```latex
%This is a comment designated to describe why I have written this document.  It is a
brief account of what the document contains and any special notes.
\documentclass[11 pt]{article}

%The portion of a TeX Document between \documentclass{} and \begin{document} is called
the Preamble.  It defines/imports the important specs for this document and allows for
flexibility.  It is here that the art of a well written document is displayed (along
with coding style).
%SET MARGINS
\setlength{\oddsidemargin}{-.45in}
%\setlength{\evensidemargin}{-.5in}
\setlength{\textwidth}{7.1in}
\setlength{\topmargin}{-0.95in}
\setlength{\textheight}{9.7in}
%UNCOMMENT NEXT LINE TO DOUBLE SPACE
%\renewcommand{\baselinestretch}{2}

%LOAD VARIOUS PACKAGES
\usepackage{amsmath, amsfonts, amsthm, graphics, latexsym, multicol}
\pagestyle{empty}

%MATH CHARACTER COMMANDS
% math
\def\N{\mathbb{N}} %Natual Bold Face
\def\Q{\mathbb{Q}} %Rational Bold Face
\def\R{\mathbb{R}} %Real Bold Face
\def\Z{\mathbb{Z}} %Integers Bold Face
\def\C{\mathbb{C}} %Complex Bold Face

%LOWER CASE VECTORS v, x, AND b IN BOLDFACE
\newcommand{\vecv}{\mathbf v}
\newcommand{\vecx}{\mathbf x}
\newcommand{\vecb}{\mathbf b}

%A plethora of options to use prearranged environments for Text Setting.  Each
different declaration
\newtheorem{theorem}{Theorem}[section]
\newtheorem{corollary}[theorem]{Corollary}
\newtheorem{lemma}[theorem]{Lemma}
\newtheorem{proposition}[theorem]{Proposition}
\newtheorem{problem}[theorem]{Problem}
\newtheorem{example}[theorem]{Example}
\newtheorem{algorithm}[theorem]{Algorithm}
\newtheorem{definition}[theorem]{Definition}
\newtheorem{conjecture}[theorem]{Conjecture}
\newtheorem{question}[theorem]{Question}
\newtheorem{remark}[theorem]{Remark}
\newtheorem{solution}[theorem]{Observations}

%Begins Document
\begin{document}

  %Exam written and administered by Professor Eric Babson at UC Davis in Math 201 A.
This course was offered in Fall 2009 in Phy Geo 130 at 9-10am.
  \noindent \textbf{Midterm Exam 1: Solutions}\\
  \textbf{November 6, 2009; 9:00am}\\
  \textbf{Solutions by Jeffrey Andersson}\\

  %Statement found on Davis PhD preliminary Exam, Winter 2003, problem 4, part ii
  \section{Problem 1}
    \begin{problem}   Show that if $X$ is the real numbers and $d(x,y)=|e^x-e^y|$ then
$(X,d)$ is a metric space which is not complete and find the
 completion.
    \end{problem}
                        (Hint: Use an isometric embedding into the real numbers with
their usual metric to find the completion.)
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%Here we see the components of this problems important in generating a solution.
These are not written in a formal proof style on purpose to give        a sense of the
thought process one might use in time constrained situations.
    \begin{solution}
      Relevant theorems and definitions to this solution include definition 1.49 and
1.51 as well as theorem 1.52 from Applied Analysis.
      \begin{enumerate}
        \item There is a big difference between $\R$ equiped with the euclidean metric
and the space $(X, d)$.
        \item The domain of both metrics is the same underlying set of objects, $\R$.
The way we measure these sets is the interesting nuance here.
        \item Any compact subset of $\R$ is complete with respect to the Euclidean
norm.
        \item Any compact subset of $\R$ is closed and bounded
        \item The natural log function is surjective onto $\R$
        \item $\lim_{x \to 0^+} ln(x) = - \infty \notin \R$
        \item $e^x$ and ln(x) are inverses of each other (id est: we have that
$e^{ln(1/n)} = 1/n$)
        \item $\{\frac{1}{n} \}_{n=1}^{\infty}$ converges to 0 from above $\Rightarrow
\{\frac{1}{n} \}_{n=1}^{\infty}$ is a Cauchy sequence with                    respect
to the metric defined by the absolute value.
        \item The function $e^x$ maps $X \rightarrow \R$ and preserves distance
      \end{enumerate}
    \end{solution}

    %Here we see a developed demonstration of the solution we achieved above.  This
usually is achieved only after meditation and practice.  For          professional
mathematicians, our livelihood depends on our ability to generate such a proof.
    \noindent \textbf{Proof:}\\
    Part 0:  We want to show that d(x, y) is a metric. We must satisfy the properties
of a metric.  To do this we can use the well ordering axioms of the     real numbers
and properties of the function $e^x$.\\

    \noindent Part I: Show $\tilde{\R}$ is incomplete.\\
      Let d(x, y) be the metric defined in the problem statement.  Let $d_2(x, y)$
denote the Euclidean metric on the real numbers.
