```
\documentclass[12point]{article}
\topmargin 0.0in
\headheight 0.0in
\headsep 0.0in
\oddsidemargin 1in
%\hoffset -1in
\textwidth 6in
\textheight 9.5in
\voffset=-.5in
\hoffset=-1in

\usepackage{color}
\usepackage{xy}
\input xy
\xyoption{all}
\usepackage{cancel}
\usepackage[mathcal]{eucal}

\usepackage{amsfonts, amstext, amsthm, amsmath, amssymb}
\usepackage{enumerate}
%\usepackage{graphics}
\usepackage{graphicx}
\usepackage{amscd}

\newenvironment{sgitem}{
\begin{enumerate}
  \setlength{\itemsep}{1pt}
  \setlength{\parskip}{0pt}
  \setlength{\parsep}{0pt}
}{\end{enumerate}}


\begin{document}

\newcommand{\red}{\textcolor{red}}
\newcommand{\blue}{\textcolor{blue}}
\newcommand{\black}{\textcolor{black}}
\newcommand{\brown}{\textcolor{brown}}
\newcommand{\green}{\textcolor{darkgreen}}
\newcommand{\yellow}{\textcolor{yellow}}
\newcommand{\pink}{\textcolor{pink}}

\newcommand{\N}{\mathbb{N}}
\newcommand{\Z}{\mathbb{Z}}
\newcommand{\Q}{\mathbb{Q}}
\newcommand{\R}{\mathbb{R}}
\newcommand{\C}{\mathbb{C}}
\newcommand{\A}{\mathcal{A}}
\newcommand{\F}{\textbf{F}}                %%%new command for fields
\newcommand{\G}{G_{\mathcal A}}
\newcommand{\D}{\mathfrak D}
\newcommand{\B}{\mathcal B}
\newcommand{\limit}[2]{lim_{#1\rightarrow #2}}
\newcommand{\abs}[1]{\left|#1\right|}    %|x|

\newcommand{\graybox}[1]{
\fcolorbox[rgb]{0.90,0.90,0.90}{0.90,0.90,0.90}{
\begin{minipage}{1\linewidth} #1
\end{minipage}
}}


\renewcommand{\proof}{\noindent \smallskip \textsc{Proof: }}
\newcommand{\soln}{\noindent \smallskip \textsc{Soln: }}


\title{MAT 201A - Analysis}
\author{Kristen Freeman}
\date{Friday, October 16, 2009}
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\maketitle

%2.2, 2.3, 2.5, 2.5, 2.6
\vspace{10pt}
\graybox{
\textbf{Ex. 2.2} \textit{ Let $f_n \in C([a,b])$ be a sequence of functions converging
 uniformly to a function $f$.  Show that
\[ \lim_{n \to \infty} \int_a^b f_n(x) dx = \int_a^b f(x)dx. \]
Give a counterexample to show that the pointwise convergence of continuous functions $
f_n$ to a continuous functions $f$ does not imply the convergence of the corresponding
 integrals.
}}

\vspace{.5cm}
\proof
First by Theorem 2.3 $f$ is continuous.  Next it is important to note that we can take
 the integrals of these functions because $[a,b]$ is a compact metric space and so by
Theorem 1.68 these functions attain their maximum and minimum so their integrals will
be defined.
Now, the limit above is equivalent to showing that for any $\epsilon >0$ there exists
$N$ such that $n > N$ implies
\[ \abs{ \int_a^b f_n(x) dx - \int_a^b f(x)dx} < \epsilon. \]

By basic rules of integrals (and Riemann Sum definitions),
\[ \abs{ \int_a^b f_n(x) dx - \int_a^b f(x)dx} = \abs{ \int_a^b f_n(x) - f(x)dx}  \leq
 \int_a^b \abs{f_n(x) - f(x)} dx. \]
Now using the fact that $f_n(x)$ converges to $f(x)$ uniformly, for any $\epsilon > 0$
 there exists $N$ such that $n > N$ implies $\sup_{a \leq x \leq b} \abs{f_n(x) - f(x)
} < \epsilon/(b-a)$.  Then for $n>N$ we have
\[ \int_a^b \abs{f_n(x) - f(x)} dx < \int_a^b \frac{\epsilon}{(b-a)} dx = \frac{\epsil
on}{(b-a)} (b-a) = \epsilon.\]  Thus the limit of the integral of the sequence of func
tion is the integral of the limit of the sequence of functions.
