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The stabilization problem for Heegaard splittings of Seifert fibered spaces

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Abstract

The maximal number of stabilizations required to obtain equivalent Heegaard splittings from Heegaard splittings of an orientable Seifert fibered space with orientable base space is 1.

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1. Introduction

To this day we do not know of examples of manifolds which possess distinct Heegaard splittings which remain distinct after 1 stabilization. Thus it is reasonable to conjecture that 1 stabilization always suffices. Rubinstein and Scharlemann recently proved that there is a linear bound on the number of stabilizations required in the case of non-Haken manifolds. Johannson has given a polynomial bound and later Rubinstein–Scharlemann a quadratic bound on the number of stabilizations required in the case of Haken manifolds. In this paper I show that for orientable Seifert fibered spaces with orientable base spaces, 1 stabilization suffices.

2. Preliminaries

For basic definitions pertaining to Heegaard splittings and Seifert fibered spaces see for instance [11]. Below, we repeat the rather lengthy definition of vertical Heegaard

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splittings in the simpler case of closed Seifert fibered spaces. Note that for product manifolds the stabilization problem is trivial since the Heegaard splitting is unique (see [10]).

Definition 1. Let (T_1, T_2) be the genus 1 Heegaard splitting of S^3. A *stabilization* of the Heegaard splitting (H_1, H_2) of M is the (unique) Heegaard splitting of M obtained by taking the connected sum of triples (M, H_1, H_2) # (S^3, T_1, T_2). We denote the stabilization of (H_1, H_2) by $S(H_1, H_2)$ and the handlebodies in the Heegaard splitting $S(H_1, H_2)$ by $S(H_i)$. The stabilization of $S(H_1, H_2)$ will be denoted by $S^2(H_1, H_2)$ and so forth. When referring to Heegaard splittings (H_1^1, H_2^1) and (H_1^2, H_2^2) of M of different genus, say g^1 and g^2 respectively, with $g^1 < g^2$, we say that 1 *stabilization suffices* if $S^{g^2-g^1+1}(H_1^1, H_2^1) = s(H_1^2, H_1^2)$.

Roughly speaking, vertical Heegaard splittings are those for which each exceptional fiber appears as the core of a 1-handle in one of the handlebodies. In order to distinguish the various vertical splittings we need to be more precise.

Let M be a closed orientable Seifert fibered space with orientable base space P and k exceptional fibers $e_1, \ldots, e_k$ with $k \geqslant 2$. Let p be the natural projection map on M. Let D_i be a closed regular neighborhood in P of the projection $p(e_i)$ of the ith exceptional fiber e_i. Let B be the closure of $P - (D_1 \cup \cdots \cup D_k)$. And let x_0 be a point in $\operatorname{int}(B)$. For $i = 1, \ldots, k$, let d_i be the union of ∂D_i and a simple arc α_i in $B \subset P$ connecting x_0 to ∂D_i. Let δ_i be the union of α_i and a radius of D_i. Further, let $a_1, b_1, \ldots, a_g, b_g$ (here g is the genus of P) be a collection of arcs based at x_0 which cuts P into a disk. We may assume that (except for $\delta_i \cap d_i = \alpha_i$), all arcs chosen are disjoint except at x_0.

Definition 2. A *vertical Heegaard splitting* of M is a Heegaard splitting obtained by the following construction, for some nonempty collection of distinct indices $(i_1, \ldots, i_j) \subset (1, \ldots, k-1)$ and possibly stabilization. Let $(l_1, \ldots, l_{k-j-1})$ be the complementary set in $\{1, \ldots, k-1\}$ to $(i_1, \ldots, i_j)$. Let

$$\Omega(i_1, \ldots, i_j)$$

be a 1-complex in M obtained by lifting

$$a_1 \cup b_1 \cup \cdots \cup a_g \cup b_g \cup \delta_{i_1} \cup \cdots \cup \delta_{i_j} \cup d_{l_1} \cup \cdots \cup d_{l_{k-j-1}}$$

to M. Let V_i be the preimage under p of D_i. And set:

$$H_1(i_1, \ldots, i_j) = N(\Omega(i_1, \ldots, i_j)) \cup V_{i_1} \cup \cdots \cup V_{i_j},$$
$$H_2 = \operatorname{closure}(M - H_1).$$

