

Problem 2.13

Consider the scalar initial value problem,

$$\begin{aligned} \dot{u}(t) &= |u(t)|^\alpha \\ u(0) &= 0 \end{aligned} \tag{1}$$

Show that the solution is unique if $\alpha \geq 1$, but not if $0 \leq \alpha < 1$

- Initial value problem:

$$\dot{u} = f(t, u), \quad u(t_0) = u_0 \tag{2}$$

- Theorem 2.26 : Suppose that $f(t, u)$ is continuous in the rectangle

$$R = \{(t, u) \mid |t - t_0| \leq T, |u - u_0| \leq L\}, \tag{3}$$

and that

$$|f(t, u)| \leq M \quad \text{if } (t, u) \in R \tag{4}$$

Let $\delta = \min(T, L/M)$. If $u(t)$ is a solution of (2) then

$$|u(t) - u_0| \leq L \quad \text{when } |t - t_0| \leq \delta$$

Suppose, in addition, that f is a Lipschitz continuous function of u , uniformly in t , meaning there is a constant C such that

$$|f(t, u) - f(t, v)| \leq C|u - v| \quad \text{for all } (t, u) \in R$$

Then the solution of (2) is unique in the interval $|t - t_0| \leq \delta$

- Lemma 2.18 : Suppose that $f : C \rightarrow \mathbb{R}$ is a continuously differentiable function on an open, convex subset C of \mathbb{R}^n , and that the partial derivatives of f are bounded on C . Then, for all $x, y \in C$ we have

$$|f(x) - f(y)| \leq M\|x - y\|$$

where $\|\cdot\|$ denotes the Euclidean norm and

$$M = \sup_{z \in C} \|\nabla f(z)\|$$

Since $\dot{u}(t) = |u(t)|^\alpha \geq 0$ for all t , we know that $u(t)$ is non decreasing for all t . Since $u(0) = 0$ and it is non decreasing, this implies that $u(t) \geq 0$ for all $t \geq 0$. We also notice that $u(t) = 0$ is a solution for all t and for all $\alpha > 0$. Also, since $u(0) = 0$ and 0^0 is not defined there can be no solutions when $\alpha = 0$.

If we have

$$\dot{u}(t) = u(t)^\alpha \quad t \geq 0 \implies \frac{d}{dt}u(t) = u(t)^\alpha \implies \frac{du(t)}{u(t)^\alpha} = dt$$

then we can integrate both sides to get a solution (technique of separation of variables). We break it into different cases of α :

1. If $\alpha = 1$ then we have $\frac{du(t)}{u(t)} = dt \implies \ln u(t) = t + C \implies u(t) = C_1 e^t$. Where C is a constant of integration (combined from both integrations) and C_1 is a constant based on C . Using initial conditions $t_0 = 0$, $u(t_0) = 0 \implies C_1 = 0$. So $u(t) = 0$, which is the solution we already have.

2. If $0 < \alpha < 1$ or $\alpha > 1$ then

$$\frac{du(t)}{u(t)^\alpha} = dt \implies \frac{1}{1-\alpha} u(t)^{1-\alpha} = t + C \implies u(t)^{1-\alpha} = (t+C)(1-\alpha) \implies u(t) = ((t+C)(1-\alpha))^{\frac{1}{1-\alpha}}$$

Now we use the initial conditions to determine C :

- If $\alpha > 1$ then $\frac{1}{1-\alpha} < 0$ and the equation will become $0 = \frac{1}{C(1-\alpha)^{\frac{1}{\alpha-1}}}$ and thus there is no C that satisfies the initial conditions. So this is not a solution.
- If $0 < \alpha < 1$ then $u(0) = 0 \implies C = 0$. Then we have $u(t) = (t(1-\alpha))^{\frac{1}{1-\alpha}}$ for $t \geq 0$. $u(t)$ is a valid solution for $t \geq 0$, and we can extend the solution by letting $u(t) = 0$ for $t < 0$. This shows that in this case there is not a unique solution since we noted that $u(t) = 0$ is also a solution.