\qed\\

\textit{Counterexample:}  Consider the following sequence of functions:
\[ f_n(x) = \left\{
\begin{array}{l l}
  4n^2(x-a), & \quad a \leq x \leq (a+ {1 \over 2n})\\
  -4n^2(x-a) + 4n, & \quad (a + 1/2 n) < x \leq (a + {1 \over n})\\
  0, & \quad \text{ otherwise.}\\
\end{array} \right.
\]
Graphically $f_n(x)$ is a triangle with area 1 whose base is getting smaller and heigh
t is getting proportionally bigger for each $n$ and is equal to zero otherwise.
First, it is clear this sequence is in $C([a,b])$ as long as $b > a + {1 \over n}$.
This sequence converges pointwise to the function $f(x) = 0$, because for any $x \in [
a,b]$ and $\epsilon >0$ we can find $N$ such that $f_n(x) = 0 < \epsilon$ for all $n >
 N$.  However, the corresponding integrals do not converge since $\int_a^b f_n(x) dx =
 1$ for all $n$ but $\int_a^b 0 dx = 0$.


\newpage
%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%
\vspace{10pt}
\graybox{\textbf{Ex. 2.3 } \textit{ Suppose that $f: G \to \R$ is a uniformly continuo
us function defined on an open subset $G$ of a metric space $X$.  Prove that $f$ has a
 unique extension to a continuous function $\overline{f}: \overline{G} \to \R$ defined
 on the closure $\overline{G}$ of G.  Show that such a extension need not exist if $f$
 is continuous but not uniformly continuous on $G$.
}}

\vspace{.5cm}

\proof \textbf{(Part A)}
Let $\overline{f}(x) = \lim_{n \to \infty} f(x_n)$ where $x_n \to x$ in $\overline{G}$
.  We need to show that $\lim_{n \to \infty} f(x_n)$ exists and that this function is
well-defined.  Since $\overline{f}(x) = f(x)$ for all $x \in G$, and we already know $
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f$ is uniformly continuous on $G$ we really only need to consider how the function beh
aves on the limit points of $G$; i.e. on the set $\overline{G} \backslash G$. Let us c
onsider $\{x_n\} \in G$ such that $x_n \to x$ where $x \in \overline{G} \backslash G$.

 \textit{Claim 1.:} $\{f(x_n)\}$ is Cauchy.

We want to show that for any $\epsilon >0$ there is some $N$ such that $m,n >N$ implie
s $\abs{f(x_n) - f(x_m)} < \epsilon$.  Since $f(x)$ is uniformly continuous there exis
ts $\delta >0$ such that if $d_X(x_n,x_m) < \delta$ then $\abs{f(x_n) - f(x_m)} < \eps
ilon$.  Then because $x_n \to y$ there exists an $N'$ such that if $n > N'$ then $d_X(
x_n,y) < \delta/2$.  By the triangle inequality it follows that if $n,m > N'$ then $d_
X(x_n,x_m) \leq d_X(x_n,y)+ d_X(y,x_m) < \delta/2 + \delta/2 = \delta$.  Thus $N = N'$
.


Since $\{f(x_n)\}$ is Cauchy and $\R$ is complete it follows that the limit exists.

\textit{Claim 2.:} $\overline{f}(x)$ is well-defined.