Then (H_1, H_2) is a Heegaard splitting with splitting surface

$$F(i_1, \ldots, i_j) = H_1(i_1, \ldots, i_j) \cap H_2(i_1, \ldots, i_j).$$

Now let M be an orientable Seifert fibered space with orientable base space P and no exceptional fibers (respectively, one exceptional fiber). Then a *vertical Heegaard* *splitting* of M is a Heegaard splitting obtained by the following construction and possibly stabilizations: Let $a_1, b_1, \ldots, a_g, b_g$ and x_0 be chosen as above and let Ω be a 1-complex in M obtained by lifting

$$a_1 \cup b_1 \cup \cdots \cup a_g \cup b_g$$

to M. Let V be a closed regular neighborhood of the regular fiber which projects to x_0 (respectively a closed regular neighborhood, containing the regular fiber projecting to x_0, of the unique exceptional fiber).

And set:

$$H_1 = N(\Omega) \cup V,$$
$$H_2 = \text{closure}(M - H_1).$$

Then (H_1, H_2) is a Heegaard splitting with splitting surface

$$F = H_1 \cap H_2.$$

Note that the minimal genus vertical Heegaard splitting is unique in this case.

Remark 3. To see that $F(i_1, \ldots, i_j)$ is well-defined, apply [10, Theorem 3.1] to $F(i_1, \ldots, i_j)$ regarded as a Heegaard splitting of $M - \eta(e_1 \cup \cdots \cup e_k)$.

Remark 4. The genus of a vertical Heegaard splitting is $2g + k - 1$ when $k \geqslant 2$ and $2g + 1$ otherwise.

Roughly speaking, horizontal Heegaard splittings are those for which the splitting surface is transverse to the Seifert fibration except in a regular neighborhood of a given exceptional fiber.

Definition 5. Let e be an exceptional fiber of the closed orientable Seifert fibered space M with orientable base space. The following procedure yields a Heegaard splitting for some of the Seifert fibered spaces under consideration. If it does, then we call the resulting Heegaard splitting a *horizontal Heegaard splitting* (rel e of M). Here $\widehat{M} = M - \eta(e)$ fibers as a surface bundle over S^1. The fiber in this fibration is a compact orientable surface Q with one boundary component. If $I_1 \cup I_2$ is a partition of S^1 into two intervals, then $\widehat{M} = (Q \times I_1) \cup (Q \times I_2)$. Note that $(Q \times I_i)$ is a handlebody. When we set

$$H_1 = (Q \times I_1) \cup N(e)$$

and

$$H_2 = (Q \times I_2),$$

we may or may not have constructed a Heegaard splitting for M, since H_1 may or may not be a handlebody. Here H_1 is constructed by glueing a solid torus to an annulus in the boundary of the handlebody $(Q \times I_1)$. Thus H_1 is itself a handlebody if and only if $\pi_1(H_1)$ is free. But $\pi_1(H_1)$ is free if and only if e is parallel to a generator of the annulus on $\partial N(e)$ along which $N(e)$ is glued to $(Q \times I_1)$. To see this condition expressed in terms of the Seifert invariants of M, see [7, 1.4].

The Main Theorem in [11] states that all Heegaard splittings of orientable Seifert fibered spaces with orientable base spaces and nonempty boundary are vertical. Thus in considering horizontal Heegaard splittings our aim will be to show that their stabilizations are vertically reducible in the following sense.

Definition 6. Let e be an exceptional fiber of M. If, after isotopy, F is a splitting surface of Heegaard splittings for both M and $M-\eta(e)$, then we say that F is *vertically reducible* at e.

Lemma 7. *Suppose (H_1, H_2) is a Heegaard splitting for M with splitting surface F. Suppose further that e (a loop, e.g., an exceptional fiber) lies in the interior of H_1 and is parallel to a simple curve in F which intersects an essential disk in H_1 once transversely. Then F is vertically reducible at e and hence vertical.*

Proof. This is [11, Lemma 2.2]. □

3. Stabilizing vertical Heegaard splittings

The essence of [6, Theorem 2] is that vertical Heegaard splittings constructed with different subcollections of exceptional fibers are distinct. The argument below shows that they become equivalent after one stabilization.

