DUCCI ITERATES AND SIMILAR ORDERING OF VISIBLE POINTS IN CONVEX REGIONS

JUNXIAN LI, ALBERT TAMAZYAN, AND ALEXANDRU ZAHARESCU

Abstract. Hardy, Littlewood and Pólya first introduced the notion of similar ordering of pairs of rationals, and A.E. Mayer proved that pairs of Farey fractions in $\mathcal{F}_Q$ are similarly ordered when Q is large enough. We generalize Mayer's result to Ducci iterates of Farey sequence and visible points in convex regions. We also study the distribution of generalized indices of these sequences.

1. Introduction

For a positive integer Q the Farey series $\mathcal{F}_Q$ of order Q is defined as the set of reduced fractions between 0 and 1 with denominators less than or equal to Q. Denote by $N(Q)$ the number of elements in $\mathcal{F}_Q$ and write these elements in increasing order as

$$\gamma_1 = \frac{a_1}{q_1} < \gamma_2 = \frac{a_2}{q_2} < \cdots < \gamma_{N(Q)} = \frac{a_{N(Q)}}{q_{N(Q)}}. \tag{1.1}$$

To study the basic properties of Farey series the reader is referred to the classical book by Hardy and Wright [12, Chapter III]. Hardy, Littlewood and Pólya [11] introduced the notion of similar ordering for pairs of rational numbers as follows: two fractions $\gamma = \frac{a}{q}$ and $\gamma' = \frac{a'}{q'}$ are called similarly ordered if $(a-a')(q-q') \geq 0$. Mayer [14] proved that any two neighboring Farey fractions are similarly ordered. Furthermore, he showed that for large values of Q, not immediate neighbors in Farey series are similarly ordered too. More precisely, for any positive integer k there exists a number $Q(k)$, so that for any $Q > Q(k)$, and any $1 \leq j < j' \leq N(Q)$ with $j' - j < k$, the numbers $\frac{a_j}{q_j}$ and $\frac{a_{j'}}{q_{j'}}$ are similarly ordered.

The notion of similar ordering can be naturally extended for two sequences of numbers. We say two sets of real numbers, $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ are similarly ordered if $(a_i - a_{i+1})(b_i - b_{i+1}) \geq 0$ for all $i = 1, ..., n-1$. In this term Mayer's first result states that the sets of numerators and denominators of any Farey sequence are similarly ordered. We denote by $\psi = \psi^{(1)}$ the Ducci operation on a tuple of numbers, namely $\psi\mathcal{A} = \{|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{n-1} - a_n|\}$. Also denote by $\psi^{(k)}\mathcal{A} = \psi(\psi^{(k-1)}\mathcal{A})$ the k-th iteration of the Ducci operation. The first main result of this paper is the following theorem.

Theorem 1.1. *For any positive integer k, there exists a number $Q_0 = Q_0(k)$ such that if $Q \geq Q_0$ is a positive integer, and $\mathcal{A}$ and $\mathcal{Q}$ are the sets of numerators and*

2010 *Mathematics Subject Classification.* 11B57, 11B99.

Key words and phrases. Ducci Iterates, Farey Fractions, Plane Visible Points, Similar Ordering.

denominators of the Farey series $\mathcal{F}_Q$, respectively, then the sequences $\psi^{(k)}\mathcal{A}$ and $\psi^{(k)}\mathcal{Q}$ are similarly ordered.

The next theorem of this paper shows this phenomenon in a more general setting. A lattice point on the plane is called visible if there is no other lattice point on the line segment connecting it to the origin. For a region $\Omega \subset \mathbb{R}^2$ we define $\mathcal{F}_\Omega = \{(q_1, a_1), (q_2, a_2), \dots, (q_n, a_n)\}$ as the set of visible lattice points of Ω in the order of increasing arguments. In what follows $\mathcal{F}_\Omega$ should be interpreted as a circular set, meaning that (q_1, a_1) should be considered not as the first element but rather the successor of (q_n, a_n). We also define

$$\mathcal{Q}_\Omega = \{q_1, q_2, \dots, q_n\} \text{ and } \mathcal{A}_\Omega = \{a_1, a_2, \dots, a_n\}$$

as the sets of first and second coordinates of the points in $\mathcal{F}_\Omega$, respectively. For a tuple of numbers $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ we define the analogue of the Ducci operator

$$\tilde{\psi}\mathcal{X} = \tilde{\psi}^{(1)}\mathcal{X} = \{|x_1 - x_2|, |x_2 - x_3|, \dots, |x_n - x_1|\},$$

and naturally $\tilde{\psi}^{(k)}\mathcal{X} = \tilde{\psi}(\tilde{\psi}^{(k-1)}\mathcal{X})$. Note that $\tilde{\psi}$ does not make the tuple shorter unlike ψ. The following result demonstrates the phenomenon of similar ordering for the sets $\mathcal{A}_\Omega$ and $\mathcal{Q}_\Omega$ for certain regions Ω.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be an open convex region with piecewise smooth boundary and containing the origin. Then for any positive integer k there exists a number $x_0(\Omega, k)$ so that for any $x > x_0$ the sets $\tilde{\psi}^{(k)}\mathcal{Q}_{x\Omega}$ and $\tilde{\psi}^{(k)}\mathcal{A}_{x\Omega}$ are similarly ordered.*

Since Farey fractions also correspond to visible points in the plain this theorem is in the same spirit as Theorem 1.1. However, it cannot be considered as a generalization of Theorem 1.1 since the region for which the set of visible points corresponds to $\mathcal{F}_Q$ is the triangle with vertices $(0, 0)$, $(0, Q)$ and (Q, Q) which apparently does not satisfy the requirements of Theorem 1.2.

Another aspect that we are going to study in this paper relates to the index of Farey fractions. In [10], the index of the i-th fraction $\gamma_i = \frac{a_i}{q_i}$ in $\mathcal{F}_Q$ is defined as

$$\nu_i := \nu_Q(\gamma_i) = \left\lfloor \frac{Q + q_{i-1}}{q_i} \right\rfloor = \frac{q_{i+1} + q_{i-1}}{q_i} = \frac{a_{i+1} + a_{i-1}}{a_i},$$

and it was proved in [10] that

$$\sum_i \nu_i = 3N(Q) - 1.$$

Moreover, from [6], the following

$$\sum_i \nu_i \nu_{i+j} = A(j)N(Q) + O_j(Q \log^2 Q)$$

holds, where $A(j)$ is some constant $\ll 1 + \log j$. In [13], Haynes considered the j-index of γ_i as

$$\nu_j(\gamma_i) := a_{i+j-1}q_{i-1} - a_{i-1}q_{i+j-1},$$

and showed that $\nu_2(\gamma_i) = \nu_Q(\gamma_i)$. The study of $\nu_j(\gamma_i)$ arises naturally in problems where the denominators of the fractions are restricted to an arithmetic progression with composite moduli (see [1, 2, 7]). It was shown in [13] that for any integer $j \geq 0$, there exists a real constant $B(j)$ such that

$$\frac{1}{N(Q)} \sum_{i=1}^{N(Q)} \nu_j(\gamma_i) = B(j) + O_j\left(\frac{(\log Q)^2}{Q}\right),$$

as $Q \to \infty$. Later, in [3], Badziahin and Haynes considered the distribution of $\nu_j(\gamma_i) = k$ with some divisibility constrains on the denominator of γ_i. This is fundamental in the study of gap distribution of special subsets of Farey fractions such as in [5]. We are going to consider the quantity

$$\nu_j^{(k)}(\gamma_i) = a_{i+j-1}^{(k)} q_{i-1}^{(k)} - a_{i-1}^{(k)} q_{i+j-1}^{(k)} = -\det\begin{pmatrix} a_{i-1}^{(k)} & a_{i+j-1}^{(k)} \\ q_{i-1}^{(k)} & q_{i+j-1}^{(k)} \end{pmatrix} \tag{1.2}$$

for the sequences $\psi^{(k)}(\mathcal{A})$ and $\psi^{(k)}(\mathcal{Q})$, which generalizes the j-index of elements in $\mathcal{F}_Q$. We will prove the existence of the limiting distribution of $\nu_j^{(k)}$ as $Q \to \infty$ and this can be generalized to visible points in convex regions with rectifiable boundary. It turns out that the limiting distribution is independent of the region.

2. Proof of Theorem 1.1

We need some preliminary lemmas before we prove the main theorem.

Lemma 2.1. *Let* $\frac{a_i}{q_i}, \frac{a_{i+1}}{q_{i+1}}, \ldots, \frac{a_{i+j}}{q_{i+j}}$ *be consecutive Farey fractions in* $\mathcal{F}_Q$. *Then*

$$\max\{q_i, q_{i+j}\} > \frac{Q}{j+1}.$$

Proof. Suppose $q_i, q_{i+j} \leq \frac{Q}{j+1}$ and consider the following fractions:

$$b_m = \frac{a_i + m a_{i+j}}{q_i + m q_{i+j}}, \quad m = 1, \ldots, j.$$

Then

$$b_m - b_{m-1} = \frac{a_{i+j} q_i - a_i q_{i+j}}{(q_i + m q_{i+1})(q_i + (m-1) q_{i+j})} > 0,$$

and also $q_i + m q_{i+j} \leq Q$, which gives at least j Farey fractions between $\frac{a_i}{q_i}$ and $\frac{a_{i+j}}{q_{i+j}}$, which is a contradiction. □

Let $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ be a set of real numbers. We denote by $x_i^{(k)}$ the i-th element of $\psi^{(k)}\mathcal{X}$. Clearly, $x_i^{(k)}$ is a linear combination of $x_i, x_{i+1}, \ldots, x_{i+k}$. The next lemma gives a bound on the coefficients of that combination.

Lemma 2.2. *Let $x_i^{(k)}$ be given by the linear form $L_i^{(k)}(\mathcal{X}) = \sum_{j=i}^{i+k} c_{i,j}^{(k)} x_j$. Then for any $i = 1, \ldots, n-k$ we have*

$$|c_{i,j}^{(k)}| \leq 2^{k-1}, \quad \text{for all } j = i, i+1, \ldots, i+k.$$

Proof. We proceed by induction on k. For $k = 1$ we have $L_i^{(1)}(\mathcal{X}) = x_i - x_{i+1}$ or $L_i^{(1)}(\mathcal{X}) = x_{i+1} - x_i$ so in both cases $c_{i,j}^{(1)} = \pm 1$. Suppose $|c_{i,j}^{(k-1)}| \leq 2^{k-2}$. Then

$$\begin{aligned} L_i^{(k)}(\mathcal{X}) &= |L_i^{(k-1)}(\mathcal{X}) - L_{i+1}^{(k-1)}(\mathcal{X})| = |\sum_{j=i}^{i+k-1} c_{i,j}^{(k-1)} x_j - \sum_{j=i+1}^{i+k} c_{i+1,j}^{(k-1)} x_j| \\ &= |c_{i,i}^{(k-1)} x_i + \sum_{j=i+1}^{i+k-1} (c_{i,j}^{(k-1)} - c_{i+1,j}^{(k-1)}) x_j - c_{i+1,i+k}^{(k-1)} x_{i+k}|. \end{aligned}$$

Thus, $c_{i,i}^{(k)} = \pm c_{i,i}^{(k-1)}$ and $c_{i,i+k}^{(k)} = \pm c_{i+1,i+k}^{(k-1)}$, so $|c_{i,i}^{(k)}|, |c_{i,i+k}^{(k)}| \leq 2^{k-2}$. For intermediate indices $j = i+1, \ldots, i+k-1$,

$$|c_{i,j}^{(k)}| = |c_{i,j}^{(k-1)} - c_{i+1,j}^{(k-1)}| \leq 2^{k-2} + 2^{k-2} = 2^{k-1}.$$

□

Lemma 2.3. *Let $\mathcal{X}_j = \{0, \ldots, 0, x_j, y_j, x_j - y_j, 0, \ldots, 0\}, j = 1, 2$ be two sets starting and ending with n zeros and satisfying $x_j \geq 2^n y_j$, for $j = 1, 2$. Then all the terms of $\psi^{(r)}\mathcal{X}_j, j = 1, 2$, are either of the form $x_j - \alpha y_j$ or βy_j, with $0 \leq \alpha, \beta \leq 2^r$, for $0 \leq r \leq n$. Moreover, the sequences $\psi^{(r)}\mathcal{X}_1$ and $\psi^{(r)}\mathcal{X}_2$ follow the same pattern for $0 \leq r \leq n$, meaning that if the i-th term of $\psi^{(r)}\mathcal{X}_1$ is $x_1 - \alpha y_1$, then the i-th term of $\psi^{(r)}\mathcal{X}_2$ is $x_2 - \alpha y_2$. Likewise, if the i-th term of $\psi^{(r)}\mathcal{X}_1$ is βy_1, then the i-th term of $\psi^{(r)}\mathcal{X}_2$ is βy_2. In particular, this means that $\psi^{(n)}\mathcal{X}_1$ and $\psi^{(n)}\mathcal{X}_2$ are similarly ordered.*

Proof. We proceed by induction on r. The statement is obviously true for $r = 0$. Suppose both parts of the claim are true for $r - 1$. Then any term in the r-th Ducci iteration of $\mathcal{X}_j$ has one of the following forms with $0 \leq \alpha_1, \alpha_2, \beta_1, \beta_2 \leq 2^{r-1}$:

(1) $|\beta_1 y_j - \beta_2 y_j| = |\beta_1 - \beta_2| y_j$, and $0 \leq |\beta_1 - \beta_2| \leq 2^r$,

(2) $|x_j - \alpha_1 y_j - \beta_1 y_j| = x_j - (\alpha_1 + \beta_1) y_j$, and $0 \leq \alpha_1 + \beta_1 \leq 2^r$,

(3) $|(x_j - \alpha_1 y_j) - (x_j - \alpha_2 y_j)| = |\alpha_1 - \alpha_2| y_j$, and $0 \leq |\alpha_1 - \alpha_2| \leq 2^r$.

