MAT 201a Homework # 3
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2.2 Let $f_n \in C([a,b])$ be a sequence of functions converging uniformly to a function f. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Give a counterexample to show that the pointwise convergence of continuous functions f_n to a continuous function f does not imply the convergence of the corresponding integrals.

Solution. Suppose f_n converges uniformly to f. Then we know $\forall \epsilon > 0\ \exists\ N \in \mathbb{N}$ such that $n > N$ implies

$$\sup_{x\in X} |f_n(x) - f(x)| = ||f_n - f|| < \frac{\epsilon}{b-a}.$$

Now we can use a well known property about Riemann integrals

$$\left|\left|\int_a^b [f_n(x) - f(x)]\,dx\right|\right| \leq \int_a^b ||f_n(x) - f(x)||\,dx < \int_a^b \frac{\epsilon}{b-a}\,dx = (b-a)\cdot\frac{\epsilon}{b-a} = \epsilon.$$

This shows the differences in their integrals converge to each other. That is,

$$\left|\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right|\right| < \epsilon$$

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Therefore, if f_n is a sequence of functions converges uniformly to f, then their integrals will converge to each other as well.

To show a counter-example, let $X = [0,1]$. Then we define a sequence of functions

$$f_n(x) = \begin{cases} 0 & 0 \leq x < 1/n \\ n(nx-1) & 1/n \leq x < 2/n \\ n & x = 2/n \\ -n(nx-3) & 2/n < x < 3/n \\ 0 & 3/n \leq x \leq 1 \end{cases}$$

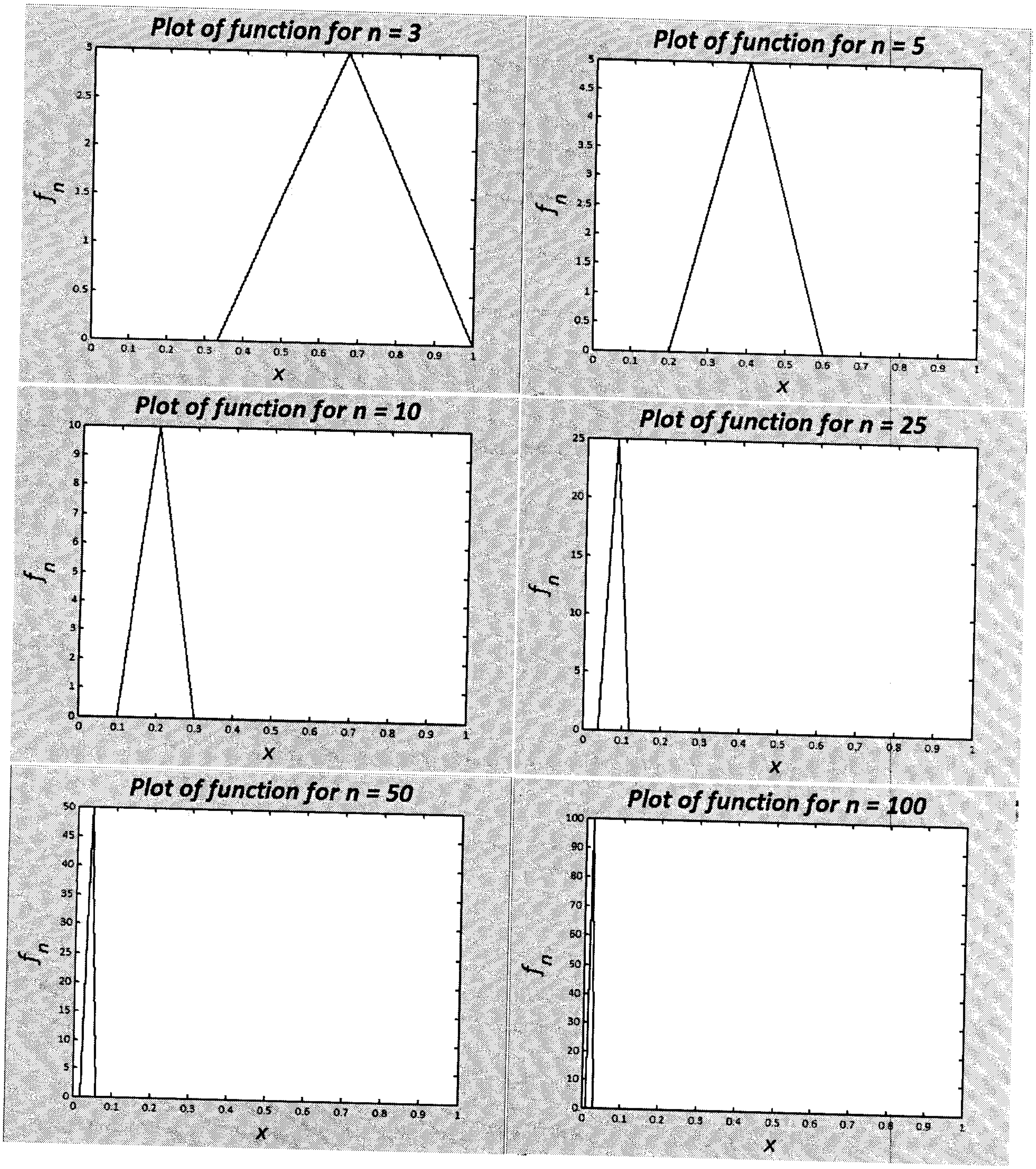
In other words, the function connects the points: $(0,0)$ to $(1/n,0)$ to $(2/n,n)$ to $(3/n,0)$ to $(1,0)$ (See figure). The limit of f_n approaches $f = 0$ because the point $x = 3/n$ goes to 0 as n goes to infinite. However, each the integral of each function is given by

