# The bridge number of a knot

Jennifer Schultens

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**Abstract**

The bridge number of a knot arose as one of the first numerical knot invariants. This chapter considers bridge number from a historical perspective, compares it to other knot invariants and reflects on related concepts.

## 1 Introduction

In the 1950s Horst Schubert set out to prove that a given knot has at most a finite number of companion knots. Companion knots are discussed in conjunction with satellite knots elsewhere in this volume. Schubert established this finiteness result with the help of a knot invariant devised for this purpose: The *bridge number*. See [24].

Given a diagram of a knot $K$, a subarc that includes an overcrossing is called a *bridge*. The number of bridges in a knot diagram is called the *bridge number of the diagram*. The minimum, over all diagrams of $K$, of the bridge numbers of the diagrams, is called the *bridge number of* $K$ and denoted by $b(K)$.

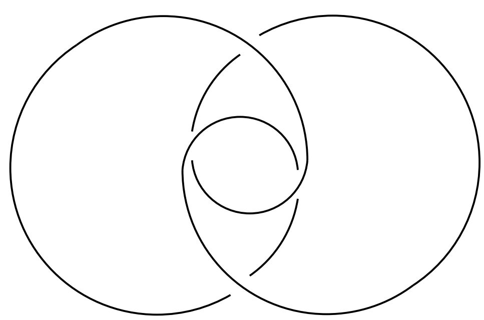

Figure 1: *A diagram of the trefoil with two bridges*

In early investigations of the bridge number of a knot, the idea of a knot lying in the plane, with a certain number of bridges venturing out of the plane, informed the discussion. In the 1980s Morse theoretic considerations and the notion of a height function on both $\mathbb{R}^3$ and $\mathbb{S}^3$ shifted our perspective. A *height function* on $\mathbb{S}^3$ is a Morse function with exactly two critical points: a maximum and a minimum. On $\mathbb{R}^3$ it is a Morse function with no critical points. More concretely, it is projection onto, say, the $z$-axis.

From this perspective it makes sense to consider the number of relative maxima of the knot $K$ with respect to a height function. We think of the plane used in a knot diagram as the $xy$-plane and our height function as projection onto $z$. Given a diagram of a knot $K$ with $b$ bridges, each subarc of the diagram that is a bridge can be converted into an arc with interior above the plane and exactly one maximum. Subarcs of the diagram that are not bridges can be concatenated and converted into arcs with interior below the plane and exactly one minimum. In this manner we construct a representative of $K$ with exactly $b$ local maxima. As we traverse $K$, we alternate between traversing arcs above the plane and arcs below the plane. It follows that the representative of $K$ also has exactly $b$ local minima.

Conversely, given a representative of $K$ with exactly $b$ local maxima and $b$ local minima, we can, by raising maxima and lowering minima, if necessary, find a horizontal plane $P$ that divides the representative of $K$ into $2b$ arcs, $b$ of which lie above $P$ and have exactly one local maximum and no other critical points and $b$ of which lie below $P$ and have exactly one local minimum. By isotoping the arcs that lie below $P$ into $P$ and viewing $P$ from above, we construct a diagram of $K$ with $b$ bridges. It follows that the bridge number of a knot equals the smallest possible number of local maxima for a representative of $K$.

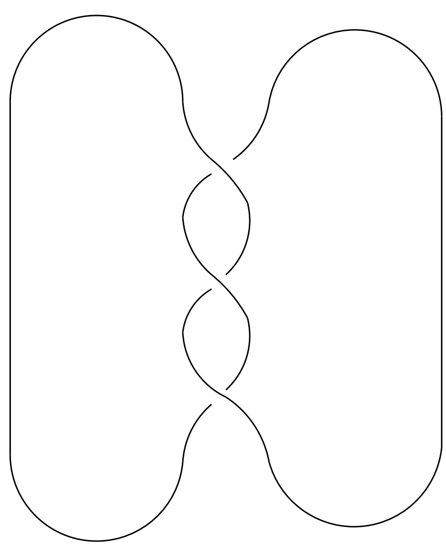

Figure 2: *A representation of the trefoil with two relative maxima*

A plane such as $P$, that lies above all local minima of $K$ and below all local maxima of $K$ is called a *bridge surface*. When we think of $K \subset \mathbb{S}^3$, then a height function on $\mathbb{S}^3$ decomposes $\mathbb{S}^3$ into level spheres together with one minimum and one maximum. A level sphere that lies above all local minima of the knot $K$ and below all local maxima of $K$ is called a *bridge sphere*.