This proves the first statement of the lemma and also shows that it is uniquely determined if a certain term of $\psi^{(r)}\mathcal{X}_j$ is of a form $x_j - \alpha y_j$ or βy_j. This in turn implies that $\psi^{(r)}\mathcal{X}_1$ and $\psi^{(r)}\mathcal{X}_2$ are similarly ordered. □

Lemma 2.4. *Let k be a fixed positive integer and $\frac{a}{q}$ be a fixed proper fraction. Then $Q > (2k+1)q$ implies that $\mathcal{F}_Q$ has the following pattern around $\frac{a}{q}$:*

$$\frac{u - ka}{v - kq} < \cdots < \frac{u - 2a}{v - 2q} < \frac{u - a}{v - q} < \frac{u}{v} < \frac{a}{q} < \frac{z}{w} < \frac{z - a}{w - q} < \cdots < \frac{z - ka}{w - kq}, \tag{2.1}$$

where $w, v \in (Q-q, Q]$, $w \equiv -\bar{a} \,(\mathrm{mod}\, q)$ *and* $v \equiv \bar{a} \,(\mathrm{mod}\, q)$.

Proof. We prove the lemma only for the right side of $\frac{a}{q}$; the argument for the left side is identical. First, if $\frac{z}{w}$ is the successor of $\frac{a}{q}$, then by the basic properties of consecutive Farey fractions, $qz - aw = 1$ and $q + w > Q$, which implies $Q - q < w \le Q$ and $w \equiv -\bar{a} \,(\mathrm{mod}\, q)$. Next, denote the i-th term to the right of $\frac{z}{w}$ by $\frac{z_i}{w_i}$, i.e., the sequence is $\frac{a}{q} < \frac{z}{w} < \frac{z_1}{w_1} < \frac{z_2}{z_2} < \dots$. We prove the lemma by induction on k. For $k = 1$, the condition is $Q > 3q$. Now, we know that $w | (w_1 + q)$ and $2w > 2(Q-q) = Q + Q - 2q > Q + q \ge w_1 + q$, hence $w = w_1 + q$, or equivalently $w_1 = w - q$. By the property of consecutive Farey fractions, z_1 is uniquely determined by z, w and w_1. Since $z_1 = z - a$ satisfies $z_1 w - z(w-q) = 1$, then $z_1 = z - a$. Next, suppose the statement is true for $k-1$, i.e., $z_i = z - ia$ and $w_i = w - iq$ for $i = 1, 2, \dots, k-1$, and show that $Q > (2k+1)q$ implies $z_k = z - ka$ and $w_k = w - kq$. Indeed, we know $w_{k-1} | (w_{k-2} + w_k)$. On one hand,

$$w_{k-1} = w - (k-1)q = w - (k-2)q - q < w_{k-2} + w_k.$$

On the other hand,

$$\begin{aligned} 3w_{k-1} &= 3(w-(k-1)q) = w - (k-2)q + 2w - (2k-1)q \\ &> w_{k-2} + Q + Q - (2k+1)q > w_{k-2} + Q \ge w_{k-2} + w_k. \end{aligned}$$

Therefore, $2w_{k-1} = w_{k-2} + w_k$, which implies

$$w_k = 2w_{k-1} - w_{k-2} = 2(w-(k-1)q) - (w-(k-2)q) = w - kq.$$

Again, z_k is uniquely determined by z_{k-1}, q_{k-1} and w_k. Since $z_k = z - kq$ satisfies $z_k w_{k-1} - z_{k-1} w_k = 1$, then the proof is complete. □

Lemma 2.5. *Let $\frac{a}{q}$ be a fixed proper fraction and suppose $Q \ge 2^{k+1} q^2$. Let $\mathcal{N}$ be the set of $2k+2$ neighbors of $\frac{a}{q}$ in $\mathcal{F}_Q$ and let $\mathcal{A}$ and $\mathcal{Q}$ be, respectively, the set of numerators and denominators of the fractions in $\mathcal{N}$. Then $\psi^{(k)}\mathcal{A}$ and $\psi^{(k)}\mathcal{Q}$ are similarly ordered.*

Proof. Notice that since $Q \ge 2^{k+1} q^2 \ge (2k+1)q$, then by Lemma 2.4 $\mathcal{N}$ is of the form (2.1). One can explicitly compute $\psi\mathcal{A}$ and $\psi\mathcal{Q}$ and see that they are indeed similarly ordered. Hence we will hereafter assume that $k \ge 2$. We also assume for concreteness that $u \ge z$; therefore also $v \ge w$. The other case is handled in a similar way. Now the second Ducci iterations of the sets $\mathcal{A}$ and $\mathcal{Q}$ have the following forms:

$$\mathcal{A}^{(2)} = \psi^{(2)}\mathcal{A} = \{\underbrace{0, \dots, 0}_{k-1}, u - 2a, u - z, z - 2a, \underbrace{0, \dots, 0}_{k-1}\},$$

$$\mathcal{Q}^{(2)} = \psi^{(2)}\mathcal{Q} = \{\underbrace{0, \dots, 0}_{k-1}, v - 2q, v - w, w - 2q, \underbrace{0, \dots, 0}_{k-1}\},$$

and it suffices to show that $\psi^{(k-2)}\mathcal{A}^{(2)}$ and $\psi^{(k-2)}\mathcal{Q}^{(2)}$ are similarly ordered. We will prove that $\mathcal{A}^{(2)}$ and $\mathcal{Q}^{(2)}$ satisfy the conditions of Lemma 2.3 and that will finish the proof.

First, we have $v > Q - q$, so $v - 2q > Q - 3q$. On the other hand, $w > Q - q$, so $v - w < q$. Thus

$$v - 2q \geq 2^{k+1}q^2 - 3q \geq (2^{k+1} - 3)q \geq 2^k q \geq 2^k(v - w),$$

which is the condition of the lemma for $\mathcal{Q}^{(2)}$.
Next, $\frac{u}{v} < \frac{a}{q}$ implies $u < \frac{a}{q}v \leq \frac{a}{q}Q$. On the other hand, $u = \frac{av-1}{q} > \frac{a}{q}(Q - q) - \frac{1}{q}$. So the following bounds for u holds:

$$\frac{a}{q}Q - a - \frac{1}{q} < u < \frac{a}{q}Q. \tag{2.2}$$

Similarly, $\frac{a}{q} < \frac{z}{w}$ implies $z > \frac{a}{q}w > \frac{a}{q}(Q-q) = \frac{a}{q}Q - a$. On the other hand, $z = \frac{aw+1}{q} \leq \frac{a}{q}Q + \frac{1}{q}$. Thus

$$\frac{a}{q}Q - a < z \leq \frac{a}{q}Q + \frac{1}{q}. \tag{2.3}$$

The bounds (2.2) and (2.3) imply $u - z \leq a + 1 \leq q$. Finally,

$$u - 2a > \frac{a}{q}Q - a - \frac{1}{q} - 2a \geq \frac{Q}{q} - 4q \geq (2^{k+1}q - 4)q \geq 2^k q \geq 2^k(u - z),$$

which is the condition of Lemma 2.3 for $\mathcal{Q}^{(2)}$. $\square$

Now we are ready to prove the main result.

Proof of Theorem 1.1. Define the function $K(k) = 2^{k+1}(k+1)^2$ for $k \geq 0$. We show by induction on k that if $Q \geq 2^{k+1}K^2(k)$, then $\psi^{(k)}\mathcal{A}$ and $\psi^{(k)}\mathcal{Q}$ are similarly ordered. The statement is trivially true for $k = 0$, and the case when $k = 1$ can be proven by elementary arguments. In fact, in these two cases the statement is true for any value of Q. Indeed, if we assume that $\psi\mathcal{A}$ and $\psi\mathcal{Q}$ are not ordered similarly, then there are three consecutive terms $\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$ in $\mathcal{F}_Q$ such that $(|a''-a|-|a-a'|)(|q''-q|-|q-q'|) < 0$. This means that $|a'' - a| - |a - a'|$ and $|q'' - q| - |q - q'|$ have opposite signs. We will show a contradiction in the case when $|a'' - a| - |a - a'| > 0$ and $|q'' - q| - |q - q'| < 0$; the other case can be done in a similar way. By the basic property of neighboring Farey fractions, $\frac{a'}{q'} = \frac{a}{q} - \frac{1}{qq'}$ and $\frac{a''}{q''} = \frac{a}{q} + \frac{1}{qq''}$, so $\left|\frac{q''a}{q} + \frac{1}{q} - a\right| > \left|a + \frac{1}{q} - \frac{q'a}{q}\right|$, hence

$$\left|q'' - q + \frac{1}{a}\right| > \left|q - q' + \frac{1}{a}\right| \quad \text{and} \quad |q'' - q| < |q - q'|.$$

The only non-trivial case when these two inequalities can hold together is when $q - q' > 0$, $q'' - q < 0$ and $|q'' - q| - |q - q'| < \frac{2}{a}$, that is when $q' - q'' < \frac{2}{a}$. Now, $q' - q'' \neq 0$, since otherwise we would have $|q'' - q| = |q - q'|$; therefore $q' - q'' = 1$, which is possible only if $a = 1$. This further implies that $q' - a'q = 1$ and $a''q - a' + 1 = 1$, i.e., $(a'' - a')q = 1$, which can't be true since $q \neq 1$.
Next, we assume that $\psi^{(k-1)}\mathcal{A}$ and $\psi^{(k-1)}\mathcal{Q}$ are similarly ordered and show that our choice of Q implies that $\psi^{(k)}\mathcal{A}$ and $\psi^{(k)}\mathcal{Q}$ are as well. Suppose the contrary holds:

assume the similarity of $\psi^{(k)}\mathcal{A}$ and $\psi^{(k)}\mathcal{Q}$ is violated at the i-th position, i.e., $a_{i+1}^{(k)}-a_i^{(k)}$ and $q_{i+1}^{(k)}-q_i^{(k)}$ have opposite signs. This means that

$$\left|(a_{i+1}^{(k)}-a_i^{(k)})-t(q_{i+1}^{(k)}-q_i^{(k)})\right|>1, \tag{2.4}$$

for any $t>0$. Let $q_j=\min\{q_i,q_{i+1},\dots,q_{i+k+1}\}$ and take $t=\frac{a_j}{q_j}$. If we had $q_j\le K(k)$, then by Lemma 2.5 $\psi^{(k)}\mathcal{A}$ and $\psi^{(k)}\mathcal{Q}$ would be similarly ordered. Thus, we should have $q_j>K(k)$. Now, for all $r=i,i+1,\dots,i+k+1$,

$$\left|\frac{a_r}{q_r}-t\right|=\left|\frac{a_r}{q_r}-\frac{a_j}{q_j}\right|\le\sum_{l=i}^{i+k}\left|\frac{a_l}{q_l}-\frac{a_{l+1}}{q_{l+1}}\right|\le\sum_{l=i}^{i+k}\frac{1}{q_lq_{l+1}}.$$

For each $l=i,i+1,\dots,i+k$ we have $\min\{q_l,q_{l+1}\}\ge\frac{Q}{2}$ and $\max\{q_l,q_{l+1}\}\ge q_j$. Hence

$$\left|\frac{a_r}{q_r}-t\right|\le\frac{2(k+1)}{q_jQ},$$

which implies $|a_r-tq_r|\le\frac{2(k+1)}{q_j}$ for all $r=i,i+1,\dots,i+k+1$. Recall that $a_i^k=\left|a_{i+1}^{(k-1)}-a_i^{(k-1)}\right|$ and $q_i^k=\left|q_{i+1}^{(k-1)}-q_i^{(k-1)}\right|$. Suppose the values of $a_i^{(k-1)}$ and $a_{i+1}^{(k-1)}$ are computed in terms of elements of $\mathcal{A}$ by the linear forms $L_i^{(k-1)}$ and $L_{i+1}^{(k-1)}$, respectively:

$$a_i^{(k-1)}=L_i^{(k-1)}(\mathcal{A})=\sum_{r=i}^{i+k-1}c_{i,r}a_r,\quad a_{i+1}^{(k-1)}=L_{i+1}^{(k-1)}(\mathcal{A})=\sum_{r=i+1}^{i+k}d_{i,r}a_r.$$

The induction hypothesis implies that the linear forms for $q_i^{(k-1)}$ and $q_{i+1}^{(k-1)}$ should have the same coefficients as those for $a_i^{(k-1)}$ and $a_{i+1}^{(k-1)}$, that is

$$q_i^{(k-1)}=L_i^{(k-1)}(\mathcal{Q})=\sum_{r=i}^{i+k-1}c_{i,r}q_r,\quad q_{i+1}^{(k-1)}=L_{i+1}^{(k-1)}(\mathcal{Q})=\sum_{r=i+1}^{i+k}d_{i,r}q_r.$$