$$\frac{1}{2}bh = \frac{1}{2}\cdot\frac{2}{n}\cdot n = 1$$

for all n. The integral of f however, is 0. Clearly, 1 does not converge to 0, so the integral of the limits of f_n do not converge to the integral of f.

□

2.3 Suppose that $f : G \to \mathbb{R}$ is a uniformly continuous function defined on an open subset G of a metric space X. Prove that f has a unique extension to a continuous function $\overline{f} : \overline{G} \to \mathbb{R}$ defined on the closure $\overline{G}$ of G. Show that such an extension need not exist if f is continuous but not uniformly continuous on G.


Plot of function for n = 3
Plot of function for n = 5
Plot of function for n = 10
Plot of function for n = 25
Plot of function for n = 50
Plot of function for n = 100
f_n
x

Solution. Since f is uniformly continuous from $G \subset X$ to $\mathbb{R}$, $\forall \epsilon > 0 \; \exists \; \delta$ such that

$$d(x, y) < \delta, \Rightarrow |f(x) - f(y)| < \epsilon.$$

We want to continue the function to the boundary of G to define $\overline{f} : \overline{G} \to \mathbb{R}$. Let $\overline{G} \setminus G = \partial G$. Since G is open, $G \cup \partial G = \overline{G}$. Since $\overline{G}$ is closed, there exists a sequence (x_k) converging to a point $x \in \overline{G}$. Since f is defined everywhere already in G, let us take a sequence (x_k) in G that converges to ∂G. Since the sequence (x_k) is convergent, it is also Cauchy (by homework 1). Then $\forall \delta > 0$, $\exists \; N \in \mathbb{N}$ such that

$$n, m > N \Rightarrow d(x_n, x_m) < \delta$$

Since f is uniformly continuous, we have that

$$d(x_n, x_m) < \delta \;\Rightarrow\; |f(x_n) - f(x_m)| < \epsilon.$$

A little technicality I should fix is this. Here, I choose an ϵ first. That forces my δ to be dependent on it, so I should write $\delta = \delta(\epsilon)$. It is after I choose this value that I choose an N. So I should write $N = N(\epsilon, \delta)$. Therefore, $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$. Let $x \in \overline{G}$, with a sequence $(x_k) \to x$. I define the extension of the uniformly continuous function as follows

$$\overline{f}(x) = \begin{cases} f(x) \text{ if } x \in G \\ \lim_{n \to \infty} f(x_n) \text{ if } x \in \partial G. \end{cases}$$

To check this function is actually a function, let two sequences (x_k) and (y_j) both in G converge to the same limit z in ∂G. Then for any $\delta/2 > 0$, there are natural numbers N_x, N_y such that if $n > N_x$ and $m > N_y$, then

$$d(x_n, z) < \delta/2 \text{ and } d(z, y_m) < \delta/2.$$

Set $N = \max\{N_x, N_y\}$. By the triangle inequality, we have

$$d(x_n, y_m) \le d(x_n, z) + d(z, y_m) < \delta/2 + \delta/2 = \delta.$$

Using the fact that these sequences both live in G, by uniform continuity we have for all $\epsilon/3 > 0$,

$$d(x_n, y_m) < \delta \;\Rightarrow |f(x_n) - f(y_m)| < \frac{\epsilon}{3}.$$

Notice that the sequence $f(x_k)$ is Cauchy by the calculation above, and that $f(y_j)$ is Cauchy as well. Let the limits of these Cauchy sequences be a and be, respectively. These live in $\mathbb{R}$ by completeness of $\mathbb{R}$. Then for all $\epsilon/3 > 0$,

$$|a - f(x_n)| < \frac{\epsilon}{3} \text{ and } |f(y_m) - b| < \frac{\epsilon}{3}.$$