## 2 Bridge numbers of torus knots

One interesting family of knots to consider in the context of bridge number consists of torus knots. A *torus knot* is an isotopy class of knots that are embedded in an unknotted torus $T$ in $\mathbb{S}^3$. The unknotted torus in $\mathbb{S}^3$ is characterized by the existence of two embedded curves called the *meridian* and *longitude* which intersect transversely in one point and bound disks in the complement of $T$. See Figure 3.

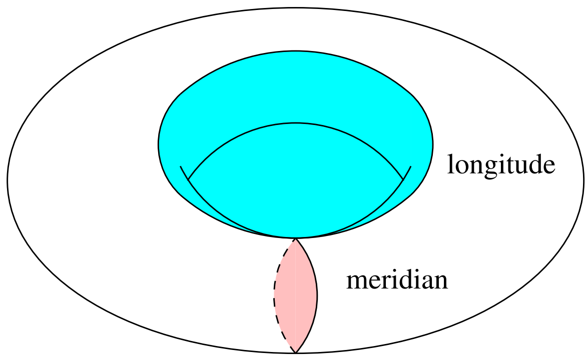


Figure 3: *A meridian/longitude pair*

An invigorating exercise shows that there is a 1-1 correspondence between the set of isotopy classes of torus knots and pairs $(p, q)$ of relatively prime integers. Given a torus knot $K$ and orientations on $K, T$, the meridian and the longitude, we take $p$ to be the oriented intersection number of the meridian with $K$ and $q$ to be the oriented intersection number of the longitude with $K$. The trefoil is an example of a torus knot, see Figure 4.

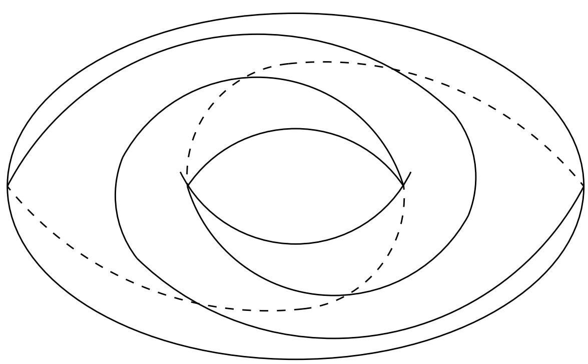

Figure 4: *The torus knot $T(3, 2)$ is also known as the trefoil*

We consider torus knots not just up to isotopy in $T$ but also as knots in $\mathbb{S}^3$. A torus knot will be the unknot in $\mathbb{S}^3$ if and only if $p = \pm 1$ or $q = \pm 1$. Knots in $\mathbb{S}^3$ are considered up to symmetry and homeomorphism. Changing relevant orientations changes $p$ to $-p$ or $q$ to $-q$. Interchanging the roles of meridian and longitude exchanges $p$ and $q$. Thus we need consider only pairs of integers $(p, q)$ such that $0 \leq q < p$ along with $(p, q) = (1, 1)$. Given $(p, q)$ with $0 \leq q < p$ or $(p, q) = (1, 1)$, we denote the corresponding torus knot by $T(p, q)$. In [24], Schubert proved the following theorem:

**Theorem 1.** *(Schubert 1954) The bridge number of $T(p, q)$ equals $q$ as long as $q > 0$.*

Recall that $T(p, q)$ is the unknot if $q \leq 1$. Of course, the unknot can be isotoped to lie entirely in a level sphere. However, counting relative maxima only makes sense for simple closed curves that are in general position with respect to a given height function. The unknot therefore, has bridge number 1. Conversely, the only knot with bridge number 1 is the unknot. Notice that this is consistent with the fact that the bridge number of $T(p, 1)$ is 1 for every $p$ since $T(p, 1)$ is the unknot for every $p$. The only knot for which $q = 0$ is $T(1, 0)$, which is also the unknot, but rather than having bridge number 0, it has bridge number 1, necessitating the hypothesis $q > 0$ in Schubert's theorem.

By considering Figure 4, we see that the bridge number of $T(3, 2)$ is less than or equal to 2. More generally, by drawing an analogous diagram for $T(p, q)$, we see that the bridge number of $T(p, q)$ is at most $q$. To see that it cannot be strictly less than $q$ requires more work and this work was carried out by Schubert in [24].

