$$|\bar f(x_0) - \bar f(x)| \le \underbrace{|\bar f(x_0) - f(z_n)|}_{\text{def } \bar f} + \underbrace{|f(z_n) - f(x_n)|}_{\text{uniform continuity (昼)}} + \underbrace{|f(x_n) - \bar f(x)|}_{\text{def } \bar f}$$

$$< \quad \varepsilon/3 \; + \; \varepsilon/3 \; + \; \varepsilon/3$$

(昼) $|f(z_n) - f(x_n)| < \varepsilon/3 \quad \Leftarrow \quad d(z_n, x_n) < \boxed{\delta}$

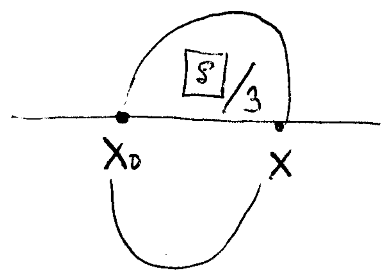
Let $\quad d(z_n, x_0) < \boxed{\delta}/3 \qquad$ since $\quad z_n \to x_0$

$\quad d(x_n, \cancel{\ }) < \boxed{\delta}/3 \qquad$ since $\quad x_n \to x$

$\quad d(x_n, x)$

$$d(x, x_0) \le d(x_0, z_n) + d(z_n, x_n) + d(x_n, x)$$

$$d(x, x_0) < \boxed{\delta}/3$$



let $\quad \delta = \boxed{\delta}/3$

$$d(x, x_0) < \boxed{\delta}/3 \quad \Leftrightarrow \quad d(z_n, x_n) < \boxed{\delta}$$

$$d(x_n, z_n) \le \underbrace{d(x_n, x)}_{\text{since } x_n \to x} + \boxed{d(x, x_0)} + \underbrace{d(x_0, z_n)}_{\text{Since } z_n \to x_0}$$

$$\qquad\qquad \boxed{\delta}/3 \qquad\quad \boxed{\delta}/3 \qquad\quad \boxed{\delta}/3$$

$$< \boxed{\delta} \quad \Rightarrow \quad |f(z_n) - f(x_n)| < \varepsilon/3$$

Opportunities for further inspection

☆ the Stone Weierstrass theorem

Exercise 2.4

We want to construct a counter example to show $f_n \to f$ in $C[0,1]$ and $\{f_n\}_{n=1}^{\infty}$ continuously differentiable but f not continuously differentiable

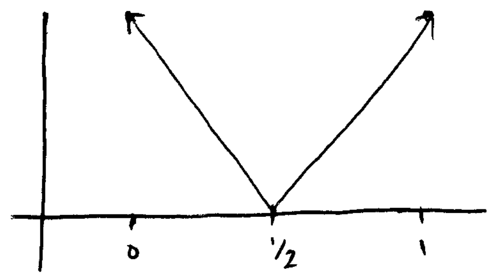
Non-differentiable $\overset{\wedge}{\text{continuous}}$ funtions on $[0,1]$

why do I not care about non-continuous functions here?

want $f_n(x) \to f(x)$ w/ respect to sup norm

thm 2.3

ex 1 $\quad f_1(x) = |x|$

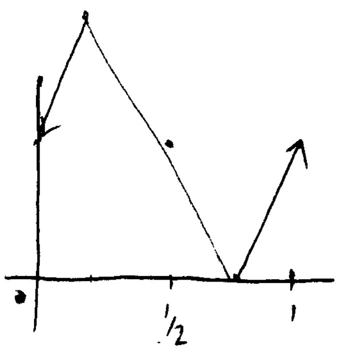


$$B_n(x, f_1) \neq \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\binom{n}{k}$$

$$= \sum_{k=0}^{n} f_1\left(\tfrac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

$f_2(x) =$

ex 2



$$B_n(x, f_2) = \sum_{k=0}^{n} f_2\left(\tfrac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

These solve this problem in a cryptic way... Why do these work

Exercise 2.4

look closely @ statement of theorem 2.9 and investigate Bernstein polynomials

what do these give you?

