Midterm Exam 1: Solutions
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1 Problem 1

Problem 1.1. *Show that if X is the real numbers and $d(x, y) = |e^x - e^y|$ then (X, d) is a metric space which is not complete and find the completion.*

(Hint: Use an isometric embedding into the real numbers with their usual metric to find the completion.)

Observations 1.2. *Relevant theorems and definitions to this solution include definition 1.49 and 1.51 as well as theorem 1.52 from Applied Analysis.*

1. *There is a big difference between $\mathbb{R}$ equiped with the euclidean metric and the space (X, d).*
2. *The domain of both metrics is the same underlying set of objects, $\mathbb{R}$. The way we measure these sets is the interesting nuance here.*
3. *Any compact subset of $\mathbb{R}$ is complete with respect to the Euclidean norm.*
4. *Any compact subset of $\mathbb{R}$ is closed and bounded*
5. *The natural log function is surjective onto $\mathbb{R}$*
6. $\lim_{x \to 0^+} ln(x) = -\infty \notin \mathbb{R}$
7. *e^x and $ln(x)$ are inverses of each other (id est: we have that $e^{ln(1/n)} = 1/n$)*
8. *$\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0 from above $\Rightarrow$ $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to the metric defined by the absolute value.*
9. *The function e^x maps $X \to \mathbb{R}$ and preserves distance*

Proof:

Part 0: We want to show that d(x, y) is a metric. We must satisfy the properties of a metric. To do this we can use the well ordering axioms of the real numbers and properties of the function e^x.

Part I: Show $\tilde{\mathbb{R}}$ is incomplete.
Let d(x, y) be the metric defined in the problem statement. Let $d_2(x, y)$ denote the Euclidean metric on the real numbers. Denote the metric space $(\mathbb{R}, d)$ as X. Denote the metric space $(\mathbb{R}, d_2)$ as $\mathbb{R}$.
Let $\{x_n\}_{n=1}^{\infty} \subset X$ be defined by $x_n = ln(1/n)$ Notice that the sequence is Cauchy by observation 8 above. However, we see that the limit point of the sequence occurs at $x^* = -\infty \notin \mathbb{R} \Rightarrow X$ is incomplete.

Part II: Find the completion of $\tilde{\mathbb{R}}$. Let $f : X \to \mathbb{R}$ be defined by $f(x) = e^x$. This is an isometric embedding from X to $\mathbb{R}$. To check this let us check first that it is an isometry and second that it is an embedding.
Isometry: To check that f is an isometry, we must show that $d(x, y) = d_2(f(x), f(y))$ as delineated in definition 1.49. Let $x, y \in X \Rightarrow d(x, y) = |e^x - e^y|$ as defined in the problem statement. Consider $d_2(f(x), f(y)) = d_2(e^x, e^y) = |e^x - e^y| = d(x, y)$. Embedding: It is worth noting that inherent in the definition of isometry are the properties of any metric function. These include continuity and positive definiteness. Using the definition of e^x, checking both of these properties breaks down into a practice applying axioms of the multiplicative and additive group structure on $\mathbb{R}$ and the well ordering axioms of the real numbers. Last, we see that the completion of is $\bar{X} = \mathbb{R} \bigcup -\infty$ since in the preimage of 0 under our isometry is $x = -\infty$.

2 Problem 2

Problem 2.1. *Assume that* (X, d) *is a compact metric space,* $y \in X$, $M > 0$ *and* $\{f_n\}_{n=1}^{\infty}$ *is an infinite sequence of Lipschitz continuous functions. Assume also that with every* $f_n \in F_M(X) \forall n \in \mathbb{N}$ *and every* $f_n(y) = 0$. *Show that there is a subsequence which converges pointwise to a continuous function.*

(Recall that $F_M(X) = \{f \in \text{Bdd}(X) | \forall x, z \in X \text{ there is } |f(x) - f(z)| \leq M d(x, z)\}$).

Observations 2.2. *Relevant theorems, definitions and examples include definition 2.10, 2.15 and theorem 2.12 from Applied Analysis. Also example 2.13 and 2.20 give wonderful insight into the keys of this problem. Last the definition of diameter of a set on page 11 and remarks directly following this definition are applicable also.*

1. *If a sequence converges uniformly, it converges pointwise.*
2. *The set of functions* $\{f_n\}_{n=1}^{\infty} \subset C(X)$
3. *A is compact* $\Leftrightarrow$ *A is sequentially compact*
4. *A is sequentially compact* $\Rightarrow$ *for any sequence, there exists a convergent subsequence*
5. *Arzela Ascoli talks about compactness of a set of functions (closed, bounded, equicontinuous)*
6. *The proof of Arzela Ascoli speaks only about bounded and equicontinuous*
7. *In this problem, we only need to show that a convergent subsequence exists, not that the limit is in A*