      Denote the metric space $(\R, d)$ as $X$.  Denote the metric space $(\R, d_2)$ as
$\R$.\\
      Let $\{ x_n \}_{n=1}^{\infty} \subset X$ be defined by $x_n = ln (1/n)$  Notice
that the sequence is Cauchy by observation 8 above.                  However, we
see that the limit point of the sequence occurs at $x^{\star} = -\infty \notin \R
\Rightarrow X$ is incomplete.\\

    \noindent Part II:  Find the completion of $\tilde{\R}$.
      Let $f: X \rightarrow \R$ be defined by $f(x) = e^x$.  This is an isometric
embedding from $X$ to $\R$.  To check this let us check first that it        is an
isometry and second that it is an embedding.\\
      Isometry: To check that $f$ is an isometry, we must show that $d(x, y) =
d_2(f(x), f(y))$ as delineated in definition 1.49.  Let $x, y \in X
   \Rightarrow  d(x, y) = |e^x-e^y|$ as defined in the problem statement.  Consider
$d_2(f(x), f(y)) = d_2(e^x, e^y) = |e^x-e^y| = d(x, y)$.
      Embedding:  It is worth noting that inherent in the definition of isometry are
the properties of any metric function. These include continuity and
positive definiteness.  Using the definition of $e^x$, checking both of these
properties breaks down into a practice applying axioms of                 the
multiplicative and additive group structure on $\R$ and the well ordering axioms of the
real numbers.
      Last, we see that the completion of is $\bar{X} = \R \bigcup {-\infty}$ since in
the preimage of 0 under our isometry is $x = -\infty$.


\newpage
  \section{Problem 2}
    \begin{problem}      Assume that $(X,d)$ is a compact metric space, $y\in X$, $M>0$
and $\{f_n\}_{n=1}^{\infty}$ is an infinite sequence of Lipschitz
   continuous functions.  Assume also that with every $f_n \in F_M(X) \forall n \in \N$
and every $f_n(y)=0$.  Show that there is a                           subsequence which
converges pointwise to a continuous function.
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    \end{problem}
                          (Recall that $F_M(X)=\{f\in \hbox{Bdd}(X)|\forall x, z \in
X\hbox{ there is }|f(x)-f(z)|\leq M d(x,z)\} $).

    \begin{solution}
      Relevant theorems, definitions and examples include definition 2.10, 2.15 and
theorem 2.12 from Applied Analysis.  Also example 2.13 and 2.20 give        wonderful
insight into the keys of this problem.  Last the definition of diameter of a set on
page 11 and remarks directly following this                    definition are applicable
also.
      \begin{enumerate}
        \item If a sequence converges uniformly, it converges pointwise.
        \item The set of functions $\{f_n\}_{n=1}^{\infty} \subset C(X)$
        \item $A$ is compact $\Leftrightarrow A$ is sequentially compact
        \item $A$ is sequentially compact $\Rightarrow$ for any sequence, there exists
a convergent subsequence
        \item Arzela Ascoli talks about compactness of a set of functions (closed,
bounded, equicontinuous)
        \item The proof of Arzela Ascoli speaks only about bounded and equicontinuous
        \item In this problem, we only need to show that a convergent subsequence
exists, not that the limit is in A
      \end{enumerate}
    \end{solution}

    \noindent \textbf{Proof:}\\
      Let $A = \{f_n\}_{n=1}^{\infty}$ be as defined in the problem statement.  Then A
is equicontinuous and bounded.\\
      Bounded:  Since X is compact $\Rightarrow$ X is closed and bounded.  Then let D =
diam(X) = $sup\{d(x,y) \vert x \in X\}$.  By definition of $A$,         we see that
$\forall n \in \N, \vert f_n(x) - f_n(y) \vert = \vert f_n(x) \vert \leq M \cdot D$. \\
      Equicontinuous:  Let $\epsilon > 0$.  Let $\delta = \frac{\epsilon}{M}
\Rightarrow \vert f_n(x) - f_n(z) \vert \leq M \cdot d(x,z)
                                                   < M \cdot \frac{\epsilon}{M} =
\epsilon$ \\
      From our claim and the proof of Theorem 2.12 we see that $A$ is precompact.  Then
$\bar{A}$ is compact $\Rightarrow \bar{A}$  sequentially compact         by Theorem
1.62.  Then there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ convergent to $f(x)
\in C(X)$.  Specifically, this means that we have        a subsequence of
$\{f_n\}_{n=1}^{\infty}$ that converges to $f$ uniformly $\Rightarrow$ our subsequence
converges pointwise.