We have shown that for $0 < \alpha < 1$ we have at least two solutions, and no solutions for $\alpha = 0$. To show that $u(t) = 0$ is a unique solution for $\alpha \geq 1$ we let $f(t, u) = |u(t)|^\alpha$, $\alpha \geq 1$ and $t_0 = 0$, $u_0 = 0$ and note that $f(t, u)$ is continuous on the rectangle

$$R = \{(t, u) \mid |t| \leq T, |u| \leq L\},$$

for some T and L . We also note that (4) is satisfied because $f(t, u)$ is continuous on the closed interval and thus reaches its maximum value. We also note that for $\alpha \geq 1$ the derivative of $f(t, u)$ is $f'(t, u) = \alpha u(t)^{\alpha-1}$ which is bounded on any rectangle (since $\alpha - 1 \geq 0$) and thus by Lemma 2.18 there exists a M such that

$$\frac{f(t, u) - f(t, v)}{u - v} \leq M$$

and thus $f(t, u)$ is Lipschitz. Then by Theorem 2.26 the solution is unique on R and since T and L can be made arbitrarily large, it is unique for all t .

Note on the case of $\alpha = 0$ we could define 0^0 to be either 0 or 1 and have a solution ($u(t) = 0$ for $0^0 = 0$ and $u(t) = t$ for $0^0 = 1$). This solution would be unique by a similar argument as above but this would be counter to the problem statement.

Case 1: Assume 0^0 is undefined. Then no solution exists since we have the initial condition $u(0) = 0$. In this case we have $\dot{u} = 1$ for all $t \neq 0$. Thus the only possible solution that satisfies this and our initial condition is $u(t) = t$. But here $\dot{u}(0) = 1$, which doesn't satisfy our conditions on \dot{u} . Thus no possible solution exists.

Case 2: Assume $0^0 = 1$. Then we have $\dot{u}(t) = 1$ for every value of $u(t)$. Thus \dot{u} is a constant function and as a consequence \dot{u} is also Lipschitz. Thus by Theorem 2.26 we know only a unique solution exists. In particular, this solution is $u(t) = t$, given our initial condition $u(0) = 0$.

Case 3: Assume $0^0 = 0$. The majority of the time only the first two cases are considered as possible definitions of 0^0 , but this case is still worth noting quickly. In this case we either have $\dot{u} = 0$ the constant function, or a discontinuous \dot{u} , which can't be solved. In the case of the constant function, we again have \dot{u} being Lipschitz, and thus a unique solution exists, $u(t) = 0$.

Exercise 2.14 Suppose that $f(t, u)$ is a continuous function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$|f(t, u) - f(t, v)| \leq K|u - v| \quad \text{for all } t, u, v \in \mathbb{R}.$$

Also suppose that

$$M = \sup\{|f(t, u_0)| \mid |t - t_0| \leq T\}.$$

Prove that the solution $u(t)$ of the initial value problem

$$\dot{u} = f(t, u), \quad u(t_0) = u_0$$

satisfies the estimate

$$|u(t) - u_0| \leq MTe^{KT} \quad \text{for } |t - t_0| \leq T.$$

Explicitly check this estimate for the linear initial value problem

$$\dot{u} = Ku, \quad u(t_0) = u_0.$$

Solution

Proof

We will use Gronwall's Inequality from Theorem 2.25 to prove this statement. However, notice that Gronwall's Inequality also holds when integrating from t_0 to t , not just from 0 to t . To show this, we would just need to substitute $r = t - t_0$ into the proof of Gronwall's Inequality, and then integrate $\phi(r)u(r)$ from 0 to r , which is precisely the same as of integrating $\phi(t)u(t)$ from t_0 to t . Thus if we end up with

$$w(t) \leq w_0 + \int_{t_0}^t \phi(s)u(s) ds$$

for $t \in [t_0, T + t_0]$ we can still use Gronwall's inequality despite the slight difference in the limits of integration.

Granted, the above argument still assumes $t \geq t_0$. However, it is enough to prove the desired inequality in only this one case. In the case where $t < t_0$ we can simply integrate from t to t_0 . And again, using variable substitution we can show this is precisely the same as integrating from 0 to r when $r = t_0 - t$. Thus showing the case where $t > t_0$ is sufficient.