Proof:
Let $A = \{f_n\}_{n=1}^{\infty}$ be as defined in the problem statement. Then A is equicontinuous and bounded.
Bounded: Since X is compact $\Rightarrow$ X is closed and bounded. Then let D = diam(X) = $sup\{d(x, y) | x \in X\}$.
By definition of A, we see that $\forall n \in \mathbb{N}, |f_n(x) - f_n(y)| = |f_n(x)| \leq M \cdot D$.
Equicontinuous: Let $\epsilon > 0$. Let $\delta = \frac{\epsilon}{M} \Rightarrow |f_n(x) - f_n(z)| \leq M \cdot d(x, z) < M \cdot \frac{\epsilon}{M} = \epsilon$
From our claim and the proof of Theorem 2.12 we see that A is precompact. Then $\bar{A}$ is compact $\Rightarrow \bar{A}$ sequentially compact by Theorem 1.62. Then there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ convergent to $f(x) \in C(X)$. Specifically, this means that we have a subsequence of $\{f_n\}_{n=1}^{\infty}$ that converges to f uniformly $\Rightarrow$ our subsequence converges pointwise.

3 Problem 4

Problem 3.1. *Show that the map of the real numbers defined by* $T(x) = \frac{\pi}{2} + x - arctan(x)$ *has no fixed points and that* $|T(x) - T(y)| < |x - y|$ *for every pair* $x \neq y$. *Why is this consistent with the contraction mapping theorem?*

Observations 3.2. *This problem is identical to a problem solved on Assignment 4. For solutions see the appropriate solution set posted on-line. A lesson in this problem is best demonstrated by the following quote: "Human reasoning processes depend more on memory retrieval and analogy than on application of formal logical rules." (Mathematics Education by L. English and G. Halford, pg 54).*

4 Problem 3

Problem 4.1. *Show that if $f \in C([0,1])$ and $0 = \int_0^1 x^n f(x)dx$ for every $n \geq 0$ then f is (the constant function) zero.*

Observations 4.2. *Relevant deinitions, theorems and exercises to this problem include definition 12.27, theorem 2.9, theorem 12.35 and exercise 2.6 from Applied Analysis.*

1. *$[0,1]$ is a compact set by the Bolzano Weierstrass theorem*
2. *$f \in C([0,1]) \Rightarrow f$ bounded, achieve min and max, uniformly continuous, and can be approximated by polynomials.*
3. *The Stone Weierstrass theorem and corollaries (including Weierstrass Approx theorem) guarantee a sequence of polynomials converging uniformly to f(x) on our compact set.*
4. *for any polynomial of finite degree m, $p(x) = \sum\limits_{j=1}^{m} a_j x^j$.*
5. *By a homework problem, uniform convergence $\Rightarrow L^1$ convergence*
6. *Mathematically 5 states $\|p_n - f\|_{unif} \to 0$ as $n \to \infty \Rightarrow \lim\limits_{n\to\infty} \int\limits_0^1 p_n(x)dx = \int\limits_0^1 f(x)dx$*

Proof:
Let the antecedents discussed in the problem statement hold. We know by a corollary of the Stone-Weierstrass theorem (known as the Weierstrass Approximation Theorem in Applied Analysis), there is a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ converging uniformly to $f(x)$. Then we have that

$$\int\limits_0^1 (f(x))^2 dx = \int\limits_0^1 \lim_{n\to\infty} p_n(x) \cdot f(x)dx = \lim_{n\to\infty} \int\limits_0^1 p_n(x) \cdot f(x)dx$$

Then using our observation about polynomials we know

$$\int\limits_0^1 p_n(x) \cdot f(x)dx = \int\limits_0^1 \sum_{j=1}^{m} a_j x^j \cdot f(x)dx = \sum_{j=1}^{m} a_j \int\limits_0^1 x^j \cdot f(x)dx = 0$$

where the last equality comes from the assumption that $0 = \int_0^1 x^n f(x)dx$ for every $n \geq 0$.

Now, we claim that for $f : [0,1] \to \mathbb{R}$ continuous, $\int\limits_0^1 (f(x))^2 dx = 0 \Rightarrow f(x) = 0$. Suppose, hoping for contradiction this was not true. Then there exists some $x^* \in [0,1]$ such that $f(x^*) \neq 0 \Rightarrow (f(x^*))^2 > 0$. Since f is continuous, we know f^2 is continuous. Letting $\epsilon = \frac{f(x^*)}{2}$ we know there is some δ such that $f(x)^2 > \frac{\epsilon}{2} \forall x \in B_{\delta/2}(x^*) \Rightarrow$

$$0 < \delta \cdot \frac{\epsilon}{2} = \int\limits_{B_{\delta/2}(x^*)} \frac{\epsilon}{2} dx < \int\limits_{B_{\delta/2}(x^*)} ((f(x))^2 dx = 0$$

Thus, we have a contradiction and our claim holds.

Remark 4.3. *To master this problem on this exam, a formal proof of our last conjecture was not necessary. However, the mathematical theory behind such an argument is well worth learning.*