Indeed, given a height function, a representative of $T(p, q)$ that realizes bridge number will necessarily lie on an unknotted torus $T$. However, $T$ could be positioned in an unusual way, folding in on itself, for instance. What Schubert accomplished in [24] was to isotope $T$ into standard position without increasing the bridge number of the representative of $T(p, q)$. For a short Morse-theoretic rendering of Schubert's proof, see [27]. Figures 5 and 6 illustrate some of the challenges involved in isotoping $T$ into standard position.

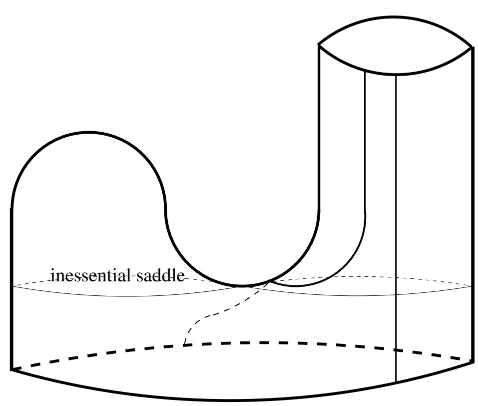


Figure 5: *An inessential saddle in $T$*

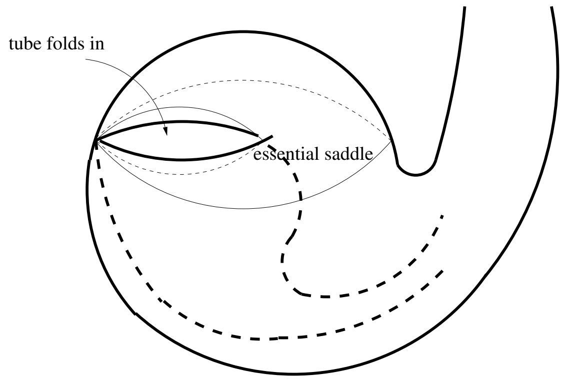


Figure 6: *An essential and nested saddle in T*

Similar reasoning applies in the setting of satellite knots and provides the theorem below:

**Theorem 2.** *(Schubert 1954) Let $K$ be a satellite knot with companion $J$ and pattern with wrapping number $k$. Then $b(K) \geq k \cdot b(J)$. If $K$ is a cabled knot, i.e., the pattern is a torus knot with wrapping number $q$, then $b(K) = q \cdot b(J)$.*

# 3 Bridge number versus genus

The *genus* of a knot $K$ is the smallest possible genus for a Seifert surface of $K$. The following theorem is due to Herbert Seifert, see [29]:

**Theorem 3.** *The genus of $T(p, q)$ is $\frac{(p-1)(q-1)}{2}$.*

Recall that $b(T(p, q))$, where $p > q \geq 0$, is $q$. Since $p$ can be arbitrarily large, we can have knots with bridge number $q$ and arbitrarily large genus.

Conversely, the doubling construction, exhibited in the case of the trefoil in Figure 7, can be performed on any knot, not just the trefoil, with a knot of high bridge number replacing the trefoil. This provides knots with genus 1 and, by Theorem 2, arbitrarily high bridge number.

We conclude that bridge number and genus are incompatible in the sense that they measure different types of complexities of a knot.

# 4 Bridge number versus hyperbolic volume

A knot $K$ is hyperbolic if its complement supports a complete finite volume hyperbolic structure. For a hyperbolic knot $K$, the volume of $K$, denoted by *volume*$(K)$, is the hyperbolic volume of $\mathbb{S}^3 - K$.

A *twist* is a succession of crossings of two strands of a knot over each other that is maximal in the sense that adjacent crossings involve other strands of the knot. See Figure 8.