Moreover, $\psi^{(k-1)}\mathcal{A}$ and $\psi^{(k-1)}\mathcal{Q}$ being similarly ordered implies that $a_{i+1}^{(k-1)}-a_i^{(k-1)}$ and $q_{i+1}^{(k-1)}-q_i^{(k-1)}$ are of the same sign. Then,

$$\begin{aligned}\left|a_i^{(k)}-tq_i^{(k)}\right|&=\left||a_{i+1}^{(k-1)}-a_i^{(k-1)}|-t|q_{i+1}^{(k-1)}-q_i^{(k-1)}|\right|\\&=\left|(a_{i+1}^{(k-1)}-a_i^{(k-1)})-t(q_{i+1}^{(k-1)}-q_i^{(k-1)})\right|\\&=\left|(a_{i+1}^{(k-1)}-tq_{i+1}^{(k-1)})-(a_i^{(k-1)}-tq_i^{(k-1)})\right|\\&=\left|\sum_{r=i+1}^{i+k}d_{i,r}(a_r-tq_r)-\sum_{r=i}^{i+k-1}c_{i,r}(a_r-tq_r)\right|.\end{aligned}$$

Now, by Lemma 2.2, $|d_{i,r}|,|c_{i,r}|\le 2^{k-2}$ for all $r=i,i+1,\dots,i+k$, and hence

$$\left|a_i^{(k)}-tq_i^{(k)}\right|\le 2(k+1)2^{k-2}\frac{2(k+1)}{q_j}=\frac{2^k(k+1)^2}{q_j}.$$

By the same argument this bound holds for $\left|a_{i+1}^{(k)} - tq_{i+1}^{(k)}\right|$ as well. Hence,

$$\begin{aligned}\left|(a_{i+1}^{(k)} - a_i^{(k)}) - t(q_{i+1}^{(k)} - q_i^{(k)})\right| &\leq \left|a_{i+1}^{(k)} - tq_{i+1}^{(k)}\right| + \left|a_i^{(k)} - tq_i^{(k)}\right| \\ &\leq \frac{2^{k+1}(k+1)^2}{q_j} < 1,\end{aligned}$$

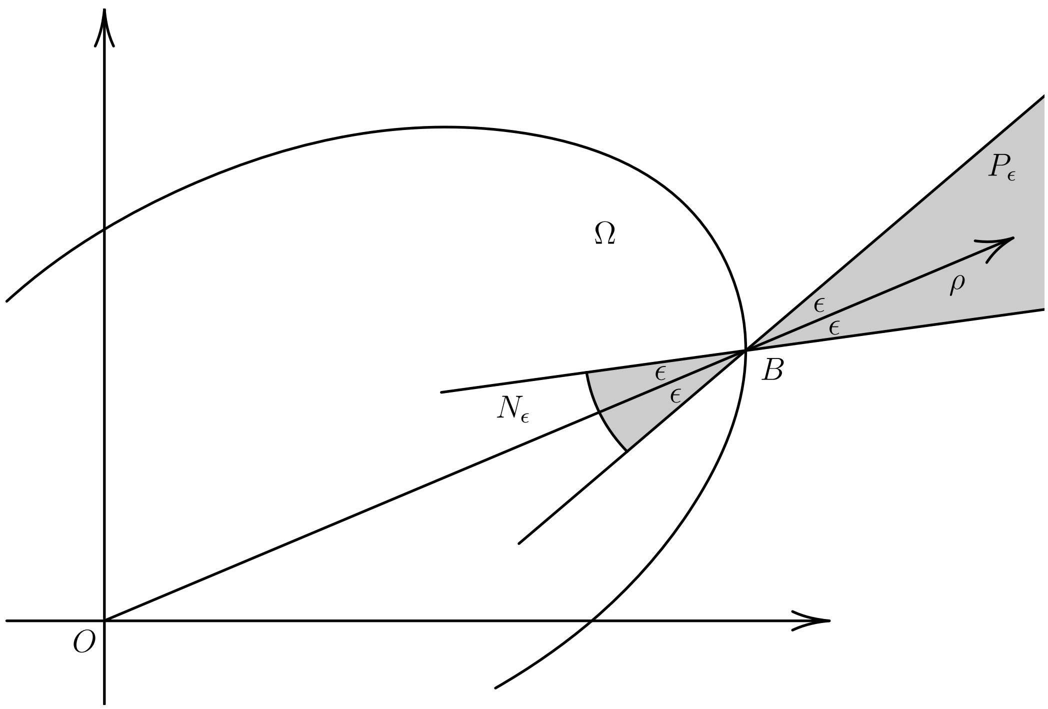
which contradicts 2.4. This completes the proof. □

3. Proof of Theorem 1.2

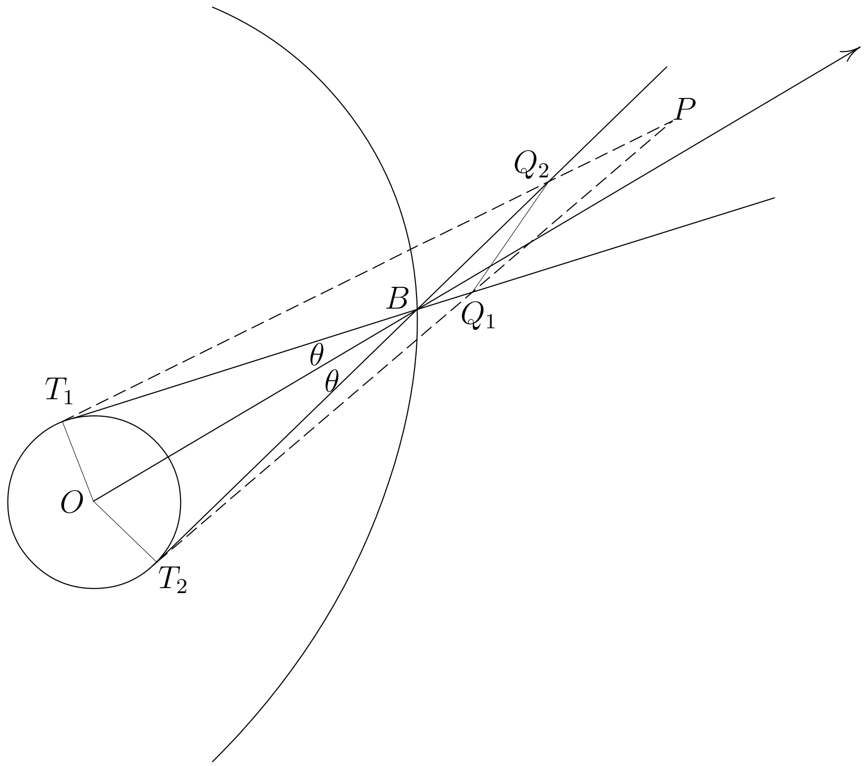
This section is devoted to the proof of Theorem 1.2, which proceeds along the same line as that of Theorem 1.1. One needs to prove the analogs of Lemmas 2.1 - 2.5 and then proceed as in the proof of Theorem 1.1. Notice that all these lemmas except Lemma 2.4 can be generalized easily in this setting. We are only left to prove the analog of Lemma 2.4, which will be shown in this section by geometrical arguments. We need several lemmas leading up to it. In the following lemmas $D_r(P)$ is used to denote the disk of radius r centered at the point P.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, convex region with a piecewise smooth boundary and containing the origin O. Then there exist numbers ϵ and h satisfying the following property: if ρ is a ray starting at the origin and intersecting the boundary of Ω at the point B, and P_ϵ and N_ϵ are the interiors of the angles at B of size 2ϵ with the bisectors coinciding with positive and negative directions of ρ, respectively, then there exist h and ϵ such that*

$$N_\epsilon \cap D_h(B) \subset \Omega \text{ and } P_\epsilon \cap \Omega = \emptyset.$$



Proof. Let the numbers R and r be such that $\Omega \subset D_R(O)$ and $D_{2r}(O) \subset \Omega$. We will show that $\epsilon := \arcsin \frac{r}{R}$ and $h := r$ satisfy the lemma. Indeed, let BT_1 and BT_2 be the tangent lines to the circle of radius r centered at O and denote $\theta = \angle T_1BO$. Since Ω is convex, then the open triangle $\triangle BT_1T_2 \subset \Omega$. Moreover, since $|BO| \geq 2r$, then $N_\theta \cap D_r(B) \subset \triangle BT_1T_2$. Finally, since $\sin\theta = \frac{r}{|BO|} \geq \frac{r}{R} = \sin\epsilon$, then $N_\epsilon \subset N_\theta$. Therefore, $N_\epsilon \cap D_r(B) \subset \Omega$. To show the second part we again use the fact that $\sin\theta \geq \sin\epsilon$, which implies that $P_\epsilon \subset P_\theta$ so it is enough to show that $P_\theta \cap \Omega = \emptyset$. Suppose contrary holds: assume there exists a point $P \in P_\epsilon \cap \Omega^c$. Then by convexity of Ω the line segments PT_1 and PT_2 lie in Ω. In particular, the points $Q_1 = PT_1 \cap T_2B$ and $Q_2 = PT_2 \cap T_1B$ are in Ω. Therefore the line segment $Q_1Q_2 \subset \Omega$ so the point $Q = Q_1Q_2 \cap \rho \in \Omega$ as well. However Q is on the continuation of the line segment OB which is a contradiction with the fact that B is a boundary point.



□

Remark. *It can be seen from the proof of lemma that if ϵ and h are the constant for region Ω, then one can take ϵ and xh as the constant for $x\Omega$.*

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a convex region and let $A_1 = (a_1, q_1), A_2 = (a_2, q_2) \in \Omega$ be consecutive points of $\mathcal{F}_\Omega$. Then*

$$\det \begin{pmatrix} a_2 & a_1 \\ q_2 & q_1 \end{pmatrix} = 1.$$

Proof. We apply Pick's theorem for the triangle OA_1A_2. There are no lattice points on the sides OA_1 and OA_2 since the points A_1 and A_2 are visible, and there are no lattice points on the side A_1A_2 and in the interior of the triangle since A_1 and A_2 are consecutive in $\mathcal{F}_\Omega$. Therefore $Area(\triangle OA_1A_2) = \frac{3}{2} - 1 = \frac{1}{2}$, and the lemma follows from

$$\det \begin{pmatrix} a_2 & a_1 \\ q_2 & q_1 \end{pmatrix} = 2Area(\triangle OA_1A_2).$$

□

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a convex region and let $(a_1, q_1), (q_2, a_2), (q_3, a_3) \in \Omega$ be consecutive points of $\mathcal{F}_\Omega$. Then $(q_3, a_3) = -(q_1, a_1) + t(q_2, a_2)$ for some integer t.*

Proof. By Lemma 3.2 we have $\det\begin{pmatrix} a_2 & a_1 \\ q_2 & q_1 \end{pmatrix} = 1$ and $\det\begin{pmatrix} a_3 & a_2 \\ q_3 & q_2 \end{pmatrix} = 1$. These two imply that $\det\begin{pmatrix} a_3+a_1 & a_2 \\ q_3+q_1 & q_2 \end{pmatrix} = 0$, which means that the first column is a multiple of the second, i.e., $(q_3+q_1, a_3+a_1) = t(q_2, a_2)$. Now we are left to show that t is an integer. If not, then $(t-\lfloor t\rfloor)(q_2,a_2)$ is lattice point on the line segment connecting the origin to (q_2,a_2), contradicting the visibility of (q_2,a_2). □

Now we are ready to prove the analog of Lemma 2.4 and thus complete the proof of Theorem 1.2.