Suppose there are two sequences $\{x_n\}$ and $\{\widetilde{x_m}\}$ which converge to
$y$, then we want to show that $\lim_{n\to \infty} f(x_n) = \lim_{n\to \infty} f(\wide
tilde{x_m})$.  This fact follows mostly from what we did above.  Since $f(x)$ is unifo
rmly continuous there exists $\delta >0$ such that if $d_X(x_n,\widetilde{x_m}) < \del
ta$ then $\abs{f(x_n) - f(\widetilde{x_m})} < \epsilon$.  Because $x_n \to y$ and $\wi
detilde{x_m} \to y$ for any $\delta > 0$ there exists an $N, N'$ such that if $n > N$
then $d_X(x_n,y) < \delta/2$ and if $m > N'$ then  $d_X(\widetilde{x_m},y) < \delta/2$
 and if we use the triangle inequality again we see that $f(x_n)$ and  $f(\widetilde{x
_m})$ are getting very close together and so they must be equal.

By definition $\overline{f}(x)$ is sequentially continuous and thus by Proposition 1.3
4 it is continuous.  To show the function is unique suppose otherwise, suppose there e
xists another continuous extension $\widetilde{f}(x)$.  The only points the two extens
ions can disagree on are the limit points of $G$.  So there must be some $y \in \overl
ine{G}\backslash G$ such that $\overline{f}(y) \neq \widetilde{f}(y)$.  But we have al
ready shown that given $\{x_n\} \in G$ such that $x_n \to y$, $\overline{f}(y) = \lim_
{n\to \infty} f(x_n) = \widetilde{f}(y)$ which is a contradiction.  Thus, the extensio
n is unique.
\qed\\



\proof \textbf{(Part B)}
Suppose $f$ is continuous but not uniformly continuous.  Then there exists $\epsilon >
0$ such that for all $\delta >0$, there are $x,y \in G$ with $d_x(x,y) < \delta$ and $
\abs{f(x) - f(y)} > \epsilon$.  Taking $\delta = 1/n$ for $n \in \N$, we find that the
re are sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that
\[ d_X(x_n,y_n) < 1/n , \quad \quad \abs{f(x_n)-f(y_n)} > \epsilon. \]
Suppose these sequences $\{x_n\}$ and $\{y_n\}$ converge to $z \in \overline{G}\backsl
ash G$.  Then the sequences converge to the same limit but the sequences $f(x_n), f(y_
n)$ either diverge or converge to different limits which would contradict the continui
ty of an extension function.  Thus an extension need not exit if $f$ is continuous but
 not uniformly continuous.

\qed



%%%%%
\vspace{10pt}
\graybox{\textbf{Ex. 2.4 } \textit{ Give a counterexample to show that $f_n \to f$ in
$C([0,1])$ and $f_n$ continuously differentiable does not imply that $f$ is continuous
ly differentiable.
}}

\vspace{.5cm}
\textit{Counterexample:}  Consider the function $f(x) = \abs{x-{1\over 2}}$.  We know
$f(x)$ is not continuously differentiable because $f'(1/2)$ does not exist. %%add more
 description here?
However, there does exist a sequence of continuously differentiable functions which co
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nverges to $f(x)$.  Consider
\[ \abs{x-{1\over 2}} = {4 \over \pi} \sum_{k=1}^\infty \frac{(-1)^{k-1}}{4k^2-1} T_{2
k}(x-{1\over 2}) + {2 \over \pi}; \;\;\; x \in (-{1\over 2}, {3 \over 2}),\]
where $T_{2k}(x-{1\over 2})$ is a Chebyshev polynomial of the first kind.  We can let
our sequence be the sequence of parital sums.  It is clear that each $f_n$ in the sequ
ence is continuous since they are each polynomials.  By that same fact they are differ
entiable and their derivative is continuous, thus the sequence is continuously differe
ntiable.  But as we already stated the limit of the $f_n$ as $n \to \infty$ is $\abs{x
}$ which is not continuously differentiable.



\newpage
%%%%%
\vspace{10pt}
\graybox{\textbf{Ex. 2.5 } \textit{ Consider the space of continuously differentiable
functions,
\[ C^1([a,b]) = \{f:[a,b] \to \R \; | \; f, f' \text{ are continuous} \}, \]
with $C^1-$norm,
\[ \| f \| = \sup_{a \leq x \leq b} \abs{f(x)} + \sup_{a \leq x \leq b} \abs{f'(x)}. \
]
Prove that $C^1([a,b])$ is a Banach space.