Now, by the FTC we have

$$\begin{aligned}
 u(t) - u_0 &= \int_{t_0}^t f(s, u(s)) \, ds \\
 \Rightarrow |u(t) - u_0| &= \left| \int_{t_0}^t f(s, u(s)) \, ds \right| \\
 &= \left| \int_{t_0}^t f(s, u(s)) - f(s, u_0) + f(s, u_0) \, ds \right| \\
 &= \left| \int_{t_0}^t f(s, u(s)) - f(s, u_0) \, ds + \int_{t_0}^t f(s, u_0) \, ds \right| \\
 &\leq \left| \int_{t_0}^t f(s, u(s)) - f(s, u_0) \, ds \right| + \left| \int_{t_0}^t f(s, u_0) \, ds \right| \\
 &\leq \left| \int_{t_0}^t f(s, u(s)) - f(s, u_0) \, ds \right| + \left| \int_{t_0}^t M \, ds \right| \\
 &= \left| \int_{t_0}^t f(s, u(s)) - f(s, u_0) \, ds \right| + M|t - t_0| \\
 &\leq \left| \int_{t_0}^t f(s, u(s)) - f(s, u_0) \, ds \right| + MT \\
 &\leq \int_{t_0}^t |f(s, u(s)) - f(s, u_0)| \, ds + MT \\
 &\leq \int_{t_0}^t K|u(s) - u_0| \, ds + MT.
 \end{aligned}$$

And so we have

$$|u(t) - u_0| \leq \int_{t_0}^t K|u(s) - u_0| \, ds + MT$$

which, if we let $w(t) = |u(t) - u_0| \geq 0$, $w_0 = MT$, and $\phi(s) = K \geq 0$ — noting both $w(t)$ and $\phi(s)$ are definitely continuous functions — is precisely the condition needed to use Gronwall's inequality (just with different limits of integration which we already showed was still a valid statement).

Applying Gronwall's Inequality we get

$$\begin{aligned} |u(t) - u_0| &\leq MT e^{\int_{t_0}^t K ds} \\ &= MT e^{K(t-t_0)} \\ &\leq MT e^{KT}. \end{aligned}$$

which is exactly what we wanted to show. \square

Explicit Check

In this case we can solve the ODE using separation of variables.

$$\begin{aligned} \dot{u}(t) &= Ku(t) \\ \frac{\dot{u}(t)}{u(t)} &= K \\ u(t) &= Ce^{Kt} \end{aligned}$$

And when we evaluate the initial value condition, this gives us

$$\begin{aligned} u(t_0) = u_0 &= Ce^{Kt_0} \\ C &= u_0 e^{-Kt_0} \\ \Rightarrow u(t) &= u_0 e^{K(t-t_0)}. \end{aligned}$$

Note also that here we have

$$M = \sup\{|f(t, u_0)| \mid |t - t_0| \leq T\} = \sup\{|Ku_0| \mid |t - t_0| \leq T\} = K|u_0|.$$

So now we can check the inequality. Note that in the following, we can assume $K > 0$ since all we care about is the magnitude of K . If it was less than zero we could just let $G = |K|$ and use G because what we really want is the Lipschitz constant of $\dot{u}(t)$ which is really $|K|$ if K is negative. Also, if $K = 0$ we just get the trivial case where u is a constant function with $u(t) = u_0$ and so we end up with $0 \leq 0$.

$$\begin{aligned} |u(t) - u_0| &= |u_0 e^{K(t-t_0)} - u_0| \\ &= |u_0| |e^{K(t-t_0)} - 1| \\ &= |u_0| |e^{K(t-t_0)} - e^0| \end{aligned}$$

But by the Mean Value Theorem we have

$$\begin{aligned} |e^{K(t-t_0)} - e^0| &= e^\gamma |K(t-t_0) - 0| \\ &= e^\gamma K |(t-t_0)| \\ &\leq e^\gamma KT \end{aligned}$$

Note that we have $0 \leq \gamma \leq K|t - t_0|$ since we are using the MVT. Using this to continue the first chain of relations,

$$\begin{aligned} |u_0| |e^{K(t-t_0)} - e^0| &= |u_0| e^{\gamma K} |(t - t_0)| \\ &\leq |u_0| e^{K|t-t_0|} K |(t - t_0)| \\ &\leq \frac{M}{K} e^{KT} KT \\ &= MT e^{KT}. \end{aligned}$$

Which is precisely what we wanted. Thus we have shown explicitly this inequality holds for the given IVP.