  %Statement found on Davis PhD preliminary Exam, Winter 2005, problem 1 and in Applied
Analysis Exercise 3.1.  This was on homework assignment 4.
  \section{Problem 4}
    \begin{problem}
      Show that the map of the real numbers defined by $T(x)= \frac{\pi}{2} + x
-\hbox{arctan}(x)$ has no fixed points and that $|T(x)-T(y)|<|x-y|$ for          every
pair $x\not= y$.  Why is this consistent with the contraction mapping theorem?
    \end{problem}

    \begin{solution}
      This problem is identical to a problem solved on Assignment 4.  For solutions see
the appropriate solution set posted on-line.  A lesson in this          problem is best
demonstrated by the following quote:
        ``Human reasoning processes depend more on memory retrieval and analogy than on
application of formal logical rules." (Mathematics Education by          L. English
and G. Halford, pg 54).
    \end{solution}

\newpage
  %Statement found on Davis PhD preliminary Exam, Winter 2009, problem 2.
  \section{Problem 3}
    \begin{problem}
      Show that if $f\in C([0,1])$ and $0=\int^1_0x^nf(x) dx$ for every $n\geq 0$ then
$f$ is (the constant function) zero.
    \end{problem}

    \begin{solution}
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    Relevant deinitions, theorems and exercises to this problem include definition
12.27, theorem 2.9, theorem 12.35 and exercise 2.6 from Applied          Analysis.
    \begin{enumerate}
      \item $[0, 1]$ is a compact set by the Bolzano Weierstrass theorem
      \item $f \in C([0,1]) \Rightarrow f$ bounded, achieve min and max, uniformly
continuous, and can be approximated by polynomials.
      \item The Stone Weierstrass theorem and corollaries (including Weierstrass
Approx theorem) guarantee a sequence of polynomials converging
uniformly to f(x) on our compact set.
      \item for any polynomial of finite degree m, p(x) = $\sum\limits_{j=1}^{m} a_j
x^j$.
      \item By a homework problem, uniform convergence $\Rightarrow L^1$ convergence
      \item Mathematically 5 states $\Vert p_n - f \Vert_{unif} \rightarrow 0$ as $n
\to \infty
            \Rightarrow \lim \limits_{n \to \infty} \int \limits_0^1 {p_n(x) dx} =
\int \limits_0^1 {f(x) dx}$
    \end{enumerate}
  \end{solution}

  \noindent \textbf{Proof:}\\
    Let the antecedents discussed in the problem statement hold.  We know by a
corollary of the Stone-Weierstrass theorem (known as the Weierstrass
Approximation Theorem in Applied Analysis), there is a sequence of polynomials
$\{p_n\}_{n=1}^{\infty}$ converging uniformly to $f(x)$.  Then we         have that
    \begin{equation*}
      \int \limits_0^1 {(f(x))^2 dx} = \int \limits_0^1 {\lim \limits_{n \to \infty}
p_n(x) \cdot f(x) dx}
                                    = \lim \limits_{n \to \infty} \int \limits_0^1
{p_n(x) \cdot f(x) dx}
    \end{equation*}
    Then using our observation about polynomials we know
    \begin{equation*}
      \int \limits_0^1 {p_n(x) \cdot f(x) dx} = \int \limits_0^1
{\sum\limits_{j=1}^{m} a_j x^j \cdot f(x) dx}
                                              = \sum\limits_{j=1}^{m} a_j \int
\limits_0^1 {x^j \cdot f(x) dx}
                                              = 0
    \end{equation*}
    where the last equality comes from the assumption that $0=\int^1_0x^nf(x) dx$ for
every $n\geq 0$.

    Now, we claim that for $f:[0, 1] \rightarrow \R$ continuous, $\int \limits_0^1{
(f(x))^2 dx} = 0 \Rightarrow f(x) = 0$.
    Suppose, hoping for contradiction this was not true.  Then there exists some $x^*
\in [0, 1]$ such that $f(x^*) \neq 0 \Rightarrow (f(x^*))^2 > 0$.       Since $f$ is
continuous, we know $f^2$ is continuous.  Letting $\epsilon = \frac{f(x^*)}{2}$ we know
there is some $\delta$ such that $f(x)^2 >          \frac{\epsilon}{2} \forall x \in
B_{\delta/2}(x^*) \Rightarrow$
    \begin{equation*}
      0 < \delta \cdot \frac{\epsilon}{2} = \int \limits_{B_{\delta/2}(x^*)}{
\frac{\epsilon}{2} dx}
                                            < \int \limits_{B_{\delta/2}(x^*)}
{((f(x))^2 dx} = 0
    \end{equation*}
    Thus, we have a contradiction and our claim holds.

  \begin{remark}
    To master this problem on this exam, a formal proof of our last conjecture was
not necessary.  However, the mathematical theory behind such            an argument
is well worth learning.
  \end{remark}
\end{document}
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