We can use these inequalities to conclude the limits of these two functions are equal

$$|a - b| \le |a - f(x_n)| + |f(x_n) - f(y_m)| + |f(y_m) - b| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

To show uniqueness, suppose there are two extensions of continuous functions, $\overline{f}$ and h. Let (x_k) converge to a limit in $x \in \overline{G}$. Consider the function $c(x_k) = \overline{f}(x_k) - h(x_k)$. Since both $\overline{f}$ and h are continuous, then c is continuous as well. Then $x_k \to x$ implies $c(x_k) \to c(x)$. Because both extensions are defined to be f if $x \in G$, we have

$$\overline{f}(x) - h(x) = c(x) = \lim_{k \to \infty} c(x_k) = \lim_{k \to \infty} \overline{f}(x_k) - h(x_k) = \lim_{k \to \infty} f(x_k) - f(x_k) = 0.$$

Therefore, the extension of a uniformly continuous real-valued function defined on an open subset of a metric space uniquely extends to its boundary.

To give a counter example, take $f(x) = \sin(1/x)$, the topologist's sine curve. Let's focus our attention on the interval $I = (0, 1/\pi)$. The function is continuous on I because f is a composition of continuous functions well-defined on I. However, near $x = 0$, the function oscillates infinitely often. Therefore, as it gets closer to the y-axis, it has no limit. Thus, f cannot be extended to the left endpoint of $\overline{I} = [0, 1/\pi]$. □

Problem 2.4

Give a counterexample to show that $f_n \to f$ in $C([0,1])$ and f_n continuously differentiable does not imply that f is continuously differentiable.

Consider the function

$$f_n(x) = \begin{cases} -(x-\frac{1}{2}) - \frac{1}{2n} & 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}(x-\frac{1}{2})^2 & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n} \\ (x-\frac{1}{2}) - \frac{1}{2n} & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$

Each $f_n(x)$ is continuous and differentiable everywhere on $[0,1]$. This is true because each piece is continuous and differentiable, and at the connection points the value of the functions are equal and the derivatives are equal. (f_n) converges to $f = |x - \frac{1}{2}|$ which is continuous everywhere on [0,1] but is not differentiable at $x = \frac{1}{2}$.

Problem 2.5

Consider the space of continuously differentiable functions,

$$C^1([a,b]) = \{f : [a,b] \to \mathbb{R} | f, f' \text{ are continuous}\}$$

with the C^1-norm,

$$\|f\| = \sup_{a \le x \le b} |f(x)| + \sup_{a \le x \le b} |f'(x)|$$

Prove that $C^1([a,b])$ is a Banach space.

We will assume that $C^1([a,b])$ is a normed linear space and all we need to show is completeness, so let (f_n) be a Cauchy sequence in $C^1([a,b])$ with respect to the $\|\cdot\|_{C^1}$ norm defined in the problem statement. Then each f_n , $f_n' \in C([a,b], \|\cdot\|_{\text{sup}})$. We know that $C([a,b])$ is complete and thus there exists $f, g \in C([a,b])$ such that $f_n \to f$, and $f_n' \to g$ (uniformly) with respect to $\|\cdot\|_{\text{sup}}$. If we let

$$F_n(x) = \int_a^x f_n(t)dt, \qquad F(x) = \int_a^x f(t)dt$$

Then $F_n \to F$ converges uniformly because :

$$\|F_n - F\|_{\text{sup}} \le \sup_{x \in [a,b]} \int_a^x |f_n(t) - f(t)|\, dt \le \|f_n - f\|_{\text{sup}} < \epsilon$$

From the fundamental theorem of calculus

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)dt \tag{1}$$

Since $f_n' \to g$ uniformly, by Problem 2.2,

$$\int_a^x f_n'(t)dt \to \int_a^x g(t)dt \tag{2}$$

Since we also know that $f_n \to f$ uniformly, we put (1) and (2) together to get

$$f(x) - f(a) = \int_a^x g(t)dt$$

which by the fundamental theorem of calculus implies that $f' = g$. So we now have that $f_n \to f$ (uniformly) and $f_n' \to g = f'$ which means that $f_n \to f \in C^1([a,b])$ with respect to $\|\cdot\|_{C^1}$. So every Cauchy sequence in $C^1([a,b])$ converges to a point inside $C^1([a,b])$ and thus we have shown that it is complete and thus a Banach space.

