Rips Theory

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April 11, 2003

Contents

1 Group action on trees

1.1 Some basic definitions

Definition 1.1 *An $\mathbb{R}$-**tree** is a geodesic metric space in which any pair of points is connected by a unique arc.*

It may be helpful to think of $\mathbb{R}$-trees as 0-hyperbolic spaces.

Definition 1.2 *Let G be a finitely-generated group. A metric tree on which G acts is called a G-**tree**. A G-tree is **trivial** if some point is fixed by the whole group. Otherwise it is **non-trivial**. A G-tree is **minimal** if it contains no G-invariant subtrees.*

Minimality is not a severe assumption in addition to non-triviality.

Lemma 1.3 (Minimal subtrees) *Let G be a finitely-generated group acting non-trivially on an $\mathbb{R}$-tree T. Then there exists a sub-tree of T denoted T_G on which G acts minimally. Further, T_G consists of a union of at most countably many lines.*

Proof: Fix a finite set of generators for G. Let T_G be the union of the axes of the hyperbolic elements of G. T_G is non-empty, because if all the generators are elliptic and no pair has hyperbolic product then each pair of fixed-point sets has non-trivial intersection, and it follows that the action of G is trivial. T_G is connected, because if a pair of hyperbolic elements have axes which do not intersect then their product is hyperbolic with an axis intersecting both of theirs. T_G is G-invariant because if g is hyperbolic and $h \in G$ then

$$h\mathrm{Axis}(g) = \mathrm{Axis}(hgh^{-1}).$$

QED

If G acts on T and S is a subtree of T, denote by $\mathrm{Fix}_G(S)$ the subgroup of elements of G which restrict to the identity on S. A subtree of T is *non-degenerate* if it is not a point.

Definition 1.4 *A non-degenerate subtree S of T is **stable** if, for every non-degenerate subtree S' of S,*

$$\mathrm{Fix}_G(S') = \mathrm{Fix}_G(S).$$

*A non-trivial action of G on T is **stable** if it is non-trivial, minimal, and every non-degenerate subtree of T contains a stable subtree.*

Remark 1.5 *If T is a stable G-tree and S_1, S_2 are stable subtrees with $S_1 \cap S_2$ non-degenerate then $S_1 \cup S_2$ is stable. In particular, every stable subtree is contained in a unique maximal stable subtree.*

2 Resolving $\mathbb{R}$-trees

2.1 Introduction

The aim of the Rips Machine is to understand group actions on $\mathbb{R}$-trees. Let T be an $\mathbb{R}$-tree, and let G be a group acting on T. Let X be a space on which G acts, and let $f : X \to T$ be a G-equivariant map. Then it might suffice to understand the action of G on the pre-images of points under f. To this end, a class of spaces is introduced with an associated foliation which is relatively easily understood and manipulated. These spaces are called band complexes, and the leaves will be pre-images of points.

2.2 Bands and their unions

Definition 2.1 *A **band** is a product $B = b \times I$ where b is a closed interval and I is the closed unit interval.*

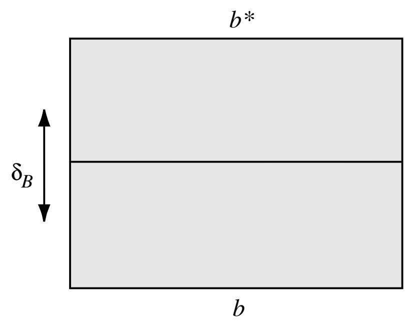

Figure 1: The band $B = b \times I$

A band has an involution δ_B given by reflection in $b \times \{\frac{1}{2}\}$, known as the *dual map*. The intervals b and $b^* = \delta_B(b)$ are the *bases* of B. Subsets of the form $\{x\} \times I$ are called *vertical fibres*, subsets contained in vertical fibres are called *vertical*, and subsets contained in sets of the form $b \times \{y\}$ are called *horizontal*.

Definition 2.2 *Let Γ be a metric graph, let $\mathcal{B}$ be a finite collection of bands, and for each base b let $f_b : b \to \Gamma$ be a length-preserving homeomorphism such that the image of the interior of b misses the vertices of Γ. Then a **union of bands** is a space of the form*

$$Y = \Gamma \cup \coprod_{B \in \mathcal{B}} B$$

quotiented by the equivalence relation generated by identifying points of b with their images under f_b.