Theorem 8. *Suppose $(i_1, \ldots, i_j)$ is a collection of indices denoting a subcollection of exceptional fibers for the orientable Seifert fibered space M with orientable base space P and that $j \geqslant 2$. Then*

$$S\big(H_1(i_1, \ldots, i_j), H_2(i_1, \ldots, i_j)\big) = S\big(H_1(i_1, \ldots, i_{j-1}), H_2(i_1, \ldots, i_{j-1})\big).$$

Remark 9. The assumption $j \geqslant 2$ ensures that $(H_1(i_1, \ldots, i_{j-1}), H_2(i_1, \ldots, i_{j-1}))$ is defined.

Proof. (Theorem 8) Let $N(e_{i_j})$ be the (maximal) closed regular neighborhood of e_{i_j} contained in $H_1(i_1, \ldots, i_j)$ and let $\widetilde{N}(e_{i_j})$ be the (maximal) closed regular neighborhood of e_{i_j} contained in $H_2(i_1, \ldots, i_{j-1})$. We may assume that $N(e_{i_j}) \subset \text{interior}(\widetilde{N}(e_{i_j}))$. Furthermore, denote the arc connecting x_0 to e_{i_k} in $\Omega(i_1, \ldots, i_j)$ by γ_j and denote the subarc of γ_j which connects x_0 to ∂D_{i_j}, and may be used in the construction of $\Omega(i_1, \ldots, i_{j-1})$ by $\widehat{\gamma}_j$. So $\widehat{\gamma}_j \subset \gamma_j$. Finally, denote the (maximal) closed regular neighborhood of γ_k (respectively $\widehat{\gamma}_j$) contained in $H_1(i_1, \ldots, i_j)$ (respectively $H_2(i_1, \ldots, i_{j-1})$) by g_k (respectively $\widehat{g}_j$). By attaching a trivial handle h to $g_j - \widehat{g}_j$ we obtain $S(H_1(i_1, \ldots, i_j), H_2(i_1, \ldots, i_j))$. See Fig. 1.

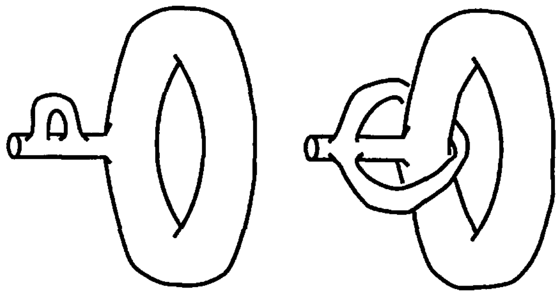

Fig. 1. Outside view.

We now perform a handleslide of h over $N(e_{i_j})$ after which $h \cup \widehat{g}_j$ is a closed regular neighborhood of the lift in $\Omega(i_1, \ldots, i_j)$ of d_{i_j}. Now $S(H_1(i_1, \ldots, i_j)) = H_1(i_1, \ldots, i_j) \cup H_1(i_1, \ldots, i_{j-1})$. Remove a shrunk copy of $H_1(i_1, \ldots, i_{j-1})$ from the interior of $H_1(i_1, \ldots, i_{j-1})$. Then the remaining manifold is a copy of $H_2(i_1, \ldots, i_{j-1})$ plus a collar. So $S(F(i_1, \ldots, i_j))$ is the splitting surface of a Heegaard splitting of a handle body and hence standard, by [3, Lemma 1.1] and Frohman's theorem [4, Lemma 1.1]. Since the genus of $S(F(i_1, \ldots, i_j))$ is one more than the genus of $F(i_1, \ldots, i_{j-1})$, the theorem follows. □

Corollary 10. *Let M be a closed orientable Seifert fibered space with orientable base space P of genus g and $k \geqslant 2$ (respectively 0 or 1) exceptional fibers. Then for every genus strictly greater than $2g + k - 1$ (respectively $2g$), M possesses exactly one vertical Heegaard splitting.*

Proof. Since $S(H_1, H_2)$ is unique for any Heegaard splitting (H_1, H_2), this follows directly from Theorem 8 and Remark 4. □

4. Stabilized horizontal Heegaard splittings

Theorem 11. *Let (H_1, H_2) be a horizontal Heegaard splitting of the closed orientable Seifert fibered space M. Then $S(H_1, H_2)$ is a vertical Heegaard splitting.*

Proof. Recall Definition 5. A meridian disk D of $N(e) \subset H_1$ intersects e once. Furthermore, since the annulus $\partial(Q \times I_1)$ (respectively $\partial(Q \times I_2)$) winds once around e, and hence intersects a meridian disk in one arc, ∂D decomposites into two arcs, α and β, where α is properly embedded in H_1 and β lies in the splitting surface F.

