{n=M+1}^N  \| x_n \|$ which goes to z
ero since $\{x_n\}$ is absolutely convergent.  Therefore $S_n$ is Cauchy and since we
are in a Banach space that means that $S_n$ converges and so $\{x_n\}$ is a convergent
 series.

($\Leftarrow$) Assume every absolutely convergent series converges.  We are given $X$
is a normed linear space.  Let $\{x_n\}$ be a Cauchy sequence.  We want to show that $
\{x_n\}$ converges. %or it is equivalent to show that $\{x_n\}$ is absolutely converge
nt by our above assumption.
Let's look at a subsequence $x_{n_i}$ of $x_n$ such that $\| x_{n_i} - x_{n_{i+1}} \|
< 1/2^i$.  So for $i=1$, choose $n_1 > N_1$ where $n_1,m > N_1 \Rightarrow \|x_{n_1}-x
_m\| < 1/2$ and for general $i$ chose $n_i>N_i\geq N_{i-1}$ where $n_i,m > N_i \Righta
rrow \|x_{n_i} -x_m \| < 1/2^i$.  It is clear that
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\[ \sum_{i=1}^\infty \|x_{n_i}-x_{n_{i+1}}\| < \sum_{i=1}^\infty \frac{1}{2^i} \]
converges.  Then by our assumption $\lim_{N\to \infty} \sum_{i=1}^N (x_{n_i}-x_{n_{i+1
}})$ converges to some $x \in X$.  Then using telescoping sums this expression is equi
valent to the following,
\begin{align*}
\lim_{N\to \infty} (x_{n_1} - x_{n_{N+1}} ) &= x \\
x_{n_1} - \lim_{N\to \infty}  x_{n_{N+1}} &= x \\
\lim_{N\to \infty}  x_{n_{N+1}} &= x_{n_1} -x
\end{align*}
and so our subsequence converges.  Then since $\{x_n\}$ is Cauchy it follows that $\{x
_n\}$ must also converge.
\end{proof}

%\newpage

%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\item[\bf{1.21}] Suppose that $X$ is a Banach space, and $(x_{mn})$ is a doubly indexe
d sequence in $X$ such that
\[\sum_{m=1}^\infty \sum_{n=1}^\infty \| x_{mn} \| < \infty.\]
Prove that
\[\sum_{m=1}^\infty \left( \sum_{n=1}^\infty  x_{mn}  \right) = \sum_{n=1}^\infty \lef
t( \sum_{m=1}^\infty  x_{mn}  \right). \]

\begin{proof}
By the previous exercise (1.20) we know that $\sum_{m=1}^\infty \left( \sum_{n=1}^\inf
ty  x_{mn}  \right)$ converges to some $x \in X$.  We want to show that
\[\lim_{N \to \infty} \sum_{m=1}^N \left( \sum_{n=1}^N  x_{mn}  \right) = \sum_{m=1}^\
infty \left( \sum_{n=1}^\infty  x_{mn}  \right). \]
To do this we must show that for a given $\epsilon > 0$ there exists $N'$ such that $N
 > N'$ implies
\[ \left\| \sum_{m=1}^N  \sum_{n=1}^N  x_{mn}  - \sum_{m=1}^\infty \sum_{n=1}^\infty
x_{mn}   \right\| < \epsilon \]
\[ \iff \left\| \sum_{m=N+1}^\infty \left( \sum_{n=N+1}^\infty x_{mn} \right) \right\|
 < \epsilon. \]
Let us rewrite this sum as a sequence of partial sums,
\[ \sum_{m=N+1}^\infty \left( \sum_{n=N+1}^\infty x_{mn} \right) = \lim_{M\to \infty}
\sum_{m=N+1}^M \left( \sum_{n=N+1}^\infty x_{mn} \right).\]
We know that $\sum_{n=N+1}^\infty x_{mn} = L_m < \infty$ for any given $m$ since it mu
st be absolutely convergent.  We can use the triangle inequality on this finite sum,
$$\left\| \lim_{M \to \infty} \sum_{m=N+1}^M \left( \sum_{n=N+1}^\infty x_{mn} \right)
 \right\| \leq
\lim_{M \to \infty}   \sum_{m=N+1}^M \left\| \sum_{n=N+1}^\infty x_{mn} \right\| \leq
\lim_{M \to \infty}  \sum_{m=N+1}^M \|L_m\|.$$

And since this last sum converges in $\R$ we know that for a given $\epsilon >0$ there
 exists some $N''$ such that the tail of our sum after $N''$ is less than $\epsilon$.
 Thus, if we let $N'=N''$ then we have shown that the limit from the beginning of the
proof is true.  Since each of the sums for $ \sum_{m=1}^N \left( \sum_{n=1}^N  x_{mn}
 \right)$ are finite then we can rearrange it and using the same proof can show that
\[\lim_{N \to \infty} \sum_{m=1}^N \left( \sum_{n=1}^N  x_{mn}  \right) = \sum_{n=1}^\
infty \left( \sum_{m=1}^\infty  x_{mn}  \right). \]
Therefore, it follows that we can interchange the summands of our series when we are w
orking in a Banach space.
\end{proof}
(Note: I'm not actually sure this proof is correct because I may have made some invali
d assumptions, particularly when I was evaluating the double sum of the tails of the s
ums.)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\item[\bf{1.24}] Let $f:X \to \R$ be a real-valued function on a set $X$.  The $\texti
t{epigraph}$ epi $f$ of $f$ is the subset $X \times \R$ consisting of points that lie
above the graph of $f$:
\[ \text{epi } f = \{ (x,t) \in X \times \R | t \geq f(x) \}.\]
Prove that a function is lower semicontinuous if and only if its epigraph is a closed
set.