A band $B = b \times I$ is an *annulus* if $f_b = f_{b^*} \circ \delta_B$. It is a *Möbius band* if $f_b = f_{b^*} \circ r_B \circ \delta_B$, where r_B is reflection in the vertical line dividing B in half. A *block* is the closure of a component of the union of the interiors of bases in Γ.

Definition 2.3 *The **weight of a base** b, denoted $w(b)$, is defined according to its band: 0 if it is an annulus, $\frac{1}{2}$ if it is a Möbius band, and 1 otherwise. The **weight of a point** $x \in \Gamma$, denoted $w(x)$, is the sum of the weights of the bases containing x. The **complexity** of a block β is*

$$\max\Big(0, -2 + \sum_{b \subset \beta} w(b)\Big)$$

for bases b. The complexity of a union of bands is the sum of the complexity of its blocks.

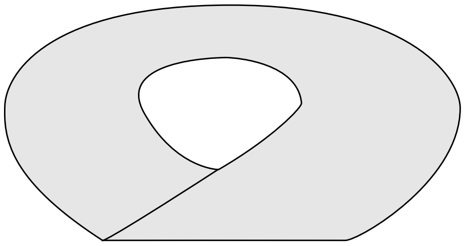

Figure 2: A Möbius band

A union of bands has a natural length metric induced by the metrics on the bands and Γ. A *leaf* of Y is an equivalence class generated by relating points which lie in the same vertical fibre of a band. A subset of Y is called *vertical* if it lies within a single leaf, and *horizontal* if it lies entirely within Γ or lies within only one band and is horizontal in that band. The *horizontal ϵ-neighbourhood* of a point z, denoted $N(z, \epsilon)$ is the horizontal set of points within ϵ of z.

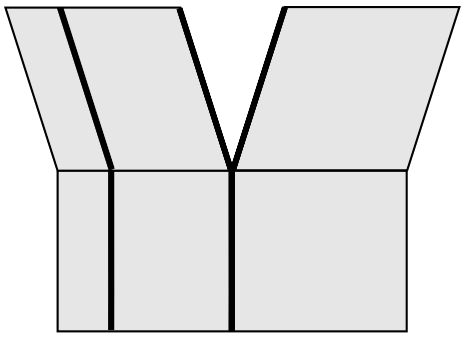

Figure 3: A union of three bands, with two leaves marked by bold lines

Lemma 2.4 (Transverse Measure) *The decomposition of Y into leaves carries a natural transverse measure.*

Proof: For a band B let $\pi : B \to b$ be projection to the first factor. A path α in B is *transversal* if $\pi \circ \alpha$ is injective; in this case define $\mu(\alpha)$ to be the length of $\pi \circ \alpha$. If the image of α is a vertical subset of B set $\mu(\alpha) = 0$. Now let α be a path in the union of bands Y. Suppose its domain can be divided into intervals I_j such that, for each j, $g|I_j$ is either transversal or vertical in some band. Then set

$$\mu(g) = \sum_j \mu(g|I_j).$$

QED

2.3 Dynamical Decomposition of Unions of Bands

Let S and S' be horizontal subsets. *S pushes into S'* if there exists a homotopy H of S into S' through horizontal subsets such that, for each $z \in S$, $H(\{z\} \times I)$ is vertical. If moreover p is a path in Y and for some $z_0 \in S$, $p(t) = H(z_0, t)$, say S pushes into S' *along* p; and if $H(S \times \{1\}) = S'$ say S pushes *onto* S'.

A subset $Z \subset Y$ is *pushing-saturated* if, whenever $p : I \to Y$ is a vertical path and $p(0) \in Z$ has a horizontal ϵ-neighbourhood which pushes along p, $p(1) \in Z$. A leaf containing a proper pushing-saturated subset is *singular*; otherwise, it is *non-singular*. A compact pushing-saturated proper subset of a leaf is called a *fault*. The union of bands shown in figure 3 has one singular leaf, containing three faults.