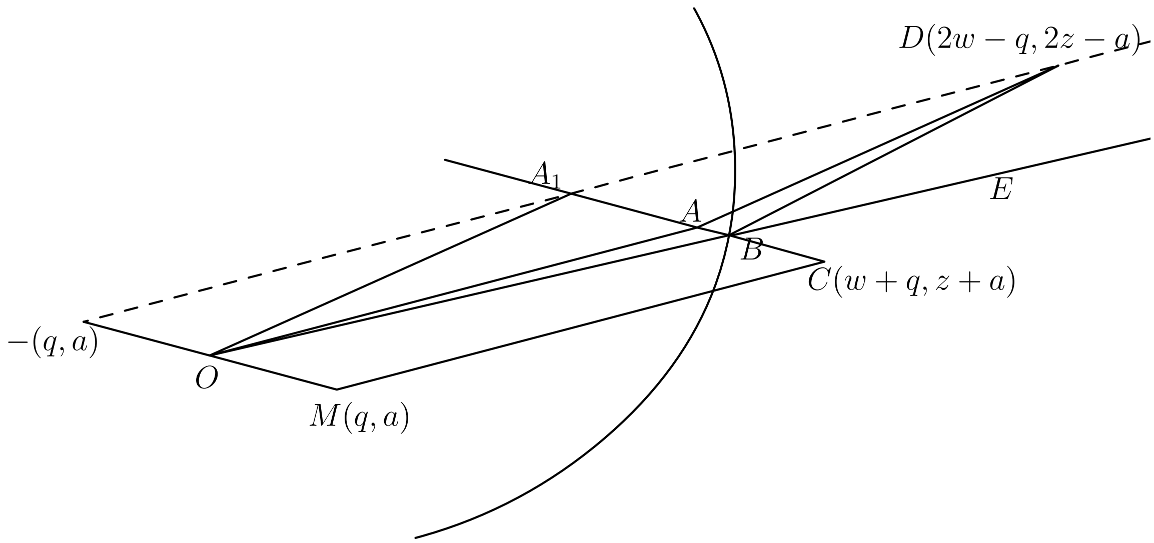
Lemma 3.4 (Analog of Lemma 2.4)**.** *Let k be a fixed positive integer and let (q,a) be a visible point in the plane. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, convex region with a piecewise smooth boundary and containing the origin O. Then there exists a number $x_0(q,\Omega)$ so that for any $x > x_0$ the $k+1$ predecessors and $k+1$ successors of (q,a) in $\mathcal{F}_{x\Omega}$ are of the following form:*

$$(v-kq, v-ka), \dots, (v-q, u-a), (v,u), (q,a), (w,z), (w-q, z-a), \dots, (w-kq, z-ka).$$

Proof. We will again prove the lemma only for the successors of (q,a), since the proof for the other part is almost identical. Let (w,z) be the successor of (q,a) in $\mathcal{F}_{x\Omega}$. Then the point $(q,a)+(w,z)$ is outside $x\Omega$, since its argument is between the arguments of (q,a) and (z,w). This means that $\|(w,z)\| \to \infty$ as $x \to \infty$. Therefore we can choose x_0 so that $x > x_0$ implies

$$\begin{aligned} &\frac{2}{\|(q,a)\|(\|(w,z)\| - (2k+2)\|(q,a)\|)} < \sin\epsilon, \\ &(k+1)\|(q,a)\| < h, \\ &\|(w,z)\| \geq (k+2)\|(q,a)\|, \end{aligned} \tag{3.1}$$

where ϵ and h are the constant of Lemma 3.1 for the region $x\Omega$. Note that we use the remark of Lemma 3.1 for the second inequality of (3.1). Denote the i-th successor of (w,z) by (w_i,z_i), i.e., we have the sequence ordered as $(q,a),(w,z),(w_1,z_1),\dots,(w_k,z_k)$. We prove by induction on k that the lemma holds true with a value of x_0 for which the conditions (3.1) are satisfied. Since the reasoning here is mostly geometrical it is convenient to make some notations. Denote the points $M := (q,a), A := (w,z)$ and $A_i := (w,z) - i(q,a)$ for $i = 1,\dots,k$. Also, denote $D := -(q,a)+2(w,z), C := (q,a)+(w,z)$ and let B be the point of intersection of the line segment AC with the boundary of Ω (see the picture below).



For $k = 1$ by Lemma 3.3 we know (w_1, z_1) is of the form $-(q, a) + t(w, z)$ for some integer t (dashed line on the picture). We will show that the point corresponding to $t = 1$ is inside and the one corresponding to $t = 2$ is outside Ω and that will prove that $(w_1, z_1) = A_1$. Since $\|A_1B\| \le \|A_1C\| = 2\|(q, a)\| < h$, and $\sin(\angle OBA_1) < \sin(\angle OAA_1) = \frac{2Area(\triangle OAA_1)}{\|AO\|\cdot\|AA_1\|} = \frac{1}{\|(w,z)\|\cdot\|(q,a)\|} \le \sin\epsilon$, then by Lemma 3.1 A_1 is inside Ω. Next, let E be a point on the continuation of the ray OB. Then

$$\angle DBE = \pi - \angle OBD$$

$$\sin(\angle OBD) = \frac{2Area(\triangle OBD)}{\|OB\|\cdot\|BD\|} \le \frac{1}{\|OB\|\cdot\|BD\|}$$

Now, two sides of $\triangle ADC$ are $\vec{AC} = (q, a)$ and $\vec{AD} = (w-q, z-a)$ so $Area(\triangle DCA) = \frac{1}{2}$. Moreover,

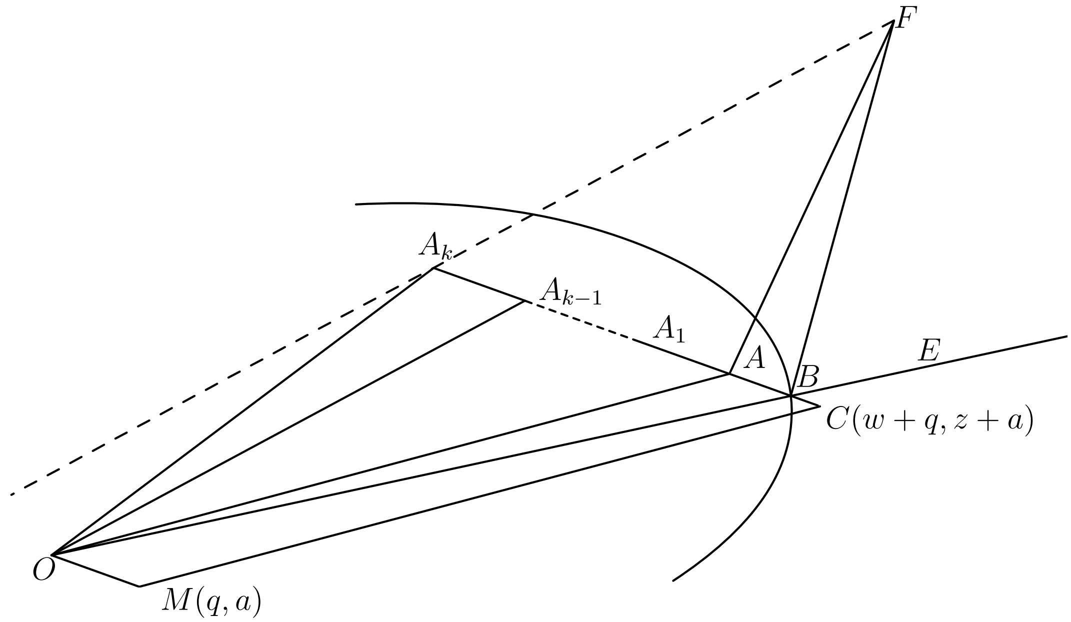
$$\|OB\| > \|OA\| - \|AB\| \ge \|(w, z)\| - \|AB\| \ge \|(w, z)\| - \|(q, a)\| \ge \|(q, a)\|,$$

$$\|DB\| > \|DA\| - \|AB\| \ge \|(w-q, z-a)\| - \|AB\| \ge \|(w, z)\| - 2\|(q, a)\|,$$

thus

$$\sin(\angle DBE) = sin(\angle OBD) \le \frac{1}{\|(q, a)\|(\|(w, z)\| - 2\|(q, a)\|)} < \sin\epsilon,$$

therefore, by Lemma 3.1 the point D is outside Ω. Next, suppose the statement is true for $k-1$, i.e., $(w_i, z_i) = (w - iq, z - ai)$ for $i = 1, \dots, k-1$ and show that the conditions (3.1) imply $(w_k, z_k) = (w - kq, z - ak)$. By Lemma 3.3 we have $(w_k, z_k) = -(w_{k-2}, z_{k-2}) + t(w_{k-1}, z_{k-1})$ for some integer t. We will show that the point corresponding to $t = 2$ (which is A_k) is inside and the one corresponding to $t = 3$ is outside Ω and that will prove the statement. Since $\|A_kB\| < \|A_kC\| = (k+1)\|(q, a)\| < h$ and $\sin(\angle OBA_k) < \sin(\angle OAA_k) = \frac{k}{\|(w,z)\|\cdot\|k(q,a)\|} < \sin\epsilon$, then by Lemma 3.1 A_k is inside Ω.



Next, let F be the point corresponding to $t = 3$, i.e., $(2w - (2k+1)q, 2z - (2k+1)z)$. Then

$$\sin(\angle FBE) = \sin(\angle FBO) < \frac{2Area(\triangle FBO)}{\|OB\|\|BF\|}$$

$$\|OB\| \geq \|(w,z)\| - \|(q,a)\| \geq (k+1)\|(q,a)\|.$$

$$\|BF\| \geq \|AF\| - \|AB\| \geq \|(w-(2k+1)q, z-(2k+1)a)\| - \|(q,a)\| \geq \|(w,z)\| - (2k+2)\|(q,a)\|$$

Moreover $Area(\triangle FBO) \leq 2(k+1)\frac{1}{2} = (k+1)$. Thus

$$\sin(\angle FBE) \leq \frac{2(k+1)}{(k+1)\|(q,a)\|(\|(w,z)\| - (2k+2)\|(q,a)\|)} < \sin\epsilon,$$

therefore by Lemma 3.1 the point F is outside Ω.

$\square$

4. Statistics of generalized index of Farey sequence

In this section, we focus on the distribution of the generalized index $\nu_j^{(k)}(\gamma_i)$ defined by (1.2). First we give a relation between $\nu_1^{(k)}(\gamma_i)$ and $\nu_2(\gamma_i)$. To do this, we will need the patterns of pairs of $(\nu_2(\gamma_i), \nu_2(\gamma_{i+1}))$. To simplify notation, we use ν_i for the index of γ_i from [10], which is the same as $\nu_2(\gamma_i)$ in [13].

Lemma 4.1. *If $\nu_i = r$ and $\nu_{i+1} = t$ are indices of two consecutive Farey fractions, then the possible combinations of (r,t) are as follows:*

$$\begin{aligned}
&r = 1, t \geq 2,\\
&r = 2, t = 1, 2, 3,\\
&r = 3, 4, t = 1, 2,\\
&r \geq 5, t = 1.
\end{aligned}$$

Proof. If $\nu_i = 1$, then $q_{i+1} = q_i - q_{i-1}$, and $q_{i+2} = \nu_{i+1}q_{i+1} - q_i = \nu_{i+1}q_{i+1} - q_{i+1} - q_{i-1} = (\nu_{i+1} - 1)q_{i+1} - q_{i-1}$, which implies that $\nu_{i+1} \geq 2$. If $\nu_i = r \geq 2$ and $\nu_{i+1} = t$, then

$$\begin{cases} r \leq \frac{q_{i-1}+Q}{q_i} < r+1 \\ t \leq \frac{q_i+Q}{rq_i - q_{i-1}} < t+1 \end{cases}, \tag{4.1}$$

which implies that

$$\frac{q_i}{q_{i-1}} \in \left(\frac{t+2}{(t+1)r-1}, \frac{t+1}{tr-2} \right] \cap \left(\frac{2}{r+1}, \frac{2}{r-1} \right].$$

To guarantee the intersection is non empty, it is equivalent to

$$\begin{cases} \frac{t+2}{(t+1)r-1} < \frac{2}{r-1} \\ \frac{t+1}{tr-2} > \frac{2}{r+1} \end{cases} \Leftrightarrow \begin{cases} t > 0 \\ t < 1 + \frac{6}{r-1} \end{cases}$$

If $r \geq 7$, then we have $t = 1$. In the cases when $r = 6, t = 2$ or $r = 5, t = 2$, we see that the system 4.1 has no solutions. This proves the lemma. $\square$

Lemma 4.2. *We have $\nu_1^{(1)}(\gamma_{i+1}) = \nu_2(\gamma_{i+1}) - 2$, where $\nu_1^{(k)}(\gamma_{i+1})$ is given by (1.2).*

Proof. Note that from the definition

$$\nu_1^{(k)}(\gamma_i) = a_{i+j-1}^{(k)}q_{i-1}^{(k)} - a_{i-1}^{(k)}q_{i+j-1}^{(k)} = -\det \begin{pmatrix} a_{i-1}^{(k)} & a_{i+j-1}^{(k)} \\ q_{i-1}^{(k)} & q_{i+j-1}^{(k)} \end{pmatrix},$$

which means

$$\nu_1^{(1)}(\gamma_{i+1}) = -\det \begin{pmatrix} |a_i - a_{i+1}| & |a_{i+1} - a_{i+2}| \\ |q_i - q_{i+1}| & |q_{i+1} - q_{i+2}| \end{pmatrix}.$$

Each pair of consecutive denominators (q_i, q_{i+1}) corresponds to an integer lattice point (x, y) with $x + y > Q$. Moreover, the quantity $\lfloor \frac{x+Q}{y} \rfloor$ gives the index of the fraction $\frac{a_{i+1}}{q_{i+1}}$. So the region

$$\Omega_k(Q) := \{(x, y) : (x + y) > Q, k \leq \frac{x+Q}{y} < k+1\} \tag{4.2}$$

corresponds to fractions with index k. Also note that in this region

$$\frac{2}{k+1} < \frac{y}{x} \leq \frac{2}{k-1}.$$

Next, the condition $(q_i - q_{i+1})(q_{i+1} - q_{i+2}) > 0$ holds if the point (x, y) is in the region

$$\mathcal{D}_k(Q) := \{(x, y) : (x - y)(x - (k-1)y) > 0\}, \tag{4.3}$$

which doesn't intersect $\Omega_k(Q)$ as long as $k \geq 3$. Therefore, for $\nu_2(\gamma_{i+1}) \geq 3$, we have $(q_i - q_{i+1})(q_{i+1} - q_{i+2}) < 0$, which gives

$$\nu_1^{(1)}(\gamma_{i+1}) = -\det \begin{pmatrix} a_{i+1} - a_i & a_{i+1} - a_{i+2} \\ q_{i+1} - q_i & q_{i+1} - q_{i+2} \end{pmatrix} = \nu_2(\gamma_{i+1}) - 2.$$

For $\nu_2(\gamma_{i+1}) = 1$, we have $q_{i+1} > q_i$ and

$$\det \begin{pmatrix} a_i^{(1)} & a_{i+1}^{(1)} \\ q_i^{(1)} & q_{i+1}^{(1)} \end{pmatrix} = \det \begin{pmatrix} a_{i+1} - a_i & a_i \\ q_{i+1} - q_i & q_i \end{pmatrix} = 1 = -\nu_2(\gamma_{i+1}) + 2.$$

For $\nu_2(\gamma_{i+1}) = 2$,

$$\det \begin{pmatrix} a_i^{(1)} & a_{i+1}^{(1)} \\ q_i^{(1)} & q_{i+1}^{(1)} \end{pmatrix} = \det \begin{pmatrix} a_i - a_{i+1} & a_{i+1} - a_i \\ q_i - q_{i+1} & q_{i+1} - q_i \end{pmatrix} = 0 = -\nu_2(\gamma_{i+1}) + 2.$$

This completes the proof. $\square$

Next we consider $\nu_1^{(2)}(\gamma_{i+1})$. We first give formulae for $a_i^{(2)}$ and $q_i^{(2)}$.