}}

\vspace{.5cm}
\proof
In order to show that $C^1([a,b])$ is a Banach space we must show that it is a complet
e. %, normed linear space with the above norm.  It is clear that $C^1([a,b])$ is a nor
med linear space.  To show that $C^1([a,b])$ is complete
To do this we must show that every Cauchy sequence converges in $C^1([a,b])$.  Let $\{
f_n(x)\}$ be a Cauchy sequence in $C^1([a,b])$.  Since the sup-norm of a function is l
ess than or equal to the $C^1-$norm it follows that $\{f_n(x)\}$ is Cauchy with respec
t to the sup-norm and likewise for $\{f_n'(x)\}$.  By the Heine-Borel Theorem we know
that $[a,b]$ is compact and then by Theorem 2.4 we know that $C([a,b])$ is complete.
Thus $\{f_n(x)\}$, $\{f_n'(x)\}$ converge to $f(x),g(x)$ respectively in $C([a,b])$.
All that is left to show is that $f'(x) = g(x)$.

By Theorem 1.67 $g(x)$ is uniformly continuous so for any $\epsilon >0$ there is a $\d
elta > 0$ such that $\abs{x-y} < \delta$ implies $\abs{g(x)-g(y)} < \epsilon$.  Fix $x
 \in [a,b]$, and $h$ such that if $\abs{x-y} < h$ then $\abs{g(x) - g(y)} < \epsilon/2
$.  Then by the Mean Value Theorem we know,
\[\frac{f_n(x+h) - f_n(x)}{h} = f_n'(c_n), \quad \text{ where } c_n \in (x,x+h). \]
The sequence $\{c_n\}$ is a bounded sequence so there exists a convergent subsequence
$c_{n_k} \to c \in (x,x+h)$.  Since $f_n'(x) \to g(x)$ we can find a $K$ such that for
 $k>K$, $\sup\abs{f_{n_k}'(x) - g(x)} < \epsilon/2$.  Then using the triangle inequali
ty we have,
\[ \abs{ f_{n_k}'(c_{n_k}) - g(c)} \leq \abs{ f_{n_k}'(c_{n_k}) - g(c_{n_k}))} + \abs{
g(c_{n_k}) - g(c)} < \epsilon/2 + \epsilon/2 = \epsilon. \]
Thus $\lim_{k\to\infty} f_{n_k}'(c_{n_k}) = g(c)$ and moreover,
\[ \lim_{k \to \infty} \frac{f_{n_k}(x+h) - f_{n_k}(x)}{h} = \lim_{k\to \infty} f_{n_k
}'(c_k) = g(c). \]
Using the definition of derivative, we let $h\to 0$ then $c \to x$ and we get $f'(x) =
 g(x)$.  Therefore  $C^1([a,b])$ is complete and furthermore a Banach space.
\qed



%%%%%
\vspace{10pt}
\graybox{\textbf{Ex. 2.6 } \textit{ Show that the space $C([a,b])$ equipped with the $
L^1-$norm $\| \cdot \|_1$ defined by
\[ \| f \|_1 = \int_a^b \abs{f(x)} dx, \]
is incomplete.  Show that if $f_n \to f$ with respect to the sup-norm $\| \cdot \|_\in
fty$, then $f_n \to f$ with respect to $\| \cdot \|_1$.  Give a counterexample to show
 the converse statement is false.
}}

\vspace{.5cm}
\proof \textbf{(Part A)}
```

```
To show that $C([a,b])$ equipped with the $L^1-$norm is incomplete we need to find a C
auchy sequence of continuous functions which does not converge with respect to the $L^
1$ norm.  Consider the following sequence:
\[ f_n(x) = \left\{
\begin{array}{l l}
  0, & \quad x \in [a,\frac{a+b}{2}]\\
  2^n(x-\frac{a+b}{2}), & \quad x \in \left(\frac{a+b}{2}, \frac{a+b}{2} + \frac{1}{2^
n}\right)\\
  1, & \quad \text{ otherwise.}\\
\end{array} \right.
