Hw 3 $\quad$ ✓ 2.2, ✓ 2.3, o 2.4, o 2.5, ✓ 2.6

* Why was Lebesgue measure and Lebesgue integration created? find examples

Exercise 2.2:

Let $f_n \in C[a,b]$

Let f_n converge uniformly to f

(i) We want to show

*dominated convergence theorem

*monotone convergence theorem

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

$$\int_a^b f_n(x)\,dx = y_n \quad \& \quad \int_a^b f(x)\,dx = y$$

$$\lim_{n\to\infty} y_n = y \iff \lim_{n\to\infty} |y_n - y| = 0$$

$\Rightarrow$ Suppose $\lim_{n\to\infty} y_n = y$

Consider $\lim_{n\to\infty} |y_n - y|$. Let $\varepsilon > 0$. $\Rightarrow \exists N$ s.t. $n \geq N \Rightarrow$

$$|y_n - y| < \varepsilon \Rightarrow |y_n - y| \to 0$$

$\Leftarrow$ Suppose $\lim_{n\to\infty} |y_n - y| = 0 \Rightarrow \forall \varepsilon > 0\ \exists N$ s.t. $n \geq N$

$$\big|\, |y_n - y| - 0 \,\big| = \big|\, |y_n - y| \,\big| = |y_n - y| < \varepsilon \ ✓$$

$$\lim_{n\to\infty} \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$$

$$= \left| \int_a^b f_n(x) - f(x)\,dx \right|$$

$$\leq \int_a^b | f_n(x) - f(x) |\,dx$$

why can we do this

$$|f_n(x) - f(x)| \overset{?}{R} \| f_n(x) - f(x) \|_{sup}$$

$$\leq \int_a^b \| f_n(x) - f(x) \|_{sup}\,dx$$

$$= (b-a) \| f_n(x) - f(x) \|_{sup}$$

why can we control this

Is it clear why we can control the size of this in $\mathbb{R}$

☆ why is this linear

Why can we bring $\int$ through $(\cdot) - (\cdot)$

binary operation

$$\int_a^b f_n(x) - f(x)\,dx = \int_a^b f_n(x) + -1 \cdot f(x)\,dx$$

$$= \int f_n(x) + \int -1 f(x)$$

$$= \int f_n(x) + -1 \cdot \int f(x)$$

scalar multiples and additive property

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum \phi_N$$

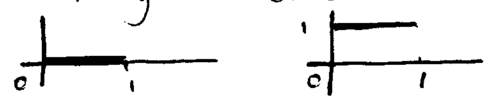
$=$

monotone convergence theorem

Exercise 2.2 ...

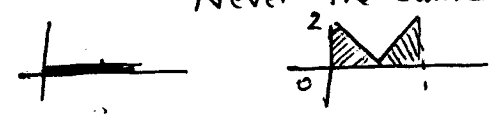
Pointwise convergence $\not\Rightarrow$ convergence of integrals

(i) what are two functions whose integrals are certainly always ~~zero~~ different



Never the same

(ii)



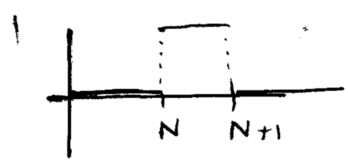
Non compact domain

$\lim_{n\to\infty} f_n(x) = f(x)$ for

choose a particular x value and check that $|f_n(x) - f(x)| < \varepsilon$ $\forall\, n \geq N$

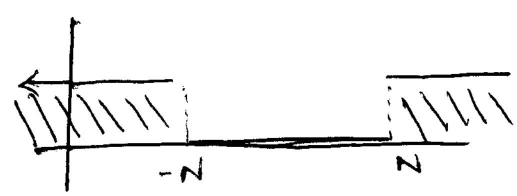
what if $f_N(x) = f(x)$
$n \geq N$ $f_n(x) = f(x)$

Two types $\boxed{1}$



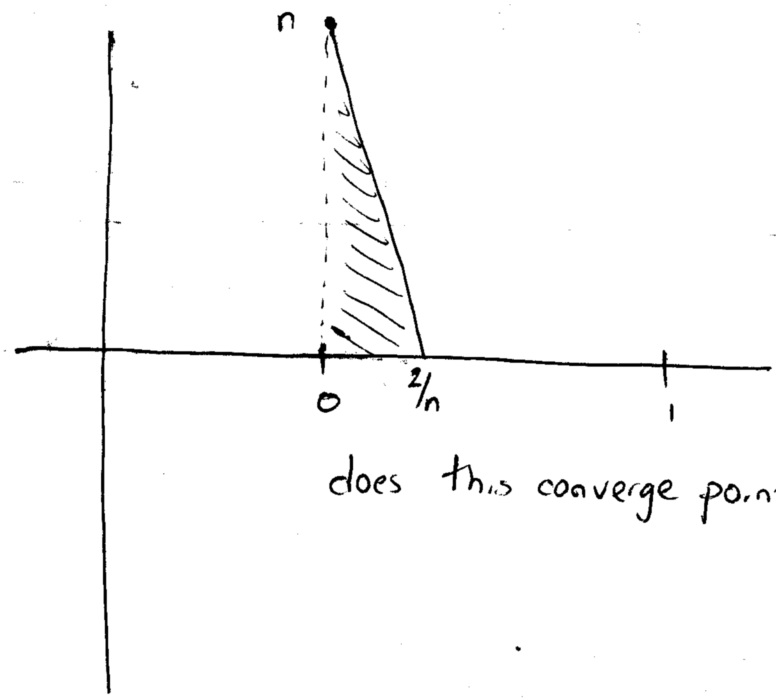
escape to infinity but still finite

$\boxed{2}$



point wise gave use freedom to choose functions that differ from zero outside of particular regions yet converge at specific points

What about



does this converge pointwise?