Lemma 4.3. *The following is true for $a_i^{(2)}$ and $q_i^{(2)}$:*

$$\begin{aligned} q_i^{(2)} &= \begin{cases} 0 & \text{if } \nu_{i+1} = 2 \\ |2q_i - \nu_{i+1} q_{i+1}| & \text{if } \nu_{i+1} \neq 2 \end{cases} \\ &= \begin{cases} q_{i+1} - 2q_i & \text{if } \nu_{i+1} = 1, \frac{q_{i+1}}{q_i} \geq 2 \\ -q_{i+1} + 2q_i & \text{if } \nu_{i+1} = 1, \frac{q_{i+1}}{q_i} \in [1, 2) \\ 0 & \text{if } \nu_{i+1} = 2 \\ \nu_{i+1} q_{i+1} - 2q_i & \text{if } \nu_{i+1} \geq 3, \frac{q_{i+1}}{q_i} \in [\frac{2}{\nu_{i+1}}, \frac{2}{\nu_{i+1}-1}] \\ 2q_i - \nu_{i+1} q_{i+1} & \text{if } \nu_{i+1} \geq 3, \frac{q_{i+1}}{q_i} \in [\frac{2}{\nu_{i+1}+1}, \frac{2}{\nu_{i+1}}], \end{cases} \\ a_i^{(2)} &= \begin{cases} 0 & \text{if } \nu_{i+1} = 2 \\ |2a_i - \nu_{i+1} a_{i+1}| & \text{if } \nu_{i+1} \neq 2 \end{cases} \\ &= \begin{cases} a_{i+1} - 2a_i & \text{if } \nu_{i+1} = 1, \frac{q_{i+1}}{q_i} \geq 2 \\ -a_{i+1} + 2a_i & \text{if } \nu_{i+1} = 1, \frac{q_{i+1}}{q_i} \in [1, 2) \\ 0 & \text{if } \nu_{i+1} = 2 \\ \nu_{i+1} a_{i+1} - 2a_i & \text{if } \nu_{i+1} \geq 3, \frac{q_{i+1}}{q_i} \in [\frac{2}{\nu_{i+1}}, \frac{2}{\nu_{i+1}-1}] \\ 2a_i - \nu_{i+1} a_{i+1} & \text{if } \nu_{i+1} \geq 3, \frac{q_{i+1}}{q_i} \in [\frac{2}{\nu_{i+1}+1}, \frac{2}{\nu_{i+1}}]. \end{cases} \end{aligned}$$

Proof. Since the sequence starting from q_i is of the form $q_i, q_{i+1}, \nu_{i+1}q_{i+1} - q_i$, then after the first iteration, we get $|q_{i+1} - q_i|, |(\nu_{i+1} - 1)q_{i+1} - q_i|$. If $\nu_{i+1} = 1$, then $|q_{i+1} - q_i| = q_{i+1} - q_i, |(\nu_{i+1} - 1)q_{i+1} - q_i| = q_i$. If $\nu_{i+1} = 2$, then it is easy to see that $q_i^{(2)} = 0$. If $\nu_{i+1} > 2$, then we have $|q_{i+1} - q_i| = q_i - q_{i+1}, |(\nu_{i+1} - 1)q_{i+1} - q_i| = (\nu_{i+1} - 1)q_{i+1} - q_i$. The proof of the second formula follows from the fact that $a_i^{(1)}$'s and $q_i^{(1)}$'s are similarly ordered for any Q. $\square$

Lemma 4.4. *The following formula is true for* $\nu_1^{(2)}(\gamma_{i+1})$*:*

$$\nu_1^{(2)}(\gamma_{i+1}) = \begin{cases} 0 & \textit{if } (\nu_{i+1}-2)(\nu_{i+2}-2)=0 \\ -\nu_{i+2}+4 & \textit{if } \nu_{i+1}=1, \nu_{i+2}\geq 3 \textit{ and } (\frac{q_{i+1}}{q_i}-\frac{\nu_{i+2}}{\nu_{i+2}-2})(\frac{q_{i+1}}{q_i}-2)\geq 0 \\ \nu_{i+2}-4 & \textit{if } \nu_{i+1}=1, \nu_{i+2}\geq 3 \textit{ and } (\frac{q_{i+1}}{q_i}-\frac{\nu_{i+2}}{\nu_{i+2}-2})(\frac{q_{i+1}}{q_i}-2)< 0 \\ -\nu_{i+1}+4 & \textit{if } \nu_{i+1}\geq 3, \nu_{i+2}=1 \textit{ and } (\frac{q_{i+1}}{q_i}-\frac{1}{\nu_{i+1}-2})(\frac{q_{i+1}}{q_i}-\frac{2}{\nu_{i+1}})\geq 0 \\ \nu_{i+1}-4 & \textit{if } \nu_{i+1}\geq 3, \nu_{i+2}=1 \textit{ and } (\frac{q_{i+1}}{q_i}-\frac{1}{\nu_{i+1}-2})(\frac{q_{i+1}}{q_i}-\frac{2}{\nu_{i+1}})< 0. \end{cases}$$

Proof. If $\nu_{i+1}=2$, or $\nu_{i+2}=2$, then $q_i^{(2)}=0$ and $a_i^{(2)}=0$ or $q_{i+1}^{(2)}=0$ and $a_{i+1}^{(2)}=0$, which gives

$$\det\begin{pmatrix} a_i^{(2)} & a_{i+1}^{(2)} \\ q_i^{(2)} & q_{i+1}^{(2)} \end{pmatrix} = 0.$$

If $\nu_{i+1}\neq 2$ and $\nu_{i+2}\neq 2$, then from Lemma 4.3,

$$\begin{aligned} \det\begin{pmatrix} a_i^{(2)} & a_{i+1}^{(2)} \\ q_i^{(2)} & q_{i+1}^{(2)} \end{pmatrix} &= \det\begin{pmatrix} |2a_i-\nu_{i+1}a_{i+1}| & |2a_{i+1}-\nu_{i+2}a_{i+2}| \\ |2q_i-\nu_{i+1}q_{i+1}| & |2q_{i+1}-\nu_{i+2}q_{i+2}| \end{pmatrix} \\ &= \det\begin{pmatrix} |2a_i-\nu_{i+1}a_{i+1}| & |\nu_{i+2}a_i-(\nu_{i+1}\nu_{i+2}-2)a_{i+1}| \\ |2q_i-\nu_{i+1}q_{i+1}| & |\nu_{i+2}q_i-(\nu_{i+1}\nu_{i+2}-2)q_{i+1}| \end{pmatrix}. \end{aligned}$$

If $\nu_{i+1}=1, \nu_{i+2}\geq 3$, then the above simplifies to

$$\begin{aligned} \det\begin{pmatrix} a_i^{(2)} & a_{i+1}^{(2)} \\ q_i^{(2)} & q_{i+1}^{(2)} \end{pmatrix} &= \det\begin{pmatrix} |2a_i-a_{i+1}| & |\nu_{i+2}a_i-(\nu_{i+2}-2)a_{i+1}| \\ |2q_i-q_{i+1}| & |\nu_{i+2}q_i-(\nu_{i+2}-2)q_{i+1}| \end{pmatrix} \\ &= \begin{cases} \nu_{i+2}-4, & \text{if } (\frac{q_{i+1}}{q_i}-\frac{\nu_{i+2}}{\nu_{i+2}-2})(\frac{q_{i+1}}{q_i}-2)\geq 0, \\ -\nu_{i+2}+4, & \text{if } (\frac{q_{i+1}}{q_i}-\frac{\nu_{i+2}}{\nu_{i+2}-2})(\frac{q_{i+1}}{q_i}-2)< 0. \end{cases} \end{aligned}$$

If $\nu_{i+2}=1$, $\nu_{i+1}\geq 3$, then

$$\begin{aligned} \det\begin{pmatrix} a_i^{(2)} & a_{i+1}^{(2)} \\ q_i^{(2)} & q_{i+1}^{(2)} \end{pmatrix} &= \det\begin{pmatrix} |2a_i-\nu_{i+1}a_{i+1}| & |a_i-(\nu_{i+1}-2)a_{i+1}| \\ |2q_i-\nu_{i+1}q_{i+1}| & |q_i-(\nu_{i+1}-2)q_{i+1}| \end{pmatrix} \\ &= \begin{cases} \nu_{i+1}-4, & \text{if } (\frac{q_{i+1}}{q_i}-\frac{1}{\nu_{i+1}-2})(\frac{q_{i+1}}{q_i}-\frac{2}{\nu_{i+1}})\geq 0, \\ -\nu_{i+1}+4, & \text{if } (\frac{q_{i+1}}{q_i}-\frac{1}{\nu_{i+1}-2})(\frac{q_{i+1}}{q_i}-\frac{2}{\nu_{i+1}})< 0. \end{cases} \end{aligned}$$

Note that by Lemma 4.1 this formula covers all cases. ◻

In the rest of the section, we consider the distribution of the values of the quantity $\nu_j^{(k)}(\gamma_{i+1}) = -\det\begin{pmatrix} a_i^{(k)} & a_{i+j}^{(k)} \\ q_i^{(k)} & q_{i+j}^{(k)} \end{pmatrix}$. We define

$$\mathbb{P}_{k,j}(t,Q) = \frac{1}{N(Q)}\#\left\{i\leq N(Q)-k-1 : \nu_j^{(k)}(\gamma_{i+1})=t\right\}, \tag{4.4}$$

as the "frequency" of $\nu_j^{(k)}$ admitting the value t. We are going to prove that the limiting distribution of $\mathbb{P}_{k,j}(t,Q)$ exists as $Q \to \infty$, and give the explicit formulae for $j = 1, k \le 2$. We need to understand the frequencies of consecutive indices of Farey sequence.

Lemma 4.5. *Let $\Omega \subset [0, R_1] \times [0, R_2]$ be a region in $\mathbb{R}^2$ with rectifiable boundary $\partial\Omega$ and let $g : \Omega \to \mathbb{R}$ be a C^1 function on Ω. Suppose $R \ge min(R_1, R_2)$, then we have*

$$\sum_{(a,b)\in\Omega\cap\mathbb{Z}^2_{vis}} g(a,b) = \frac{6}{\pi^2}\iint_{\Omega} g(x,y)dxdy + O(||Dg||_\infty area(\Omega)\log R)$$
$$+ O\left(||g||_\infty\left(R + \frac{area(\Omega)}{R} + length(\partial\Omega)\log R\right)\right),$$

where $\mathbb{Z}^2_{vis}$ is the set of visible points in $\mathbb{Z}^2$.