\begin{proof}
First, we want to show that if a function, $f$, is lower semicontinuous then its epigr
aph, denoted epi$f$, is a closed set by doing the contrapositive.  We are given that $
```

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(X,d)$ is a metric space, thus $X \times \R$ is also a metric space using the metric f
rom exercise 1.4. %%%hmmmm
Assume that epi$f$ is not closed. Then by Proposition 1.41, there exists a sequence $\
{(a_n,b_n)\} \in$ epi$f$ such that $\{(a_n,b_n)\} \to (a,b) \in (X \times \R)$ but $(a
,b) \notin$ epi$f$.  Since $(a,b) \notin$ epi$f$ but $(a,b) \in (X \times \R)$ it foll
ows that $b < f(a)$.  Using this sequence in epi$f$ we have a sequence in $X$, $\{a_n\
} \to a$.  Also, we know by definition of epigraph that $f(a_n) \leq b_n$ for all $n$.
  Therefore,
\[ \liminf_{n \to \infty} f(a_n) \leq \liminf_{n \to \infty} b_n = \lim_{n \to \infty}
 b_n = b < f(a).\]
But this means that by definition $f$ is not lower semicontinuous.  Thus, if a functio
n, $f$, is lower semicontinuous then its epigraph is a closed set.

Now let us consider the case where epi$f$ is closed and we want to show that $f$ is lo
wer semicontinuous.
%There are two methods we can use here, either straightforward or proving the contrapo
sitive.  If we want to show the contrapositive then we need to show that if $f$ is not
 lower semicontinuous there exists a sequence in epi$f$ that does not converge to some
thing in epi$f$.
Let us show the contrapositive.  Suppose $f$ is not lower semicontinuous.  Then there
exists a sequence $\{x_n\} \to x \in X$ such that $\liminf_{n \to \infty} f(x_n) < f(x
)$.  In other words $\lim_{n \to \infty} [\inf \{f(x_k) | k \geq n\}] < f(x)$ so there
 exists a sequence in $\R$ such that $y_n =  [\inf \{f(x_k) | k \geq n\}]$ and this is
 monotonically increasing.  We know that $y_n = f(x_k)$ for some $x_k \in \{x_n\}$.  L
et us consider the sequence $t_n = (x_k, f(x_k))$ where $f(x_k) = y_n$.  The first coo
rdinate is a subsequence of $\{x_n\}$ and so it must converge to $x$ and we know $\lim
_{n \to \infty} y_n$ exists and is less than $f(x)$.   Therefore we have found a seque
nce of epi$f$ that converges to $(x, \lim_{n \to \infty} [\inf \{f(x_k) | k \geq n\}])
 \notin$ epi$f$.  Thus, epi$f$ is not closed.
%\vspace{3in}
\end{proof}

%%%%%%%%%%%%%%%%%%%%%%%%%%%
\item[\bf{1.26}] Let $p: \R^2 \to \R$ be a polynomial function of two real variables.
 Suppose that $p(x,y) \geq 0$ for all $x,y \in \R$.  Does every such function attain i
ts infimum?  Prove or disprove.

Claim: Every such function does not attain its infimum.
\begin{proof}
Consider the function $p(x,y) = (xy-1)^2 + y^2$ and the sequence $(x_n, y_n) = (n^2 +
n, 1/n^2)$ (this example was found by Brian Alger).  It should be clear that $p(x,y) \
geq 0$.  The limit of the sequence $p(x_n,y_n)$ as $n$ goes to infinity is zero, $\lim
_{n\to \infty} p(x_n, y_n) = \lim_{n\to \infty} \frac{n^2+1}{n^4} = 0$.  This function
 is a sum of continuous functions so $p(x,y)$ is continuous and therefore it is also s
equentially continuous.  Since $p(x,y)$ is sequentially continuous we know that for $\
epsilon > 0$ there exists $N$ such that $n>N$ implies $\abs{p(x_n,y_n)-0} < \epsilon$.
  Therefore our function gets very close to 0, and since $p(x,y) \geq 0$, it follows t
hat zero is our infimum.  However there does not exist $(x,y)$ such that $p(x,y) = 0$
because we would need $xy =1$ and $y=0$ which isn't possible since zero multiplied by
any number in $\R$ is zero.
\end{proof}


\end{itemize}
\end{document}
```

```
\documentclass[12point]{article}
\topmargin 0.0in
\headheight 0.0in
\headsep 0.0in
\oddsidemargin 1in
%\hoffset -1in
\textwidth 6in
\textheight 9.5in
\voffset=-.5in
\hoffset=-1in

\usepackage{color}
\usepackage{xy}
\input xy
\xyoption{all}
\usepackage{cancel}
\usepackage[mathcal]{eucal}

\usepackage{amsfonts, amstext, amsthm, amsmath, amssymb}
\usepackage{enumerate}
%\usepackage{graphics}
\usepackage{graphicx}
\usepackage{amscd}

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\newcommand{\G}{G_{\mathcal A}}
\newcommand{\D}{\mathfrak D}
\newcommand{\B}{\mathcal B}
\newcommand{\limit}[2]{lim_{#1\rightarrow #2}}
\newcommand{\abs}[1]{\left|#1\right|}    %|x|

\newcommand{\graybox}[1]{
\fcolorbox[rgb]{0.90,0.90,0.90}{0.90,0.90,0.90}{
\begin{minipage}{1\linewidth} #1
\end{minipage}
}}


\renewcommand{\proof}{\noindent \smallskip \textsc{Proof: }}
\newcommand{\soln}{\noindent \smallskip \textsc{Soln: }}


\title{MAT 201A - Analysis}
\author{Kristen Freeman}
\date{Friday, October 16, 2009}
```