Say I want to construct a $\{f_n(x)\}$ s.t.

$$f_n(x) \xrightarrow{\text{uniformly}} \cancel{|x|} \; |x - 1/2| = \begin{cases} x - 1/2 & x \in [1/2, 1] \\ -x + 1/2 & x \in [0, 1) \end{cases}$$

$$f_N(x) = \sum_{n=-N}^{N} \langle f(x), e_n \rangle \, e_n$$

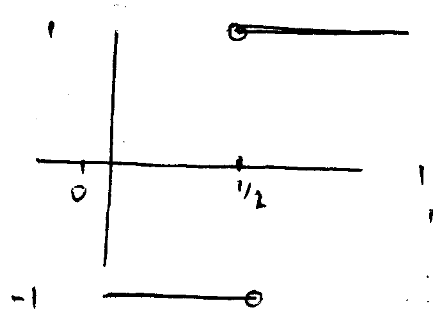
Fourier Series

- this doesn't quite work since $f : [0,1] \to \mathbb{R}$ not periodic,

can extend $f : [0,1]$ to a periodic function on $\mathbb{R}$ by natural idea..

$$f(x) = |x - 1/2|.$$

$f'(x) =$

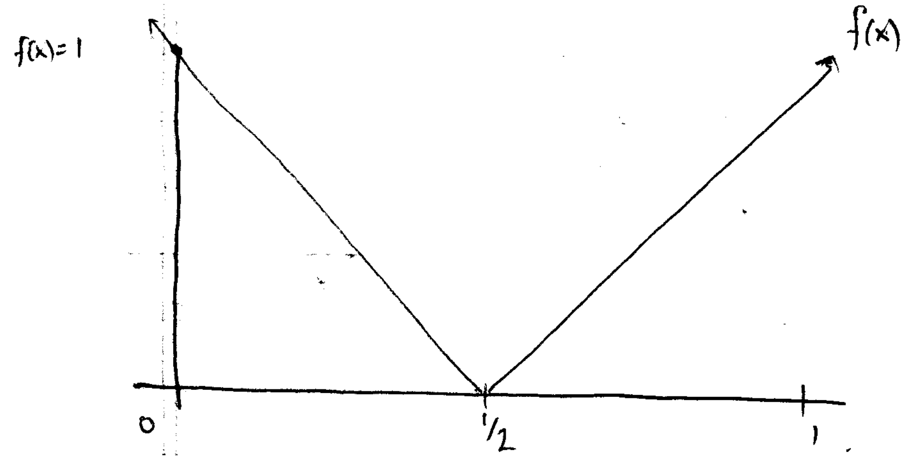


$$f'(x) = \begin{cases} -1 & x \in (0, 1/2) \\ 1 & x \in (1/2, 1) \end{cases}$$

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (x-a)^n \, f^{(n)}(a)$$

Exercise 2.4...



$f_1(x) = 1$

$f_2(x) =$

$f_n\big|_{x=0} = f_n\big|_{x=1} = 1$

$\lim_{n\to\infty} f_n(1/2) = 0$

$(1)^n = 1$

can changke w/ n

$f(x) = (x - a_n)^n + b$

$f(0) = (-a)^n$

$=$

$\lim_{n\to\infty} (-a_n)^n + b_n = 1$

$(-a_n)^n$

$\lim_{n\to\infty} (1 - a_n)^n + b_n = 1$

$\left(\frac{1}{2} - a_n\right)^n + b_n = 0$

$$f(x) = |x - 1/2|$$

$$\cancel{f_n(x)} \neq \sum_{n=0}^{\infty} \qquad f_N(x) = \sum_{n=0}^{N} a_n x^n$$

$$\lim_{N\to\infty} f_N(x) = |x - 1/2|$$

x	$\|x-1/2\|$
0	1/2

$$\lim_{N\to\infty} f_N(x) = 2|x - 1/2|$$

x	$2\|x-1/2\|$
0	1
$0+1/2$	0
1	1

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n (0)^n = 1 \qquad a_0 = 1$$

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n = 1$$

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_N \left(\frac{1}{2}\right)^n = 0$$

what is idea of Bernstein polynomials? what thaughts do these generalize?

Exercise 2.5

Let $C'[a,b]$ be the space of continuously differentiable functions

$$C'[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \in C[a,b],\ f' \in C[a,b] \}$$

Let $\|\cdot\|_{C'} = \sup_{a \le x \le b} |f(x)| + \sup_{a \le x \le b} |f'(x)|$ be a norm

We want to show $(C'[a,b], \|\cdot\|_{C'})$ is a Banach space

start with a ~~conve~~ cauchy sequence, show convergence in the space

Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(C'[a,b], C'\text{-norm})$.