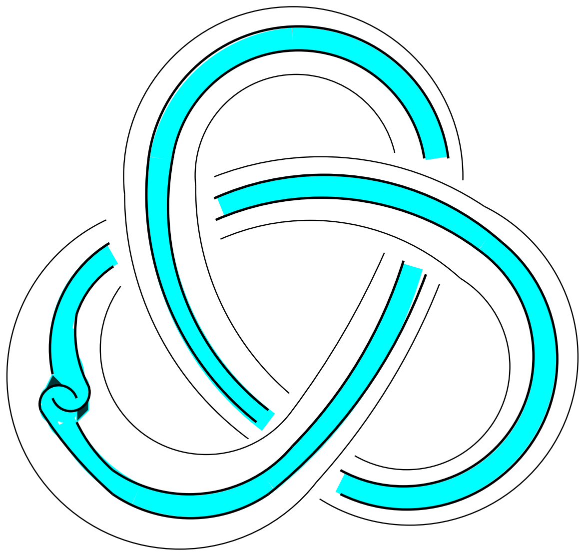

Figure 7: *The double of a trefoil*

Figure 9 schematically exhibits a family of 2-bridge knot diagrams. The boxes labeled with the numbers $t_1, \ldots, t_n$ represent twists with the given number of crossings. The numbers $t_1, \ldots, t_n$ can be chosen so that the knot diagram is alternating. The *twist number* of a knot diagram $D$, denoted by $t(D)$, is the minimal number of twists in the diagram $D$. The twist number for the diagrams of the family of 2-bridge knots represented in Figure 9 is arbitrarily high. See [28].

In [26], Schubert shows that 2-bridge knots are prime. Furthermore, Allen Hatcher and William Thurston show in [10] that 2-bridge knots are simple, *i.e.*, there are no essential tori in the complements of 2-bridge knots. Some 2-bridge knots will be torus knots, but those with more than one twist will not be torus knots. It follows that the complements of 2-bridge knots that are not torus knots support complete finite volume hyperbolic structures. For details see [10].

**Theorem 4.** *(Lackenby) Let $D$ be a prime alternating diagram of a hyperbolic link $K$ in $S^3$. Then $v_3(t(D)-2)/2 \leq volume(K) < v_3(16t(D)-16)$, where $v_3(\approx 1.01494)$ is the volume of a regular hyperbolic ideal 3-simplex.*

**Corollary 5.** *(S) There are 2-bridge knots of arbitrarily large volume.*

More generally, Jessica Purcell and Alexander Zupan prove, among other things, the following, see [22]:

**Theorem 6.** *(Purcell-Zupan) For any natural number $b$, there exists a sequence of knots $\{K_n\}$ such that $b(K_n) = b$ but $volume(K_n) \to \infty$ as $n \to \infty$.*

They also prove a partial converse:

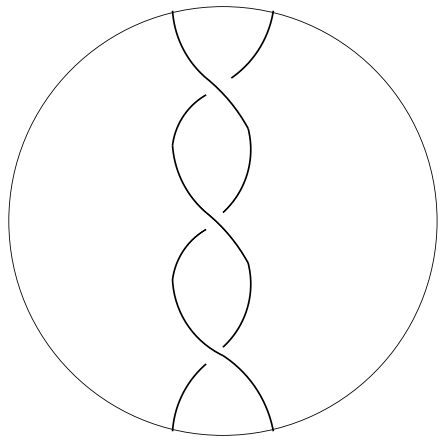

Figure 8: *A twist is a succession of crossings of two strands over each other*

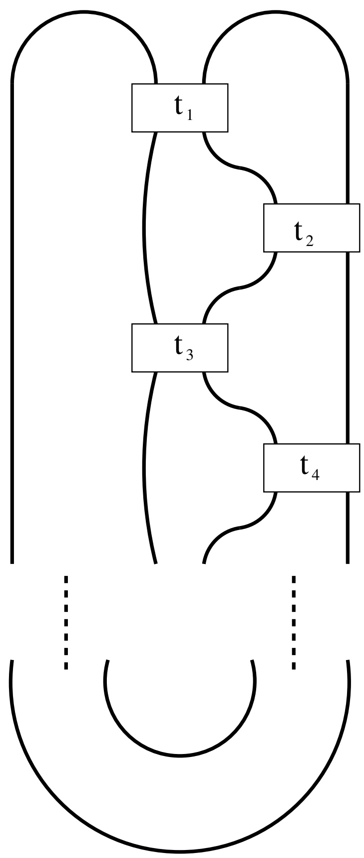


Figure 9: *Schematic of a 2-bridge knot where boxes represent twists*

**Theorem 7.** *(Purcell-Zupan) There is a constant $V > 0$ and a sequence of knots $\{K_n\}$ such that volume$(K_n) < V$ for all $n$ but $b(K_n) \to \infty$ as $n \to \infty$.*

We conclude as they do, that bridge number and volume are incompatible in the sense that they measure different types of complexities of a knot.