Exercise 2.3 Let $f: G \to \mathbb{R}$ be uniformly continuous

$:= \forall \varepsilon > 0, \forall x,y \in G \ \exists \delta$ s.t.

$d(x,y) < \delta$ ~~d(f(x)~~ $|f(x) - f(y)| < \varepsilon$

(i) WTS $\exists$ extension to continuous function $\bar{f}: \bar{G} \to \mathbb{R}$

$:= \bar{G} = \{ g \in X$ s.t. $\exists x_n \in G \mid \lim_{n\to\infty} x_n = g \}$

adherent & interior

Guess for $\bar{f}(x) = \begin{cases} f(x) & x \in G \\ \lim_{n\to\infty} f(x_n) & x \in \bar{G} \setminus G \end{cases}$ where $x_n \to x$ in $\bar{G} \subseteq X$, $x_n \subseteq G$

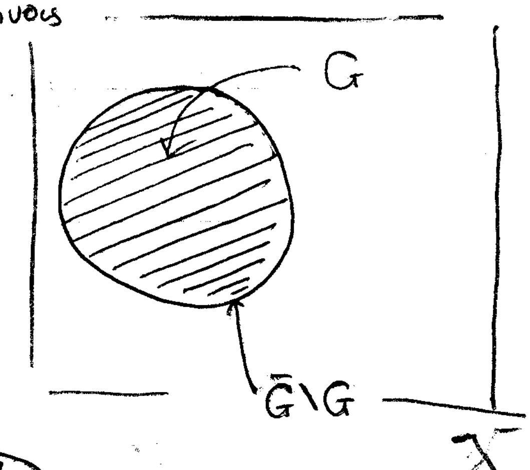
(ii) is this well defined

$x_n \to x$

is an equivalence relation

By def $\bar{f}$ is continuous on ~~f~~ G.

~~Cases~~ ideas continuous $\Leftrightarrow$ sequentally continuous

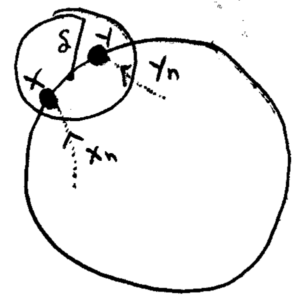


Let $\tilde{\varepsilon} > 0$.

Suppose $x,y \in \bar{G}$ s.t. $d_X(x,y) < \tilde{\delta} = \frac{\delta}{3}$

$x,y \in \bar{G} \Rightarrow \exists \{x_n\}_{n=1}^{\infty} \subseteq G \to x$ in X

$\exists \{y_n\}_{n=1}^{\infty} \subseteq G \to y$ in Y



~~$\Rightarrow \lim_{n\to\infty} |f_n(x$~~

by definition

$$x_n \to x \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(x)$$

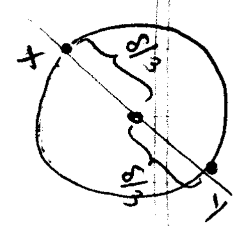
$$\Rightarrow \quad \exists\, N_x \text{ s.t. } n \geq N_x \Rightarrow |f(x_n) - f(x)| < \tilde{\varepsilon}/3$$

$$y_n \to y \quad \Rightarrow \quad \lim_{n\to\infty} f(y_n) = f(y)$$

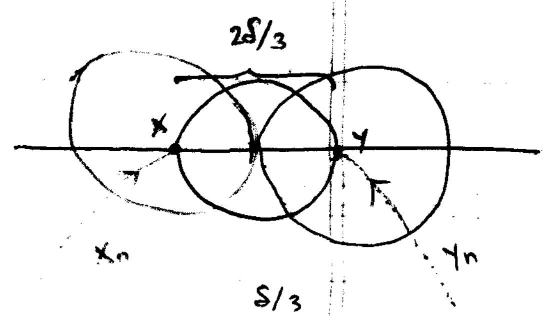
$$\Rightarrow \quad \exists\, N_y \text{ s.t. } n \geq N_y \Rightarrow |f(y_n) - f(y)| < \tilde{\varepsilon}/3$$

$d(x,y) < \dfrac{\delta}{3}$ $\qquad d(x_n,$

$$d(x,x_n) + \underbrace{d(x_n,y_n)} + d(y_n,y)$$



A function is continuous at $x_0 \in \overline{G}$
if $\quad x_0 \in G \Rightarrow f(x_0)$ continuous



uniformly continuous $\Rightarrow$ continuous

$$\delta = \tilde{\delta}$$

unif continuous δ