$\Rightarrow$ $f_n(x)$ Cauchy in $C[a,b]$ since $\|f\|_C \le \|f\|_{C'}$

$\frac{d}{dx} f_n(x)$ Cauchy in $C[a,b]$ $\|f'\|_{C[a,b]} \le \|f\|_{C'}$

$\Rightarrow$ $\exists\, f \in C[a,b]$ s.t.

$f_n(x) \to f(x)$ w/ respect to $\|\cdot\|_{C[a,b]}$

$\exists\, g \in C[a,b]$ s.t. $\frac{d}{dx} f_n(x) \to g(x)$ w/ respect to $\|\cdot\|_{C[a,b]}$

Since $C[a,b]$ is a Banach space

Theorem 2.4

Claim: $g(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x) = \frac{d}{dx} \lim_{n\to\infty} f_n(x) = \frac{d}{dx} f(x)$

We Know $\lim_{n\to\infty} f_n(x) = f(x)$ under $\|\cdot\|_C$ norm

$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x)$ pointwise

Consider $\frac{1}{h}\left(f(x+h) - f(x) \right) = \frac{1}{h} \lim_{n\to\infty} f_n(x+h) - f_n(x)$

(i) why is this differentiable

(ii) how to enterchange these limits

$= \lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} f_n(x+h) - f_n(x)$

$$\lim_{n\to\infty} \lim_{h\to 0} \frac{f_n(x+h) - f_n(x)}{h} = \lim_{n\to\infty} \frac{d f_n(x)}{dx} = g(x)$$

we know $\{f_n\}_{n=1}^{\infty}$ are differentiable

$$\left\| \frac{d}{dx} f_n(x) - g(x) \right\|_{C[a,b]} \to 0 \quad \text{as } n\to\infty$$

$$\star \quad \left\| \frac{d}{dx} f(x) - g(x) \right\|_{C[a,b]} \to 0 \quad \star$$

is $f(x)$ differentiable?

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \lim_{n\to\infty} f_n(x) \Rightarrow \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to\infty} \lim_{n\to\infty} \frac{f_n(x+h) - f_n(x)}{h}$$

Exercise 2.5...

$$\lim_{h \to 0} \frac{1}{h} \left[[f(x+h) - f(x)] - [\cancel{f_n(x+h) - f_n(x)} \lim_{n\to\infty} f_n(x+h) - f(x)] \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[[f(x+h) - \lim_{n\to\infty} f_n(x+h)] - f(x) + \lim_{n\to\infty} f_n(x) \right]$$

$$= \lim_{h \to 0} \frac{1}{h} [0 - 0]$$

$$= \frac{0}{0}$$

we want to show that

$$\lim_{n\to\infty} \frac{d}{dx} f_n(x) = g(x)$$

switch limiting processes

$$= \frac{d}{dx} \lim_{n\to\infty} f_n(x) \cancel{dx}$$

this is similar to problem 1.21

(in the sense that it asks to justify the interchange of limiting processes)

$$\lim_{n\to\infty} \left\| \frac{d}{dx} f_n(x) - g(x) \right\|_{C[a,b]} < \varepsilon$$

$$\lim_{h\to\infty} \left\| \lim_{n\to\infty} \frac{f_n(x+h) - f_n(x)}{n} - d\,\frac{f(x+h) - f(x)}{h} \right\|$$

It is not clear to me that $\frac{d}{dx} f(x)$ exists?

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cancel{dx} = \frac{\lim_{h\to 0} f(x+h) + -1 \cdot \lim_{h\to 0} f(x)}{\lim_{h\to 0} h}$$

$$= \frac{f(x) - f(x)}{0}$$

$$= \frac{0}{0}$$

$$|f(x) - f(y)| = |f'(\xi)||x-y| \qquad \text{some } \xi \in [a,b]$$

$f'(\xi)$

We have that

$f_n(x)$ continuous on a compact set

$f_n'(x)$ continuous on a compact set

$$\Rightarrow \max_{x\in[a,b]} |f_n'(x)| < \infty$$

$$\cancel{\lim_{x\in[a,b]}} \quad \lim_{n\to\infty} |f_n'(x)| = g(x) \quad \text{continuous on a compact set}$$