Let $\widetilde{\alpha}$ be a pushoff, fixing endpoints, of α into $\eta(e)$. Stabilize (H_1, H_2) by attaching a handle $h = N(\widetilde{\alpha})$ to H_2. Now slide $\widetilde{\alpha}$ across D so that h turns into $N(\beta)$. Then $\widetilde{D} = D \cap S(H_1)$ intersects e once. Since e is parallel into $S(F)$ (e was parallel into F), it follows from Lemma 7 that $S(H_1, H_2)$ is vertically reducible. □

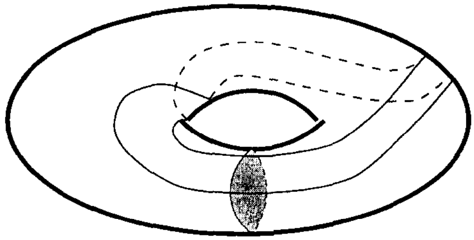

Fig. 2. Outside view.

Corollary 12. *An orientable circle bundle over an orientable surface has a unique irreducible Heegaard splitting.*

Proof. This follows directly from [7, Theorem 0.6] and Theorem 11. For base space a closed orientable surface of genus g the Heegaard genus is $2g$ for Euler number ± 1 and is realized by the unique horizontal Heegaard splitting which becomes the unique vertical Heegaard splitting after one stabilization. For Euler number different from ± 1 there are no horizontal Heegaard splittings. Here the Heegaard genus of the bundle is $2g + 1$ and is realized by the unique vertical Heegaard splitting. □

Combining [7, Theorem 0.1] and [12, Corollary 2] yields the result that an irreducible Heegaard splitting of an orientable Seifert fibered space over an orientable base space is either horizontal or vertical. This enables the case by case analysis in the proof of the following theorem.

Theorem 13. *Heegaard splitting of a closed orientable Seifert fibered space with orientable base space are equivalent after one stabilization.*

Proof. Suppose F and F' are the splittings surfaces of Heegaard splittings of the same orientable Seifert fibered space with orientable base space. We may assume that both Heegaard splitting are irreducible, for the general case will then follow. Let g be the genus of F and g' the genus of F'. We may assume that $g \leqslant g'$. Let g_v, denote the genus of a vertical Heegaard splitting of M. We wish to show that $S^{g'-g+1}(F) = S(F')$.

Case 1: $g < g_v$. For most Seifert fibered spaces, the Heegaard genus is achieved by a vertical Heegaard splitting. The only manifolds in which the minimal genus is not achieved by a vertical Heegaard splitting, are those described in terms of their invariants by

$$S(0, e_0 \mid 1/2, 1/2, \ldots, \beta_k/2\lambda + 1),$$

where k is even and greater than four, $\lambda > 0$ and either $e_0 = \pm 1/2(2\lambda + 1)$ or $\beta_k = \pm(\lambda) \bmod (2\lambda + 1)$. In these cases it is achieved by a horizontal Heegaard splitting of genus $k - 2$ (see [2,7]), i.e., $g_v - 1$.

It follows from Theorem 11 that for the Seifert fibered spaces mentioned above the minimal genus vertical Heegaard splitting (for uniqueness of the vertical Heegaard splitting of these manifolds see [6, 2.8]) is a stabilization of any of the horizontal Heegaard splittings of genus $k - 2$. More generally, it follows from [7, Theorem 0.1] and [12, Corollary 2] that any Heegaard splitting is a stabilization of one of the horizontal Heegaard splittings of genus $k - 2$. Thus, since we are considering only irreducible Heegaard splittings, $g = g' = k - 2$ and both $S^{g'-g+1}(F) = S(F)$ and $S(F')$ are the unique vertical Heegaard splitting of genus $k - 1 = g + 1$ of one of the manifolds above.

Case 2: $g \geqslant g_v$. By [7, Theorem 0.1] and [12, Corollary 2] the Heegaard splittings under consideration are either horizontal or vertical. It now follows directly from Corollary 10 and Theorem 11 that $S^{g'-g+1}(F)$ and $S(F')$ are the unique vertical Heegaard splitting of genus $g' + 1$ (regardless of whether F and F' were horizontal or vertical). □

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