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\maketitle

%2.2, 2.3, 2.5, 2.5, 2.6
\vspace{10pt}
\graybox{
\textbf{Ex. 2.2} \textit{ Let $f_n \in C([a,b])$ be a sequence of functions converging
 uniformly to a function $f$.  Show that
\[ \lim_{n \to \infty} \int_a^b f_n(x) dx = \int_a^b f(x)dx. \]
Give a counterexample to show that the pointwise convergence of continuous functions $
f_n$ to a continuous functions $f$ does not imply the convergence of the corresponding
 integrals.
}}

\vspace{.5cm}
\proof
First by Theorem 2.3 $f$ is continuous.  Next it is important to note that we can take
 the integrals of these functions because $[a,b]$ is a compact metric space and so by
Theorem 1.68 these functions attain their maximum and minimum so their integrals will
be defined.
Now, the limit above is equivalent to showing that for any $\epsilon >0$ there exists
$N$ such that $n > N$ implies
\[ \abs{ \int_a^b f_n(x) dx - \int_a^b f(x)dx} < \epsilon. \]

By basic rules of integrals (and Riemann Sum definitions),
\[ \abs{ \int_a^b f_n(x) dx - \int_a^b f(x)dx} = \abs{ \int_a^b f_n(x) - f(x)dx}  \leq
 \int_a^b \abs{f_n(x) - f(x)} dx. \]
Now using the fact that $f_n(x)$ converges to $f(x)$ uniformly, for any $\epsilon > 0$
 there exists $N$ such that $n > N$ implies $\sup_{a \leq x \leq b} \abs{f_n(x) - f(x)
} < \epsilon/(b-a)$.  Then for $n>N$ we have
\[ \int_a^b \abs{f_n(x) - f(x)} dx < \int_a^b \frac{\epsilon}{(b-a)} dx = \frac{\epsil
on}{(b-a)} (b-a) = \epsilon.\]  Thus the limit of the integral of the sequence of func
tion is the integral of the limit of the sequence of functions.
\qed\\

\textit{Counterexample:}  Consider the following sequence of functions:
\[ f_n(x) = \left\{
\begin{array}{l l}
  4n^2(x-a), & \quad a \leq x \leq (a+ {1 \over 2n})\\
  -4n^2(x-a) + 4n, & \quad (a + 1/2 n) < x \leq (a + {1 \over n})\\
  0, & \quad \text{ otherwise.}\\
\end{array} \right.
\]
Graphically $f_n(x)$ is a triangle with area 1 whose base is getting smaller and heigh
t is getting proportionally bigger for each $n$ and is equal to zero otherwise.
First, it is clear this sequence is in $C([a,b])$ as long as $b > a + {1 \over n}$.
This sequence converges pointwise to the function $f(x) = 0$, because for any $x \in [
a,b]$ and $\epsilon >0$ we can find $N$ such that $f_n(x) = 0 < \epsilon$ for all $n >
 N$.  However, the corresponding integrals do not converge since $\int_a^b f_n(x) dx =
 1$ for all $n$ but $\int_a^b 0 dx = 0$.


\newpage
%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%
\vspace{10pt}
\graybox{\textbf{Ex. 2.3 } \textit{ Suppose that $f: G \to \R$ is a uniformly continuo
us function defined on an open subset $G$ of a metric space $X$.  Prove that $f$ has a
 unique extension to a continuous function $\overline{f}: \overline{G} \to \R$ defined
 on the closure $\overline{G}$ of G.  Show that such a extension need not exist if $f$
 is continuous but not uniformly continuous on $G$.
}}

\vspace{.5cm}

\proof \textbf{(Part A)}
Let $\overline{f}(x) = \lim_{n \to \infty} f(x_n)$ where $x_n \to x$ in $\overline{G}$
.  We need to show that $\lim_{n \to \infty} f(x_n)$ exists and that this function is
well-defined.  Since $\overline{f}(x) = f(x)$ for all $x \in G$, and we already know $
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