# 5 Bridge number versus rank and meridional rank

Recall that a bridge sphere is a level sphere of a height function that lies above all local minima and below all local maxima of a knot $K$. Thus a bridge sphere $S$ separates $\mathbb{S}^3$ into two 3-balls, $B_1$ and $B_2$, each containing a collection of subarcs of $K$. If the height function realizes the bridge number $b$ of $K$, then it has exactly $b$ local maxima and $b$ local minima. Denote the subarcs of $K$ in $B_i$ by $a_1^i, \ldots, a_b^i$. Within $B_i$, the twists lying above (respectively, below) the bridge surface can be "untwisted" to reveal that $a_1^i, \ldots, a_b^i$ are unknotted in $B_i$, meaning that there are pairwise disjoint disks $D_1^i, \ldots, D_b^i$ such that $\partial D_j^i$ is partitioned into two subarcs, one lying in $\partial B_i$ and the other equal to $a_j^i$.

The disks $D_1^i, \ldots, D_b^i$ cut $B_i$ into a 3-ball $\hat{B}_i$. Thus the complement of $K$ in $B_i$ can be constructed from the 3-ball $\hat{B}_i$ by identifying the remnants of the disks $D_1^i, \ldots, D_b^i$ in the boundary of $\hat{B}_i$. This tells us that $B_i - \eta(K)$, for $\eta(K)$ an open regular neighborhood of $K$, retracts to a wedge of $b$ circles and hence its fundamental group is the free group on $b$ generators.

To compute the fundamental group of the complement of $K$ we choose a basepoint $x$ in the bridge sphere. The bridge sphere retracts to a wedge of $2b$ circles and hence its fundamental group is the free group on $2b$ generators. The generators of the bridge sphere pair up to coincide in $\pi_1(B_i - K)$. It follows that $b$ is an upper bound for the *rank*, *i.e.*, the minimal number of generators needed to generate $\pi_1(\mathbb{S}^3 - K)$:

$$rank(K) := rank(\pi_1(\mathbb{S}^3 - K)) \leq b(K)$$

On the other hand, recall that the bridge number of $T(p,q)$ is $q$. However, regardless of $p, q$, the complement of $T(p,q)$ is the union of two solid tori along an annulus. It follows that $rank(T(p,q)) = 2$. Thus

$$rank(T(p,q)) < b(K) \; for \; q > 2$$

Moreover,

$$rank(T(p,q)) - b(K)$$

is arbitrarily large.

An interesting variation on the rank arises if we restrict our presentations of $\pi_1(\mathbb{S}^3 - K)$ by requiring each generator to be freely homotopic to the meridian. The minimum number of generators required in such a presentation of $\pi_1(\mathbb{S}^3 - K)$ is called the *meridional rank* of $K$. The above argument shows that

$$meridional\ rank(K) \leq b(K)$$

Equality holds for several classes of knots, *e.g.*, generalized Montesinos links and iterated torus knots. See [2], [4], [5], [15] and [23]. Whether or not equality holds is currently unknown.

# 6 Recognizing and computing

In [31], Robin Wilson proved that, subject to certain technical conditions, every bridge sphere is isotopic to a *meridional almost normal* sphere. We will not be interested in a precise definition of meridional almost normal spheres here, suffice it to say that there is an algorithm to detect meridional almost normal spheres. The converse is not true: A meridional almost normal sphere in a knot complement need not be a bridge sphere.

In [11], William Jaco and Jeffrey Tollefson exhibit an algorithm to determine whether or not a given 3-manifold is a 3-ball minus a collection of unknotted arcs. Since meridional almost normal spheres can be detected algorithmically, it is tempting to think that applying Jaco and Tollefson's algorithm to the two components of the complement of the meridional almost normal spheres in a knot complement should provide an algorithm to detect bridge spheres and thereby the bridge number of the knot. However, in toroidal knot complements, *i.e.*, for satellite knots, this method breaks down in the sense that the process need not terminate due to the possible existence of infinitely many meridional almost normal spheres of a given Euler characteristic.

An alternative to the strategy outlined above for computing bridge numbers of knots rests on a result of William Thurston. He proved that for any prime knot $K$, one of the following holds: 1) $K$ is a torus knot: 2) $K$ is a satellite knot; 3) $K$ is a hyperbolic knot. This trichotomy result proved to be a special case of the geometrization of 3-manifolds. See [19], [20], [21]. In [13], Greg Kuperberg proved a computational analogue of geometrization of 3-manifolds. Kuperberg's work provides an algorithm to determine whether or not a given knot is a torus knot, a satellite knot or a hyperbolic knot. For torus knots, Schubert's theorem tells us the bridge number and for hyperbolic knots Alex Coward exhibits an algorithm to recognize the bridge number, see [3].