Proof. This is Lemma 2.1 in [6]. □

Lemma 4.6. *If $c \in (0, 1]$, then*

$$\#\{\gamma \in \mathcal{F}_Q : \gamma \le c\} = \#\left\{(q_1, q_2) : {}^{1\le q_1,q_2\le Q, q_1+q_2>Q, \gcd(q_1,q_2)=1,}_{\exists 0\le a_1<q_1, 0\le a_2<q_2, s.t.\ a_2q_1-a_1q_2=1, a_2/q_2\le c}\right\}$$

Proof. This is Corollary 1.2 in [4]. □

Lemma 4.7. *Let $P_Q(r,t)$ be the probability of (r,t) appearing as a pair of consecutive indices in the Farey sequence $\mathcal{F}_Q$, and let $P(r,t)$ be the limit of $P_Q(r,t)$ as $Q \to \infty$. Then $P(1,2) = P(2,1) = P(2,3) = \frac{1}{15}$, $P(1,3) = P(3,1) = \frac{8}{105}$, $P(1,4) = P(4,1) = \frac{2}{35}$, $P(2,2) = \frac{1}{5}$, $P(3,2) = \frac{2}{35}$, $P(4,2) = \frac{1}{105}$ and*

$$P(r,t) = \begin{cases} \frac{8}{t(t+1)(t+2)}, & r = 1, t \ge 5, \\ \frac{8}{r(r+1)(r+2)}, & r \ge 5, t = 1. \end{cases}$$

Proof. From Lemma 4.6, we see that $\mathcal{F}_Q$ corresponds to visible points in the triangle defined by the points $(0, Q), (Q, 0), (Q, Q)$. From (4.2), we see that

$$\#\{\gamma \in \mathcal{F}_Q\} = \#\{(q_{i-1}, q_i) : 1 \le q_{i-1}, q_i \le Q, q_{i-1} + q_i > Q, \gcd(q_{i-1}, q_i) = 1\}.$$

Applying Lemma 4.5, we have $N(Q) = \frac{3}{\pi^2}Q^2 + O(Q\log Q)$. The condition $\nu_i = r$ and $\nu_{i+1} = t$ is given by (4.1). Thus

$$\#\{\gamma_i \in \mathcal{F}_Q : \nu_i = r, \nu_{i+1} = t\}$$
$$= \#\left\{(q_{i-1}, q_i) : {}^{1\le q_{i-1},q_i\le Q, q_{i-1}+q_i>Q, \gcd(q_{i-1},q_i)=1,}_{r\le \frac{q_{i-1}+Q}{q_i}<r+1, t\le \frac{q_i+Q}{rq_i-q_{i-1}}<t+1}\right\}.$$

If we define the region

$$\mathcal{R}(r,t) := \left\{(x,y) : {}^{0\le x,y\le 1, x+y>1,}_{r\le\frac{x+1}{y}<r+1, t\le\frac{y+1}{ry-x}<t+1}\right\}, \tag{4.5}$$

Then from Lemma 4.5,

$$\#\{\gamma_i \in \mathcal{F}_Q : \nu_i = r, \nu_{i+1} = t\} = \frac{6Q^2}{\pi^2}area(\mathcal{R}(r,t)) + O(Q\log Q),$$

which gives

$$P(r,t) = 2area(\mathcal{R}(r,t)).$$

Now, the problem is reduced to computing the areas of $\mathcal{R}(k,t)$.
$\mathcal{R}(1,2)$ is the triangle with vertices

$$(0,0), \left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{1}{3}, 1\right),$$

so the area is $\frac{1}{30}$. $\mathcal{R}(1,3)$ is the quadrilateral with vertices

$$\left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{2}{7}, \frac{5}{7}\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{3}, 1\right),$$

so the area is $\frac{4}{105}$. For $t \geq 5$, $\mathcal{R}(1,t)$ is the quadrilateral with vertices

$$\left(\frac{t-3}{t+1}, \frac{t+2}{t+1}\right), \left(\frac{t-2}{t+2}, \frac{t+3}{t+2}\right), \left(\frac{t-1}{t+1}, 1\right), \left(\frac{t-2}{t}, 1\right),$$

which has area $\frac{4}{t(t+1)(t+2)}$. Similarly, for $r \geq 5$, $\mathcal{R}(r,1)$ is the quadrilateral with vertices

$$\left(\frac{r-1}{r+1}, \frac{2}{r+1}\right), \left(\frac{r}{r+2}, \frac{2}{r+2}\right), \left(1, \frac{2}{r+1}\right), \left(1, \frac{2}{r}\right),$$

which has area $\frac{4}{r(r+1)(r+2)}$. Next, $\mathcal{R}(2,1)$ is the triangle with vertices

$$\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{5}, \frac{3}{5}\right), (1,1),$$

which has area $\frac{1}{30}$. $\mathcal{R}(2,2)$ is the quadrilateral with vertices

$$\left(\frac{2}{5}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{4}{5}\right), (1,1),$$

which has area $\frac{1}{10}$. $\mathcal{R}(2,3)$ is the triangle with vertices

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{2}{3}\right), \left(1, \frac{4}{5}\right),$$

which has area $\frac{1}{30}$. $\mathcal{R}(3,1)$ is the quadrilateral with vertices

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{4}{7}, \frac{3}{7}\right), \left(1, \frac{3}{5}\right), \left(1, \frac{2}{3}\right),$$

which has area $\frac{4}{105}$. $\mathcal{R}(3,2)$ corresponds to the quadrilateral with vertices

$$\left(\frac{4}{7}, \frac{3}{7}\right), \left(\frac{3}{5}, \frac{2}{5}\right), \left(1, \frac{1}{2}\right), \left(1, \frac{3}{5}\right),$$

which has area $\frac{1}{35}$. $\mathcal{R}(4,1)$ corresponds to the quadrilateral with vertices

$$\left(\frac{3}{5}, \frac{2}{5}\right), \left(\frac{2}{3}, \frac{1}{3}\right), \left(1, \frac{3}{7}\right), \left(1, \frac{1}{2}\right),$$

which has area $\frac{1}{35}$. Finally, $\mathcal{R}(4,2)$ corresponds to the triangle with vertices

$$\left(\frac{2}{3},\frac{1}{3}\right),\left(1,\frac{2}{5}\right),\left(1,\frac{3}{7}\right),$$

which has area $\frac{1}{210}$. □

Finally, we are able to provide the values of limiting probabilities for some particular values of k, j and t.

Theorem 4.8. *We have the following limits:*

$$\lim_{Q\to\infty} \mathbb{P}_{0,1}(1,Q) = 1, \tag{4.6}$$

$$\lim_{Q\to\infty} \mathbb{P}_{1,1}(-1,Q) = \frac{1}{3}, \tag{4.7}$$

$$\lim_{Q\to\infty} \mathbb{P}_{1,1}(t,Q) = \frac{8}{(t+2)(t+3)(t+4)}, t \geq 0, \tag{4.8}$$

$$\lim_{Q\to\infty} \mathbb{P}_{2,1}(t,Q) = \begin{cases} \frac{61}{105}, & \text{if } t=0,\\ \frac{1}{14}, & \text{if } t=1,\\ \frac{11}{70}, & \text{if } t=-1,\\ \frac{8}{(|t|+4)(|t|+5)(|t|+6)}, & \text{if } |t|\geq 2, \end{cases} \tag{4.9}$$

where $\mathbb{P}_{k,j}(t,Q)$ is defined by (4.4).

Proof. For $k=0$, we have $\nu_1^{(0)}(\gamma_{i+1}) = -\det\begin{pmatrix} a_i & a_{i+1}\\ q_i & q_{i+1}\end{pmatrix} = 1$, so (4.6) holds trivially.

For $k=1$, by Lemma 4.2, $\nu_1^{(1)}(\gamma_{i+1}) = \nu_2(\gamma_{i+1}) - 2$, thus (4.7) and (4.8) follow from Lemma 4.7.

For $k=2$, by Lemma 4.4, we have

$$\nu_1^{(2)}(\gamma_{i+1}) = 0$$

if and only if $(\nu_{i+1}-2)(\nu_{i+2}-2)(\nu_{i+1}-4)(\nu_{i+2}-4)=0$. Thus by Lemma 4.7,

$$\lim_{Q\to\infty} \mathbb{P}_{2,1}(0,Q) = P(1,2)+P(2,1)+P(2,2)+P(2,3)+P(3,2)+P(1,4)+P(4,1)+P(4,2)$$
$$= \frac{1}{15}+\frac{1}{15}+\frac{1}{5}+\frac{1}{15}+\frac{2}{35}+\frac{2}{35}+\frac{2}{35}+\frac{1}{105} = \frac{61}{105}.$$

If $\nu_{i+1}=1, \nu_{i+2}=3$, then

$$\nu_1^{(2)}(\gamma_{i+1}) = \begin{cases} 1, & \text{if } \frac{q_{i+1}}{q_i} \geq 3,\\ -1, & \text{if } 2 \leq \frac{q_{i+1}}{q_i} \leq 3. \end{cases}$$

The first case corresponds to the triangle with vertices

$$\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{3},1\right),$$

the area of which is $\frac{1}{120}$ and the second case corresponds to the quadrilateral with vertices

$$\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{3},1\right),\left(\frac{1}{2},1\right),\left(\frac{2}{7},\frac{5}{7}\right),$$

the area of which is $\frac{5}{168}$.

If $\nu_{i+1}=1,\nu_{i+2}=r,\ r\geq 5$, then

$$\nu_1^{(2)}(\gamma_{i+1})=\begin{cases}-r+4, & \text{if } \frac{q_{i+1}}{q_i}\leq\frac{r}{r-2},\\ r-4, & \text{if } \frac{r}{r-2}\leq\frac{q_{i+1}}{q_i}\leq 2.\end{cases}$$

The first case corresponds to the triangle with vertices

$$\left(\frac{r-2}{r+2},\frac{r}{r+2}\right),\left(\frac{r-1}{r+1},1\right),\left(\frac{r-2}{r},1\right),$$

with area $\frac{2}{r(r+1)(r+2)}$. The second case corresponds to the triangle with vertices

$$\left(\frac{r-3}{r+1},\frac{r-1}{r+1}\right),\left(\frac{r-2}{r+2},\frac{r}{r+2}\right),\left(\frac{r-2}{r},1\right),$$

with area $\frac{2}{r(r+1)(r+2)}$.

If $\nu_{i+1}=3,\nu_{i+2}=1$, then

$$\nu_1^{(2)}(\gamma_{i+1})=\begin{cases}1, & \text{if } \frac{q_{i+1}}{q_i}\leq\frac{2}{3},\\ -1, & \text{if } \frac{2}{3}\leq\frac{q_{i+1}}{q_i}\leq 1.\end{cases}$$

The first case corresponds to the triangle with vertices

$$\left(\frac{3}{4},\frac{1}{2}\right),\left(1,\frac{3}{5}\right),\left(1,\frac{2}{3}\right),$$

whose area is $\frac{1}{120}$. The second region is the quadrilateral

$$\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{4}{7},\frac{3}{7}\right),\left(\frac{3}{4},\frac{1}{2}\right),\left(1,\frac{2}{3}\right),$$

whose area is $\frac{5}{168}$.

If $\nu_{i+1}=r,\nu_{i+2}=1,\ r\geq 4$, then

$$\nu_1^{(2)}(\gamma_{i+1})=\begin{cases}r-4, & \text{if } \frac{q_{i+1}}{q_i}\leq\frac{2}{r},\\ -r+4, & \text{if } \frac{q_{i+1}}{q_i}\geq\frac{2}{r}.\end{cases}$$

The first case correspond to the triangle with vertices

$$\left(\frac{r}{r+2},\frac{2}{r+2}\right),\left(1,\frac{2}{r+1}\right),\left(1,\frac{2}{r}\right),$$

the area of which is $\frac{2}{r(r+1)(r+2)}$. The second case corresponds to the triangle with vertices

$$\left(\frac{r-1}{r+1},\frac{2}{r+1}\right),\left(\frac{r}{r+2},\frac{2}{r+2}\right),\left(1,\frac{2}{r}\right),$$

the area of which is $\frac{2}{r(r+1)(r+2)}$.

The case $\det = -1$ comes from the pairs $(1,3),(3,1),(5,1)$ and $(1,5)$ which give a total area of $\frac{1}{120}+\frac{1}{120}+\frac{1}{105}+\frac{1}{105}=\frac{1}{28}$. The case $\det = 1$ comes from $(1,3),(3,1),(5,1)$ or $(1,5)$ which give a total area of $\frac{5}{168}+\frac{5}{168}+\frac{1}{105}+\frac{1}{105}=\frac{11}{140}$. Thus (4.9) follows from the fact that the Farey sequence correspond to the triangle $(0,1),(1,0),(1,1)$ with area $\frac{1}{2}$. □

Theorem 4.9. *If $t \geq 2^{23}$, then*

$$\lim_{Q\to\infty} \mathbb{P}_{3,1}(-t,Q) = \frac{8}{(t+4)(t+5)(t+6)}, \tag{4.10}$$

$$\lim_{Q\to\infty} \mathbb{P}_{3,1}(t,Q) = \frac{16}{(t+4)(t+5)(t+6)}. \tag{4.11}$$

Proof. From Lemma 1 in [3], we know that if $\nu_{i+1} \geq 9$, then $\nu_i = \nu_{i+2} = 1$ and $\nu_{i-1} = \nu_{i+3} = 2$. This allows us to compute the probability of large values of $\nu_1^{(3)}(\gamma_i)$. If $r \geq 9$, then there are three cases for the triple $(\nu_{i+1},\nu_{i+2},\nu_{i+3})$. We first give the probability of such triples' appearance.

The probability of $(q_i,q_{i+1}) \in \mathcal{F}_Q^2$ such that $(\nu_{i+1},\nu_{i+2},\nu_{i+3}) = (2,1,r)$ corresponds to the region defined by

$$\begin{cases} 2 \leq \frac{x+1}{y} < 3 \\ 1 \leq \frac{y+1}{2y-x} < 2 \\ r \leq \frac{2y-x+1}{y-x} \leq r+1 \end{cases},$$

which is a quadrilateral with vertices $(\frac{r-5}{r+1},\frac{r-3}{r+1}),(\frac{r-4}{r+2},\frac{r-2}{r+2}),(\frac{r-3}{r+1},\frac{r-1}{r+1}),(\frac{r-4}{r},\frac{r-2}{r})$ with area $\frac{4}{r(r+1)(r+2)}$.

The probability of $(q_i,q_{i+1}) \in \mathcal{F}_Q^2$ such that $(\nu_{i+1},\nu_{i+2},\nu_{i+3}) = (1,r,1)$ corresponds to the region defined by

$$\begin{cases} 1 \leq \frac{x+1}{y} < 2 \\ r \leq \frac{y+1}{y-x} \leq r+1 \\ 1 \leq \frac{y-x+1}{(r-1)y-rx} < r+1 \end{cases},$$

which is a quadrilateral with vertices $(\frac{r-3}{r+1},\frac{r-1}{r+1}),(\frac{r-2}{r+2},\frac{r}{r+2}),(\frac{r-1}{r+1},1),(\frac{r-2}{r},1)$ which has area $\frac{4}{r(r+1)(r+2)}$.