Problem 2.6

Show that the space $C([a,b])$ equipped with the L^1-norm $\|\cdot\|_1$ defined by

$$\|f\|_1 = \int_a^b |f(x)|dx$$

is incomplete. Show that if $f_n \to f$ with respect to the sup-norm $\|\cdot\|_\infty$, then $f_n \to f$ with respect to $\|\cdot\|_1$. Give a counterexample to show that the converse statement is false.

1. Show that $\|f\|_1$ is incomplete. We need to find a Cauchy sequence that does not converge with respect to $\|\cdot\|_1$

   Let $f_n(x) = \begin{cases} 0 & x \in [a, \frac{b-a}{2}) \\ nx - n\frac{b-a}{2} & x \in [\frac{b-a}{2}, \frac{b-a}{2} + \frac{1}{n}) \\ 1 & x \in [\frac{b-a}{2} + \frac{1}{n}, b] \end{cases}$. This converges to the step function $f(x) = \begin{cases} 0 & x \in [a, \frac{b-a}{2}) \\ 1 & x \in [\frac{b-a}{2}, b] \end{cases}$ which is not continuous. If we look at the sequence $\|f_n - f\|_1$ then this is a Cauchy sequence but it converges to a function outside of $C([a,b])$. So $(C([a,b], \|\cdot\|_1)$ is incomplete.

2. Show that if $f_n \to f$ with respect to the sup-norm $\|\cdot\|_{\text{sup}}$, then $f_n \to f$ with respect to $\|\cdot\|_1$:

$$\|f_n(x) - f(x)\|_{\text{sup}} = \lim_{n\to\infty} \sup_{x\in[a,b]} |f_n(x) - f(x)|$$

   Now if we look at

$$\|f_n(x) - f(x)\|_1 = \int_a^b |f_n(x) - f(x)|\, dx$$

   We know that continuous functions will reach their maximum and minimum on a closed interval and since $f_n(x)$ and $f(x)$ are both continuous, their difference is also continuous and therefore will reach its maximum and minimum. By definition this is $\|f_n(x) - f(x)\|_{\text{sup}}$ and we can use this to bound the integral. Since, by assumption, we can make $\|f_n(x) - f(x)\|_{\text{sup}} < \epsilon$

$$\int_a^b |f_n(x) - f(x)|\, dx \leq (b-a)\|f_n(x) - f(x)\|_{\text{sup}} < \epsilon$$

   But this then implies that $\|f_n(x) - f(x)\|_1$ converges to 0 and that $f_n \to f$ with respect to $\|\cdot\|_1$.

3. Give a counterexample to show that the converse statement is false.

   We can look at the sequence $(f_n(x))$ where $f_n(x) = x^n$ on the interval [0,1]. Each $f_n(x)$ is continuous on the entire interval. Then $f_n(x)$ goes towards 0 on [0,1) and $f_n(1) = 1$. We would like $f(x)$ to be continuous so we set $f(x) = 0$ for $x \in [0,1]$. Then our integral

$$\int_0^1 |f_n(x) - f(x)|dx = \int_0^1 |f_n(x)|dx$$

   is well defined everywhere on $x \in [0,1]$ because f_n is bounded on the interval. We have that

$$\|f_n(x) - f(x)\|_1 = \int_0^1 |f_n(x)|dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1} < \epsilon$$

   Where the last inequality is because we can always find an $n > N$ to sastify it for any $\epsilon > 0$. Thus $\lim_{n\to\infty} \|f_n(x) - f(x)\|_1 = 0$ and thus $f_n \to f$ with respect to $\|\cdot\|_1$.

Now we look at

$$\|f_n(x) - f(x)\|_{\text{sup}} = \lim_{n\to\infty} \sup |f_n(x) - f(x)|$$

This does not converge because $\|f_n(x) - f(x)\|_{\text{sup}} = 1$ as $n \to \infty$ because $f(1) = 0$ and each $f_n(1) = 1$. So f_n does not converge to f with respect to $\|\cdot\|_{\text{sup}}$.