Exercise 1.21

Let X be a Banach Space.

Let $\{x_{m,n}\}_{m,n=1}^{\infty}$ be a double indexed sequence s.t.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|x_{mn}\| < \infty$$

We want to show

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{mn}. \qquad (\ast)$$

Existence:

For a fixed $\tilde{m} \in \mathbb{P}$, consider the sequence $\{x_{\tilde{m}n}\}_{n=1}^{\infty}$. We know that $\sum_{n=1}^{\infty}\|x_{\tilde{m}n}\| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|x_{mn}\| < \infty$

$\Rightarrow$ $\{x_{\tilde{m}n}\}_{n=1}^{\infty}$ absolutely converges

$\Rightarrow$ $\{x_{\tilde{m}n}\}_{n=1}^{\infty}$ converges by problem 1.20.

Let $\chi_m = \lim_{n\to\infty} x_{mn}$ for a fixed $m \in \mathbb{P}$

$$\Rightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn} = \sum_{m=1}^{\infty}\chi_m$$

By assumption $\sum_{m=1}^{\infty}\|\chi_m\| < \infty \Rightarrow \{\chi_m\}_{m=1}^{\infty}$ converges as a series by problem 1.20.

Thus $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}$ converges (ie the limit exists).

Let us call this limit x. By a similar argument we can show $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{mn}$ converges

Exercise 1.21 continued

Now we would like to show we have $(\overline{X}')$.

Consider the double sequence $\{x_{mn}\}_{m,n=1}^{\infty}$ as an infinite dimensional (square) matrix with coefficients defined by the sequence.

For any finite $K \in \mathbb{P}$ define $S_K = \sum_{m=1}^{K}\sum_{n=1}^{K} x_{mn}$

(this would be the sum of the coefficients $K \times K$ submatrix ~~of~~ defined by $\{x_{mn}\}_{m,n=1}^{\infty}$

$$\left[\begin{array}{cccc|c} x_{11} & x_{12} & \cdots & x_{1K} & \cdots \\ x_{22} & x_{22} & \cdots & x_{2K} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ x_{K1} & x_{K2} & \cdots & x_{KK} & \cdots \\ \hline \vdots & \vdots & & \vdots & \ddots \end{array}\right]$$

Notice that $S_K = \sum_{m=1}^{K}\sum_{n=1}^{K} x_{mn} = \sum_{n=1}^{K}\sum_{m=1}^{K} x_{mn}$

Claim $\lim_{K\to\infty} S_K = x$ from before.

Indeed consider

$$\left\| \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn} - x \right\| = \left\| \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn} - \sum_{m=1}^{\infty} x_m + \sum_{m=1}^{\infty} x_m - x \right\|$$

$$= \left\| \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} x_{mn} - x_m \right) + \sum_{m=1}^{\infty} x_m - x \right\|$$

$$\leq \sum_{m=1}^{\infty} \left\| \sum_{n=1}^{\infty} x_{mn} - x_m \right\| + \left\| \sum_{m=1}^{\infty} x_m - x \right\|$$

by continuity of $\|\cdot\|$ & triangle inequality applied multiple times.

Exercise 1.21 continued

Since $\{\chi_m\}_{m=1}^{\infty} \to X$ as a series $\Rightarrow \exists M$ s.t. $\tilde{m} \geq M \Rightarrow \left\| \sum_{m=1}^{\tilde{m}} \chi_m - X \right\| < \varepsilon/2$.

Since $\sum_{n=1}^{\infty} x_{mn} \to \chi_m$ $\exists N \in \mathbb{N}$ s.t. $\tilde{n} \geq N$

$$\Rightarrow \left\| \sum_{n=1}^{\tilde{n}} x_{mn} - \chi_m \right\| < \frac{\varepsilon}{2} \cdot \frac{1}{\tilde{m}}$$

Let $K = \max\{\cancel{\tilde{m}}\ M, N\}$.

$$\| S_K - X \| = \left\| \sum_{m=1}^{K} \sum_{n=1}^{K} x_{mn} - X \right\|$$

$$< \varepsilon \quad \text{by construction.}$$

We notice $\lim_{K \to \infty} S_K = (\frac{1}{X})$. This is what we wanted to show.