Definition 2.5 *A union of bands Y is* ***minimal*** *if every pushing saturated subset is dense in Y; it is* ***simplicial*** *if every leaf of Y is compact.*

Note that these two properties are mutually exclusive if Y contains any non-singleton horizontal subsets.

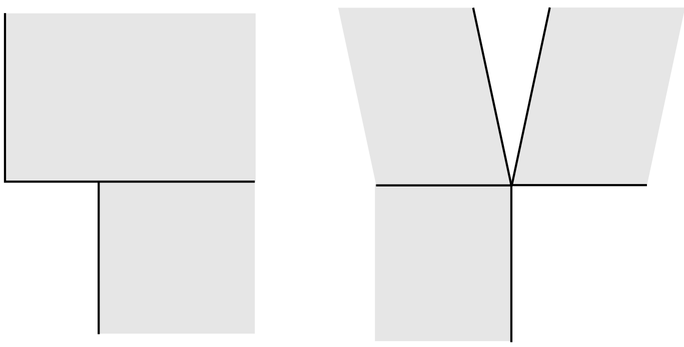

Figure 4: A dead end on the left and a cut point on the right

How can singular leaves occur? Let L be a leaf, and let $S \subset L$ be a proper pushing-saturated subset. Let $x \in L$ be an end-point of S. If S intersects a band B at x and x is not a corner of B then S contains the whole of the vertical fibre of B containing x. So it suffices to consider the following situation. Let b be a base of a band B, let $L \cap b = \{x\}$ and suppose S contains $\{x\} \times I$. There are two cases: x lies in the interior of b, and x is an end-point of b. In the first case, the only way in which S can be a proper subset of L is if there exists another band $C = c \times I$ with $c \subset b$ and x is an end-point of c. Such a point x is called a *dead end*. In the second case, S is a proper pushing-saturated subset of L if x locally separates Y. Such a point x is called a *cut point*. Cut points and dead ends are known collectively as *critical points*, and their existence is a necessary (but not sufficient) condition for singularity of a leaf.

Proposition 2.6 *Let Y be a union of bands. There are only finitely many singular leaves and finitely many faults. Suppose Y contains no faults. Then:*

1. *each component of Y is either simplicial or minimal;*
2. *each simplicial component is an I-bundle over a leaf;*
3. *only finitely many leaves are singular.*

Proof: By the above discussion a singular leaf contains a critical point, and a fault has a critical point at its boundary. Critical points only arise as the corners of bands, so there are only finitely many critical points and hence only finitely many faults.

Let L be a compact non-singular leaf. Then all nearby leaves are also compact and non-singular. Let $U(L)$ be the maximal connected neighbourhood of L consisting of compact non-singular leaves. Then $U(L)$ is an interval bundle over L. The boundary of $U(L)$ must consist of compact singular leaves, which contain faults, so $U(L)$ is a component of Y. Let U be the union of all such $U(L)$. Choose a component M of $Y - U$. Let S be a pushing-saturated subset of M. Then the aim is to show that S is dense in M. Without loss of generality, S is contained in an infinite leaf L.

The first claim to be made is that, for some $\delta \geq 0$ depending on S, for every base b, $b - \overline{S}$ has length at least δ. To see this, choose δ such that, for x an end-point of a base b, $d(x, b^o \cap S) \geq \delta$. Suppose that c is a component of $b - \overline{S}$ of length $\lambda < \delta$. Since the accumulation set of a pushing-saturated set is also pushing-saturated, for x an end-point of c, $P(x)$ is an infinitely long vertical set intersecting b infinitely often. So there exist $y, z \in P(x) \cap b$ satisfying the following conditions:

1. $d(y, z) < \lambda$;
2. a non-degenerate subinterval $[x, u]$ of c pushes into $[y, z]$ along the vertical path p connecting c to y.

Let $x' \in c$ be such that $d(x, x') = \lambda$. Because x' does not lie in $\overline{L}$, $[x, x']$ does not push into $[y, z]$ along p. Decompose p as a composition of paths $p_1 \star \ldots \star p_k$; for some j, $[x, x']$ pushes along $p'' = p_1 \star \ldots \star p_{j-1}$ but does not push along $p' = p_1 \star \ldots \star p_j$. Let x'' be the point closest to x such that $[x, x'']$ does not push along p'.