\]
Graphically $f_n(x)$ is made up of horizontal line segments at 0 and 1 and another lin
e segment connecting the two pieces. It is clear that $f_n(x)$ is in $C([a,b])$.  Also
 for $m>n$,
\[ \int_a^b \abs{f_n(x) -f_m(x)} dx = \frac{1}{2}\left(\frac{1}{2^n} - \frac{1}{2^m} \
right). \]
Since $\frac{1}{2^n}$ is Cauchy it follows that the above sequence is Cauchy with resp
ect to the $L^1$ norm.  However $f_n(x)$ converges to something that is not continuous
,
\[ \lim_{n\to \infty} f_n(x) = f(x) = \left\{
\begin{array}{l l}
  0, & \quad x \in [a,\frac{a+b}{2}]\\
  1, & \quad \text{ otherwise.}\\
\end{array} \right.
\]
Thus $f_n(x)$ converges to something not contained in $C([a,b])$, so $C([a,b])$ is inc
omplete when equipped with the $L^1$ norm.

\qed\\

\proof \textbf{(Part B)}
Suppose $f_n \to f$ with respect to the sup-norm $\| \cdot \|_\infty$.  Then for all $
\epsilon > 0$ there exists $N$ such that $n > N$ implies $\sup_{a \leq x \leq b} \abs{
f_n(x) - f(x)} < \frac{\epsilon}{(b-a)}$.  Let $g_n(x) = f_n(x) - f(x)$.  Since $f_n(x
) \in C([a,b])$, by Theorem 1.68 each $f_n(x)$ is bounded and attains its maximum and
minimum.  And since $f_n \to f$ with respect to the sup-norm we know that $f$ is likew
ise bounded, furthermore $g_n(x)$ is bounded so we can consider $\| g_n \|_1$.  We wan
t to show that there exists $N$ such that for $n>N$, $\| g_n \|_1 < \epsilon$.  We hav
e the following,
\begin{align*}
\| g_n \|_1 &=  \int_a^b \abs{f_n(x) - f(x)} dx \leq  \int_a^b  \|f_n(x) - f(x)\|_{\su
p}  dx =\\
&= \|f_n(x) - f(x)\|_{\sup} \int_a^b dx = \|f_n(x) - f(x)\|_{\sup} \abs{b-a}.
\end{align*}
Thus if we use the same $N$ as we used for the sup-norm then $\|g_n\|_1 < \epsilon$.
Thus f $f_n \to f$ with respect to the sup-norm $\| \cdot \|_\infty$, then $f_n \to f$
 with respect to $\| \cdot \|_1$.
\qed\\

\textbf{(Part C)} \textit{Counterexample:}
Consider the following sequence of functions:
\[ f_n(x) = \left\{
\begin{array}{l l}
  2n^3(x-a), & \quad a \leq x \leq (a+ \frac{1}{ 2n^2})\\
  -2n^3(x-a-\frac{1}{n^2}), & \quad (a+ \frac{1}{ 2n^2}) \leq x \leq (a + {1 \over n^2
})\\
  0, & \quad \text{ otherwise.}\\
\end{array} \right.
\]
Graphically $f_n(x)$ is a triangle at the beginning of the interval $[a,b]$ and 0 afte
rwards.  Note $b > (a + {1 \over n^2})$, otherwise just start the sequence when this i
nequality holds.  Now $f_n(x) \to f(x) = 0$.  So $\| f_n(x) - f(x) \| = \int_a^b f_n(x
) dx = \frac{1}{2n}$.  Thus it is clear $f_n(x)$ converges to $f(x)$ with respect to t
he $L^1$ norm.  However $f_n(x)$ does not converge to $f(x)$ with respect to the sup-n
orm since $\sup_{a \leq x \leq b} \abs{f_n(x) - f(x)} = n$ which is not approaching 0.
```