The probability of $(q_i,q_{i+1}) \in \mathcal{F}_Q^2$ such that $(\nu_{i+1},\nu_{i+2},\nu_{i+3}) = (r,1,2)$ corresponds to the region defined by

$$\begin{cases} r \leq \frac{x+1}{y} < r+1 \\ 1 \leq \frac{y+1}{ry-x} \leq 2 \\ 2 \leq \frac{ry-x+1}{(r-1)y-x} < 3 \end{cases},$$

which is a quadrilateral with vertices $(\frac{r-1}{r+1},\frac{2}{r+1}),(\frac{r}{r+2},\frac{2}{r+2}),(1,\frac{2}{r+1}),(1,\frac{2}{r})$, with area $\frac{4}{r(r+1)(r+2)}$.

Now we consider the value of $\nu_1^{(3)}(\gamma_{i+1})$ in each of the three cases above.

If $\nu_{i+1} = r, \nu_{i+2} = 1, \nu_{i+3} = 2$, then

$$\nu_1^{(3)}(\gamma_{i+1}) = \begin{cases} r-4, & \text{if } \frac{2}{r+2} \leq \frac{q_{i+1}}{q_i} \leq \frac{2}{r}, \\ -r+4, & \text{if } \frac{2}{r} \leq \frac{q_{i+1}}{q_i} \leq \frac{2}{r-1}, \end{cases}$$

and both of them have probability $\frac{4}{r(r+1)(r+2)}$.

If $\nu_{i+1} = 2, \nu_{i+2} = 1, \nu_{i+3} = r$, then

$$\nu_1^{(3)}(\gamma_{i+1}) = \begin{cases} r-4, & \text{if } \frac{r-2}{r-4} \leq \frac{q_{i+1}}{q_i} \leq \frac{2}{r-1}, \\ -r+4, & \text{if } \frac{2}{r+1} \leq \frac{q_{i+1}}{q_i} \leq \frac{r-2}{r-4}, \end{cases}$$

and both of them have probability $\frac{4}{r(r+1)(r+2)}$.

If $\nu_{i+1} = 1, \nu_{i+2} = r, \nu_{i+3} = 1$, then

$$\nu_1^{(3)}(\gamma_{i+1}) = r-4,$$

and this has probability $\frac{8}{k(k+1)(k+2)}$.

To compute $\nu_1^{(3)}(\gamma_{i+1})$, we need $\gamma_i, \gamma_{i+1}, \nu_{i+1}, \nu_{i+2}, \nu_{i+3}$. If $\nu_{i+1}, \nu_{i+2}, \nu_{i+3} \leq 8$, then

$$|\nu_1^{(3)}(\gamma_i)| \leq 2 \cdot (4\nu_{i+1}\nu_{i+2}\nu_{i+3})^2 \leq 2^{23}.$$

Thus, if $\nu_1^{(3)}(\gamma_{i+1}) = t$, for some $|t| > 2^{23}$, there must exist $j \in \{i+1, i+2, i+3\}$ such that $\nu_j \geq 9$, which will be in one of the three cases discussed above. Thus the theorem follows by combining all three cases. □

Note that the bound for t is not optimal, and numerical computations suggest that the formula holds when $t \geq 5$.

Next we show the existence of $\lim_{Q\to\infty} \mathbb{P}_{k,j}(n, Q)$. First we note that there is an algebraic relation between $\nu_j(\gamma_i)$ and $\nu_2(\gamma_t)$, where $t = i, \ldots, i+j-2$.

Lemma 4.10. *For any $\gamma_i \in \mathcal{F}_Q$,*

$$\nu_j(\gamma_i) = \left(\frac{2j-1}{2}\right) K_{j-1}(-\nu_2(\gamma_i), \nu_2(\gamma_{i+1}), \ldots, (-1)^{j-1}\nu_2(\gamma_{i+j-2})),$$

where $\left(\frac{\cdot}{2}\right)$ is the Kronecker symbol and $K_j(x_1, \ldots, x_j)$ are known as the convergent polynomials (continuant polynomials), and they appear in the study of continued fractions (See [9]*, Section 6.7). In fact, these polynomials are defined inductively by $K_j(x_1, \ldots, x_j) = x_j K_{j-1}(x_1, \ldots, x_{j-1}) + K_{j-2}(x_1, \ldots, x_{j-2})$, $K_0(\cdot) = 1$ and $K_1(x) = x$.*

Proof. This is Theorem 1 in [13]. □

Theorem 4.11. *For any k, j, n, $\lim_{Q\to\infty} \mathbb{P}_{k,j}(n, Q)$ exists.*

Proof. For any k, we know that $q_i^{(k)} = \sum_{r=0}^{k} c_{i,r} q_{i+r}$ is a linear form. Replacing $q_{i+r} = \nu_{i+r-1} q_{i+r-1} - q_{i+r-2}$, we can see that $q_i^{(k)} = g_{k,i}(q_i, q_{i+1}, \nu_{i+1}, \ldots, \nu_{i+k}) = \lambda_{q_i,q_{i+1}}(\vec{v}_k) q_i + \eta_{q_i,q_{i+1}}(\vec{v}_k) q_{i+1}$, which is linear in q_i and q_{i+1} for a fixed k-tuple $(\nu_{i+1}, \nu_{i+2}, \ldots, \nu_{i+k})$.

Next, given a k-tuple, depending on $\frac{q_{i+1}}{q_i}$, there are at most 2^{k+2} choices for λ and η. Similarly, $q_{i+j}^{(k)} = \tilde{g}_{k+j,i}(q_i, q_{i+1}, \nu_{i+j+1}, \dots, \nu_{i+j+k}) = \tilde{\lambda}_{q_i,q_{i+1}}(\vec{v}_k)q_i + \tilde{\eta}_{q_i,q_{i+1}}(\vec{v}_k)q_{i+j+1}$. Thus, for a fixed $(k+j)$ tuple $\vec{v} = (\nu_1, \dots, \nu_{k+j})$, there are at most 2^{k+2} lines $\alpha_s(\vec{v})x + \beta_s(\vec{v})y = 0, s \leq 2^{k+2}$ that divide the region $\mathcal{R}(\nu_1, u)$ defined in (4.5) into at most $2^{k+2}+1$ parts, such that $\nu_j^{(k)}(\gamma_{i+1})$ is a constant on each region. Denote these regions by $\mathcal{R}_s(\vec{v}), s \leq 2^{k+2}+1$, and let the constant be $f(\vec{v}, s)$. Then given $\vec{v} \in \mathbb{Z}^{k+j}$, there exists a unique index $s_{q_i,q_{i+1}}$ such that $(\frac{q_i}{Q}, \frac{q_{i+1}}{Q}) \in \mathcal{R}_{s_{q_i,q_{i+1}}}(\vec{v})$. Thus

$$\begin{aligned}
\mathbb{P}_{k,j}(n, Q) &= \frac{1}{N(Q)} \sum_{\vec{v} \in \mathbb{Z}^k} \sum_s \sum_{\substack{(\frac{q_i}{Q}, \frac{q_{i+1}}{Q}) \in \mathcal{R}_s(\vec{v}) \\ \gcd(q_i, q_{i+1})=1}} \mathbb{1}(f(\vec{v}, s) = n) \\
&= \frac{6Q^2}{\pi^2 N(Q)} \sum_{\vec{v} \in \mathbb{Z}^k} \sum_s \mathbb{1}(f(\vec{v}, s) = n) \iint_{\mathcal{R}_s(\vec{v})} 1 dx dy + O\left(\frac{\log Q}{Q}\right) \\
&= 2 \sum_{\vec{v} \in \mathbb{Z}^k} \sum_s \mathbb{1}(f(\vec{v}, s) = n) \iint_{\mathcal{R}_s(\vec{v})} 1 dx dy + O\left(\frac{\log Q}{Q}\right),
\end{aligned}$$

Since for a given n, there are finitely many $\vec{v}$ and s such that $f(\vec{v}, s) = n$, then $\lim_{Q \to \infty} \mathbb{P}_{k,j}(n, Q)$ exists.

$\square$

5. Statistics of generalized index of visible points in convex region

In this section we generalize the statistics of $\nu_j^{(k)}(\gamma_{i+1}) = -\det \begin{pmatrix} a_i^{(k)} & a_{i+j}^{(k)} \\ q_i^{(k)} & q_{i+j}^{(k)} \end{pmatrix}$, where (q_i, a_i) belongs to a convex region Ω with rectifiable boundary. Let

$$\mathbb{P}_{k,j,\Omega}(t, Q) = \frac{1}{N_\Omega(Q)} \# \left\{ i \leq N_\Omega(Q) + k - 1 : \nu_j^{(k)}((q_{i+1}, a_{i+1})) = -\det \begin{pmatrix} a_i^{(k)} & a_{i+j}^{(k)} \\ q_i^{(k)} & q_{i+j}^{(k)} \end{pmatrix} = t \right\},$$

where (q_i, a_i) is the i-visible point in the region $Q\Omega$, $(q_i^{(k)}, a_i^{(k)})$ is obtained from k iterations of ψ and $N_\Omega(Q)$ is the number of visible points in $Q\Omega$. First we consider a convex region $T_{\alpha,\beta,Q}$ bounded by $y = x \tan \alpha$, $y = x \tan \beta$ and $y = Q$, where $0 \leq \alpha < \beta < \frac{\pi}{2}$ and $x, y \geq 0$. From Lemma 4.5, we know that there are approximately $\frac{Q^2(\cot \alpha - \cot \beta)}{2\zeta(2)}$ visible points in $T_{\alpha,\beta,Q}$. In fact, the visible points in $T_{\alpha,\beta,Q}$ correspond to the Farey fractions with denominators $\leq Q$ which belong to the interval $[\cot \beta, \cot \alpha]$.

By Lemma 4.6, we see that

$$\#\{\gamma \in \mathcal{F}_Q : \gamma \leq \cot\alpha\} = \#\left\{(q_1, q_2) : {}^{1\leq q_1, q_2 \leq Q, q_1+q_2>Q, \gcd(q_1,q_2)=1,}_{\exists 0\leq a_1<q_1, 0\leq a_2<q_2, s.t.\ a_2q_1-a_1q_2=1, a_2/q_2\leq \cot\alpha}\right\}$$
$$= \sum_{q_2=1}^{Q} \sum_{0<a_2\leq q_2\cot\alpha} \sum_{\substack{Q-q_2<q_1\leq Q \\ \gcd(q_1,q_2)=1}} \mathbb{1}(q_1a_2 \equiv 1 \,(\mathrm{mod}\, q_2))$$

We will show the existence of $\lim_{Q\to\infty} \mathbb{P}_{k,j,T_{\alpha,\beta,1}}(n, Q)$ for any fixed k, j, n. To prove this, we need the following lemma which is closely related to Kloosterman sum results, proved by Estermann in [8], using Weil's estimates of exponential sums over prime fields (See [15]).

Lemma 5.1. *Suppose that $\mathcal{I}_1$ and $\mathcal{I}_2$ are subintervals of $[1, q]$, then*

$$\#\{(x, y) \in \mathcal{I}_1 \times \mathcal{I}_2 : xy = 1 \,(\mathrm{mod}\, q)\} = \frac{\varphi(q)}{q^2}|\mathcal{I}_1||\mathcal{I}_2| + O\left(\sigma_0(q)\sigma_{\frac{1}{2}}(q)q^{\frac{1}{2}}\log^2 q\right),$$

where $\sigma_c(n) = \sum_{d|n} d^c$ and $\varphi(n) = \sum_{d\leq n, \gcd(d,n)=1} 1$ is the Euler Phi function.