Coward's argument continues a line of investigation begun by Wolfgang Haken in the 1960s to solve recognition problems in low-dimensional topology. However, rather than merely using normal or almost normal surface theory, Coward uses partially flat angled ideal triangulations as described by Marc Lackenby in [14]. Coward's argument also breaks down for knots that are not hyperbolic, again because of the presence of tori. The problem of algorithmically computing bridge number is hence still open.

# 7 Generalized bridge number

One can consider the number of relative maxima of the knot $K$ not just with respect to a height function on $\mathbb{S}^3$ but with respect to any self-indexing Morse function on $\mathbb{S}^3$. A level surface of minimal Euler characteristic of such a Morse function is a *bridge surface* for $K$ if it lies above all relative minima and below all relative maxima of $K$. The *g-bridge number* of $K$ is the least number of relative maxima $K$ will exhibit with respect to a Morse function with bridge surfaces of genus $g$. Additivity in the sense of Theorem 2 fails. For more on properties of the generalized bridge number, see Helmut Doll's investigation, [7].

# 8 Bridge distance

For any compact connected orientable surface $\Sigma$, the curve complex of $\Sigma$ is defined as follows:

- Vertices of $C(\Sigma)$ correspond to isotopy classes of essential simple closed curves in $\Sigma$;
- Edges correspond to pairs of vertices admitting disjoint representatives;
- $C(\Sigma)$ is a flag complex;
- The distance between two vertices is the least number of edges in an edge path between the two vertices.

We will be interested in bridge spheres of knots. Let $S$ be a bridge sphere for $K$ and set $\Sigma = S - \eta(K)$. If $\Sigma$ is a four times punctured sphere (in the case where $K$ a 2-bridge knot) the definition above yields a complex that is disconnected. By convention, the definition of the edges for this curve complex is adjusted (requiring two points of intersection rather than requiring disjointness), in order to guarantee connectedness.

The bridge sphere $S$ separates $\mathbb{S}^3$ into balls $B_1$ and $B_2$. Denote by $\mathcal{D}_i$ the collection of isotopy classes of essential disks in $B_i - \eta(K)$ with boundary in $S - \eta(K)$. Denote the collection of boundaries of disks in $\mathcal{D}_i$ by $\partial_i$. The *bridge distance of* $S$, denoted by $d(S)$, is given by

$$d(S) = \min\{d(c_1, c_2) \mid c_i \in \partial_i\}$$

The *bridge distance of* $K$, denoted by $d(K)$, is the greatest possible bridge distance of a bridge sphere of $K$ that meets $K$ in exactly $2b(K)$ points. To understand the subtleties concerning how this gives us a well-defined integer, see [30]. For more on the topic of bridge distance, see [12] and [32]. For a natural generalization of bridge distance, see [16].

# 9 Bridge number versus distortion

In [8], Mikhail Gromov studied embeddings of manifolds and defined a notion called *distortion*. In the specialized setting of knots we have the following: Given a knot $K$ in $\mathbb{S}^3$, a smooth representative $\gamma$ of $K$ and two points $p, q \in \gamma$, the distance between $p$ and $q$ can be measured in two ways: 1) As the distance between $p$ and $q$ in $\mathbb{S}^3$, which we denote by $d_s(p,q)$; 2) As the length of the (shorter) subarc of $\gamma$ from $p$ to $q$, which we denote by $d_\gamma(p,q)$. The *distortion* of a knot $K$, denoted by $\delta(K)$, is then given by:

$$\delta(\gamma) = \sup_{p,q\in\gamma} \frac{d_\gamma(p,q)}{d_s(p,q)}$$

and

$$\delta(K) = \inf_{\gamma\in K} \delta(\gamma)$$

In [6], Elizabeth Denne and John Sullivan proved that the distortion of a nontrivial knot is bounded below by $\frac{5\pi}{3}$. Having observed that the distortion of a knot remains constant under connected sum, Gromov asked in [8] whether there is a universal upper bound on the distortion of a knot. The answer to this question is "no". In [18], John Pardon, building on work of Makoto Ozawa, see [17], used the bridge number as a tool to prove that torus knots provide a family with arbitrarily high distortion. See also [9].

**Theorem 8.** *(Pardon)*

$$\delta(T(p,q)) \geq \frac{1}{160}\min(p,q)$$

Recent work announced by Ryan Blair, Marion Campisi, Scott Taylor and Maggy Tomova suggests that the distortion of a knot is far more closely related to bridge number than other invariants. See [1].

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Department of Mathematics
1 Shields Avenue
University of California, Davis
Davis, CA 95616
USA