$\Rightarrow$ bounded

$$\Rightarrow |f_n'(x)| \text{ uniformly bounded}$$

$$\Rightarrow \lim_{n\to\infty} |f_n(x) - f_n(y)|$$

$$\lim_{n\to} \qquad |f_n(x) - f_n(y)| \leq M$$

Mean value theorem $\quad |f_n(x) - f_n(y)| = |f_n'(\xi)||x-y|$

Exercise 2.6

Let $(C[a,b], \|\cdot\|_1)$ be a metric space

where $\|f\|_{L^1} := \int_a^b |f(x)|\, dx$

We want to show

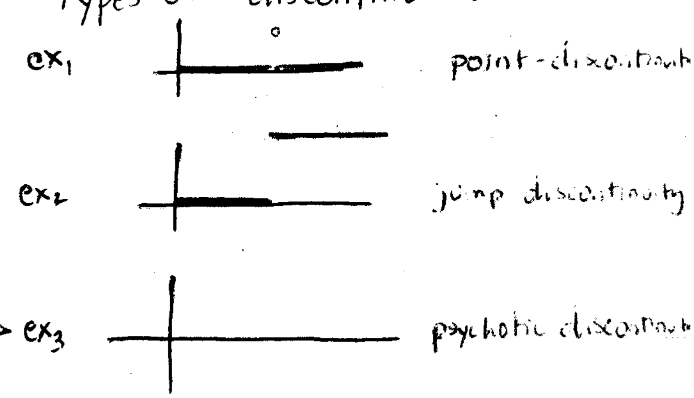
(i) $(C[a,b], \|\cdot\|_1)$ Not complete

(ii) $f_n \to f$ uniformly $\Rightarrow f_n \to f$ w/ respect to L^1-norm

(iii) $\exists \{f_n\} \to f$ in L^1 s.t. $f_n \not\to f$ in $\|\cdot\|_{sup}$

(i) Want a sequence of continuous functions $\{f_n(x)\}_{n=1}^{\infty}$ that converges to a non continuous function

Types of discontinuities



Lebesgue integral/Lebesgue measure

Related problem: $(C[0,1], \|\cdot\|_{L^1})$ Not complete

Type$_1$:

$f_n(x) = x^n$

$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x \in \{1\} \end{cases}$

pointwise limit

under equality almost every where

$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

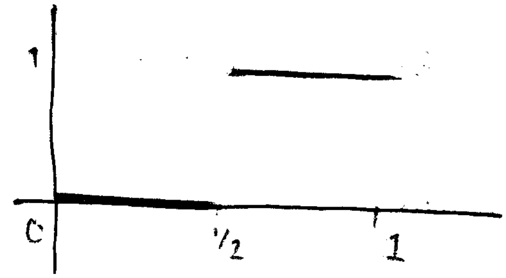
is identified with the zero function (equivalence classes of functions)

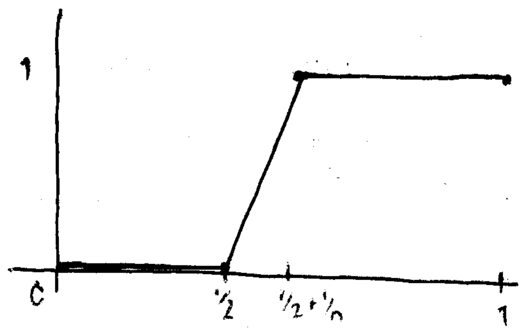
We Know by calculus

$\lim_{n\to\infty} x^n = 0, \ \forall |x| < 1$ 1

★ find the N that makes this true

type 2:





what needs to be done

(i) generalize to $C[a,b]$

(ii) show explicity convergence under L^1-norm

Opportunities for further inspection:

(a) ~~meas~~ what is measure theory

(b) what is Lebesgue measure

(c) what is the definition of $\int f\,d\mu$ that deals with previous issues

(ii) Suppose that $f_n \to f$ uniformly

Consider

$g(x) = f_n(x) - f(x)$

$$\int_a^b |g(x)|\,dx = \int_a^b |f_n(x) - f(x)|\,dx$$

$$\leq \int_a^b \|f_n(x) - f(x)\|_{sup}\,dx$$

$$= \|f_n(x) - f(x)\|_{sup} \int_a^b dx$$

$$= \|f_n(X) - f(X)\|_{sup}\,(b-a)$$

(iii) ✱ why do examples (i) & (ii) take care of this issue