Proof. This is Lemma 1.7 in [4], which employs incomplete Kloosterman sums to estimate the error term. □

Lemma 5.2. *Suppose f is piecewise C^1 on $[a, b]$, Then*

$$\sum_{\substack{a\leq k\leq b \\ \gcd(k,d)=1}} f(k) = \frac{\varphi(q)}{q}\int_a^b f + O\left(\sigma_0(q)\left(||f||_\infty + \int_a^b |f'|\right)\right), \tag{5.1}$$

$$\sum_{a\leq k\leq b} \frac{\varphi(k)}{k} f(k) = \frac{1}{\zeta(2)}\int_a^b f + O\left(\log b\left(||f||_\infty + \int_a^b |f'|\right)\right). \tag{5.2}$$

Proof. These are Lemmas 2.2 and 2.3 in [4]. □

Lemma 5.3. *For any k, j, n, the limit $\lim_{Q\to\infty} \mathbb{P}_{k,j,T_{\alpha,\beta,1}}(t, Q) = \rho_{k,j}(t)$, exists, where $\rho_{k,j}(t)$ is a piecewise smooth function.*

Proof. Applying Lemma 5.1 and 5.2,

$$\begin{aligned}
\#\{\gamma \in \mathcal{F}_Q : \gamma \le \cot\alpha\} &= \#\left\{(q_1,q_2) : \substack{1\le q_1,q_2\le Q, q_1+q_2>Q, \gcd(q_1,q_2)=1, \\ \exists 0\le a_1<q_1, 0\le a_2<q_2, s.t.\ a_2q_1-a_1q_2=1, a_2/q_2\le \cot\alpha}\right\} \\
&= \sum_{q_2=1}^{Q} \sum_{0<a_2\le q_2\cot\alpha} \sum_{\substack{Q-q_2<q_1\le Q \\ \gcd(q_1,q_2)=1}} \mathbb{1}(q_1a_2 \equiv 1 \pmod{q_2}) \\
&= \sum_{q=1}^{Q}\left(\frac{\varphi(q)}{q^2}q^2\cot\alpha + O\left(\sigma_0(q)\sigma_{\frac12}(q)q^{\frac12}\log^2 q\right)\right) \\
&= \frac{\cot\alpha}{2\zeta(2)}Q^2 + O(Q\log Q) + \sum_{q=1}^{Q}\left(O\left(\sigma_0(q)\sigma_{\frac12}(q)q^{\frac12}\log^2 q\right)\right) \\
&= \frac{\cot\alpha}{2\zeta(2)}Q^2 + O\left(\sigma_0(Q)\sigma_{\frac12}(Q)Q^{\frac32}\log^2 Q\right)
\end{aligned}$$

This also shows that

$$\#\{\gamma \in \mathcal{F}_Q : \cot\beta \le \gamma \le \cot\alpha\} = \frac{\cot\alpha-\cot\beta}{2\zeta(2)}Q^2 + O\left(\sigma_0(Q)\sigma_{\frac12}(Q)Q^{\frac32}\log^2 Q\right).$$

We now prove the existence of $\lim_{Q\to\infty}\mathbb{P}_{k,j,T_{\alpha,\beta,1}}(n,Q)$. From the proof of Theorem 4.11, we see that for a fixed $\vec{v}\in\mathbb{Z}^k$ and s, for any $\left(\frac{q_i}{Q},\frac{q_{i+1}}{Q}\right)\in\mathcal{R}_s(\vec{v})$, $\nu_j^{(k)}((q_{i+1},a_{i+1}))$ is a constant denoted by $f(\vec{v},s)$. Also,

$$\begin{aligned}
&\#\left\{(q_1,q_2) : \substack{1\le q_1,q_2\le Q, q_1+q_2>Q, \gcd(q_1,q_2)=1, \\ \exists 0\le a_1<q_1, 0\le a_2<q_2, s.t.\ a_2q_1-a_1q_2=1, a_2/q_2\le\cot\alpha \\ (\frac{q_i}{Q},\frac{q_{i+1}}{Q})\in\mathcal{R}_s(\vec{v})}\right\} \\
&= \sum_{q_2=1}^{Q}\sum_{0<a_2\le\cot\alpha q_2}\sum_{\substack{Q-q_2<q_1\le Q \\ \gcd(q_1,q_2)=1 \\ (\frac{q_i}{Q},\frac{q_{i+1}}{Q})\in\mathcal{R}_s(\vec{v})}} \mathbb{1}(q_1a_2\equiv 1 \pmod{q_2}) \\
&= \cot\alpha\sum_{q=1}^{Q}\left(\frac{\varphi(q)}{q}f_{\mathcal{R}_s(\vec{v})}(q)Q + O\left(\sigma_0(q)\sigma_{\frac12}(q)q^{\frac12}\log^2 q\right)\right) \\
&= \frac{\cot\alpha Q}{\zeta(2)}\int_1^Q f_{\mathcal{R}_s(\vec{v})}(q)dq + O(Q\log Q) + \sum_{q=1}^{Q}\left(O\left(\sigma_0(q)\sigma_{\frac12}(q)q^{\frac12}\log^2 q\right)\right) \\
&= \frac{\cot\alpha}{\zeta(2)}Q^2\iint_{\mathcal{R}_s(\vec{v})} 1 + O\left(\sigma_0(Q)\sigma_{\frac12}(Q)Q^{\frac32}\log^2 Q\right),
\end{aligned}$$

where $f_{\mathcal{R}_s(\vec{v})}(q)$ is the length of line segment of $(1-q/Q,1]\cap R_s(\vec{v})$. Thus,

$$\begin{aligned}\mathbb{P}_{k,j,T_{\alpha,\beta,1}}(n,Q) &= \frac{1}{N_{T_{\alpha,\beta,1}}(Q)}\sum_{\vec{v}\in\mathbb{Z}^k}\sum_{s}\sum_{\substack{(\frac{q_i}{Q},\frac{q_{i+1}}{Q})\in\mathcal{R}_s(\vec{v})\cap T_{\alpha,\beta,1}\\ \gcd(q_i,q_{i+1})=1}}\mathbb{1}(f(\vec{v},s)=n)\\ &= \frac{6Q^2(\cot\alpha-\cot\beta)}{\pi^2N(Q)(\cot\alpha-\cot\beta)}\sum_{\vec{v}\in\mathbb{Z}^k}\sum_{s}\mathbb{1}(f(\vec{v},s)=n)\iint_{\mathcal{R}_s(\vec{v})}1+O\left(\sigma_0(Q)\sigma_{\frac{1}{2}}(Q)Q^{-\frac{1}{2}}\log^2 Q\right)\\ &= 2\sum_{\vec{v}\in\mathbb{Z}^k}\sum_{s}\mathbb{1}(f(\vec{v},s)=n)\iint_{\mathcal{R}_s(\vec{v})}1+O\left(\sigma_0(Q)\sigma_{\frac{1}{2}}(Q)Q^{-\frac{1}{2}}\log^2 Q\right),\end{aligned}$$

which proves the existence of $\lim_{Q\to\infty}\mathbb{P}_{k,j,T_{\alpha,\beta,1}}(n,Q)$ for fixed k,j,n. We also see that the limit is independent of α,β, which completes the proof. $\square$

Theorem 5.4. *Given any convex region Ω with rectifiable boundary and any integers k,j,t,*

$$\lim_{Q\to\infty}\mathbb{P}_{k,j,\Omega}(t,Q)=\rho_{k,j}(t).$$

Proof. We are going to use $T_{\alpha,\beta,h}$ to approximate Ω. It is enough to consider $\Omega^* := \Omega\cap\{(x,y): y\ge x, y\ge 0, x\ge 0\}$. Denote

$$E_{\alpha,\beta,m,M,Q} := T_{\alpha,\beta,MQ}\setminus T_{\alpha,\beta,mQ},$$

hence

$$area(E_{\alpha,\beta,m,M,Q})=\frac{1}{2}Q^2(M^2-m^2)(\cot\alpha-\cot\beta).$$

Let $\Delta=(\pi/4=\alpha_0<\alpha_1<\cdots<\alpha_{n+1}=\pi/2)$ be a partition of $[\pi/4,\pi/2]$ with norm $||\Delta||=\max_{0\le k\le n}(\alpha_{k+1}-\alpha_k)$ and let θ_k, $\xi_k\in[\alpha_k,\alpha_{k+1}]$, $0\le k\le n$. Suppose the boundary of Ω^* is defined by the curve $y(\Omega)$. Let $y_\Omega(\theta_k)=m_k=\min_{\alpha\in[\alpha_k,\alpha_{k+1}]}y(\alpha)$ and $y_\Omega(\xi_k)=M_k=\max_{\alpha\in[\alpha_k,\alpha_{k+1}]}y_\alpha$. Let $E(\Omega,\Delta,Q)$ be the difference of the region $Q\Omega$ and $\bigcup_{k=0}^n T_{\alpha_k,\alpha_{k+1},m_kQ}$, then

$$\begin{aligned}area(E(\Omega,\Delta,Q)) &\le \sum_{k=0}^n area(E_{\alpha_k,\alpha_{k+1},m_k,M_k,Q})\\ &= \frac{1}{2}\sum_{k=0}^n(\cot\alpha_k-\cot\alpha_{k+1})|M_k^2-m_k^2|\\ &\le \frac{1}{2}\sum_{k=0}^n(\cot\alpha_k-\cot\alpha_{k+1})\left(|y_\Omega(\xi_k)^2-y_\Omega(\alpha_k)^2|+|y_\Omega(\alpha_k)^2-y_\Omega(\theta_k)^2|\right)\end{aligned}$$

When $Q^2||\Delta||\to 0$, $area(E(\Omega,\Delta,Q))\ll Q^2||y_\Omega||_\infty||y'_\Omega||_\infty||\Delta||\to 0$. Thus, we fix a large Q such that after k-iterations, the sets of first and second coordinates of the visible points are similarly ordered. Then fix n large enough and Δ so that $||\Delta||\ll\frac{1}{n}$, thus the visible points in $Q\Omega$ and $\bigcup_{r=0}^n T_{\alpha_r,\alpha_{r+1},m_rQ}$ are ordered the same way, except for $O(Q^2/n)$ terms. Then after k-iteration, the sequence $\{(q^{(k)},a^{(k)}):(q,a)\in Q\Omega\}$

and $\bigcup_{r=0}^{n}\{(q^{(k)}, a^{(k)}) : (q,a) \in T_{\alpha_r,\alpha_{r+1},m_rQ}\}$ differ by at most $O(k(nQ+Q^2/n))$ terms. Thus,

$$\begin{aligned}\mathbb{P}_{k,j,\Omega}(t,Q) &= \frac{1}{N_\Omega(Q)}\#\left\{i \le N_\Omega(Q)+k-1 : \nu_j^{(k)}((q_{i+1},a_{i+1})) = t, (q_i,a_i) \in Q\Omega\right\}\\ &= \frac{1}{N_\Omega(Q)}\left(\#\left\{i : \nu_j^{(k)}((q_{i+1},a_{i+1})) = t, (q_i,a_i) \in \bigcup_{r=0}^{n} T_{\alpha_r,\alpha_{r+1},m_rQ},\right\} + O(knQ+kQ^2/n)\right)\\ &= \frac{\sum_{r=0}^{n}\mathbb{P}_{k,j,T_{\alpha_r,\alpha_{r+1},m_rQ}}(t,Q)N_{T_{\alpha_r,\alpha_{r+1},m_r}}(Q)}{N_\Omega(Q)} + O(kn/Q+k/n)\end{aligned}$$

Choose $n = Q^{1/2}$, then we see as $Q \to \infty$,

$$\lim_{Q\to\infty}\mathbb{P}_{k,j,\Omega}(t,Q) = 2\sum_{\vec{v}\in\mathbb{Z}^k}\sum_{s}\mathbb{1}(f(\vec{v},s)=t)\iint_{\mathcal{R}_s(\vec{v})}1,$$

since

$$\lim_{Q\to\infty}\mathbb{P}_{k,j,T_{\alpha_r,\alpha_{r+1},m_rQ}}(t,Q) = 2\sum_{\vec{v}\in\mathbb{Z}^k}\sum_{s}\mathbb{1}(f(\vec{v},s)=t)\iint_{\mathcal{R}_s(\vec{v})}1$$

and

$$\frac{\sum_{r=0}^{n}N_{T_{\alpha_r,\alpha_{r+1},m_r}}(Q)}{N_\Omega(Q)} = 1 + O\left(\frac{1}{n}\right).$$

□

References

[1] E. Alkan, A. H. Ledoan, M. Vâjâitu, and A. Zaharescu. Discrepancy of fractions with divisibility constraints. *Monatsh. Math.*, 149(3):179–192, 2006.

[2] E. Alkan, A. H. Ledoan, M. Vâjâitu, and A. Zaharescu. Discrepancy of sets of fractions with congruence constraints. *Rev. Roumaine Math. Pures Appl.*, 51(3):265–276, 2006.

[3] D. A. Badziahin and A. K. Haynes. A note on Farey fractions with denominators in arithmetic progressions. *Acta Arith.*, 147(3):205–215, 2011.

[4] F. P. Boca, C. Cobeli, and A. Zaharescu. Distribution of lattice points visible from the origin. *Comm. Math. Phys.*, 213(2):433–470, 2000.

[5] F. P. Boca, B. Heersink, and P. Spiegelhalter. Gap distribution of Farey fractions under some divisibility constraints. *Integers*, 13:Paper No. A44, 15, 2013.

[6] F. P. Boca, A. Zaharescu, and R. N. Gologan. On the index of Farey sequences. *Q. J. Math.*, 53(4):377–391, 2002.

[7] C. Cobeli and A. Zaharescu. On the Farey fractions with denominators in arithmetic progression. *J. Integer Seq.*, 9(3):Article 06.3.4, 26, 2006.

[8] T. Estermann. On Kloosterman's sum. *Mathematika*, 8:83–86, 1961.

[9] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company, Reading, MA, second edition, 1994. A foundation for computer science.

[10] R. R. Hall and P. Shiu. The index of a Farey sequence. *Michigan Math. J.*, 51(1):209–223, 2003.

[11] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.

[12] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, Oxford, sixth edition, 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

[13] A. K. Haynes. Numerators of differences of nonconsecutive Farey fractions. *Int. J. Number Theory*, 6(3):655–666, 2010.
[14] A. E. Mayer. A mean value theorem concerning Farey series. *Quart. J. Math., Oxford Ser.*, 13:48–57, 1942.
[15] A. Weil. On some exponential sums. *Proc. Nat. Acad. Sci. U. S. A.*, 34:204–207, 1948.

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA.

E-mail address: jli135@illinois.edu

E-mail address: tamazya2@illinois.edu

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, United States and Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail address: zaharesc@illinois.edu