1. The point u arrived at by pushing x'' along p'' is not in $\overline{S}$; if it were, it could be pushed into $[y, z]$.
2. u is an end-point of a base.
3. $0 < d(c, S) = \lambda < \delta$ since the entire interval $[x, x'']$ pushes along p'' and so its image is disjoint from $\overline{S}$.

This contradicts the choice of δ.

Now consider, for a base $b \subset M \cap \Gamma$, $b \cap \overline{S}$. Let x be a point of the boundary of $b \cap \overline{S}$ in b which is not an end-point of b, and let $K = P(x)$. Let $c \subset b$ be an interval with end-point x such that $c^0 \cap \overline{S} = \emptyset$. Choose δ depending on K according to the previous claim, and let $y, k \in K$ be such that:

1. $0 < d(y, z) < \delta$;
2. if p is a path connecting x to y then p pushes a non-degenerate subinterval $[x, x']$ into $[y, z]$.

Then $[y, z] \subset \overline{K} \subset \overline{S}$. But $[x, x']$ pushes into $[y, z]$ and $(x, x'] \cap \overline{S} = \emptyset$. This is a contradiction. *QED*

2.4 Band Complexes

Using unions of bands, there is too little control over the fundamental group to resolve $\mathbb{R}$-trees as desired. A union of bands is homotopic to a graph (by collapsing the bands) so has free fundamental group. In general the leaves may need to contain 2-cells.

Definition 2.7 *A **band complex** X is a relative CW 2-complex based on a union of bands Y (that is, Y with 1- and 2-cells attached) subject to the following conditions:*

1. *the 1-cells of X meet Y in a subset of Γ;*
2. *each component of the intersection of a 2-cell with Γ is a point;*
3. *each component of the intersection of a 2-cell with a band is vertical.*

Throughout this document, X is a band complex, Y is its underlying union of bands and Γ its underlying simplicial graph. Likewise X', Y', Γ' and X_n, Y_n and Γ_n.

X is *minimal* or *simplicial* if Y is.

A path $\alpha : I \to X$ is *transverse* if I can be divided into finitely many intervals I_j such that, for all j, either $\alpha|I_j$ is transversal or vertical in Y, or $\alpha(I_j)$ is contained in the closure of $X - Y$. In the second case set $\mu(g|I_j) = 0$. This defines transverse measure on X.

The *leaves* of X are given by equating the end-points of paths in X of measure 0. Likewise the leaves of X define leaves in the universal cover $\tilde{X}$, by setting as equivalent the end-points of a path if it projects to a path of measure 0 in X.

The *complexity* of a band complex is the complexity of its underlying union of bands.

Definition 2.8 *A **resolution** of an action of a finitely-generated group G on an $\mathbb{R}$-tree T is a band complex X with fundamental group G and a G-equivariant map*

$$f : \tilde{X} \to T$$

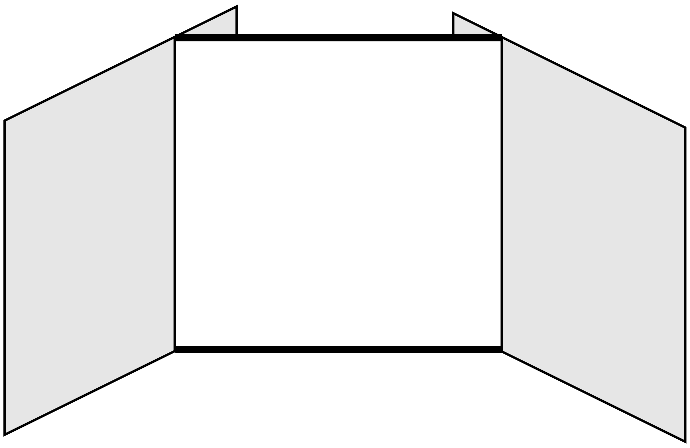

Figure 5: A band complex with two bands (in grey), two 1-cells (in bold) and one 2-cell (unshaded).

satisfying the following conditions:

1. *the image of a leaf of $\tilde{X}$ in T is a point;*
2. *each base can be broken into finitely many subintervals, the lifts of which are embedded isometrically in T by f.*

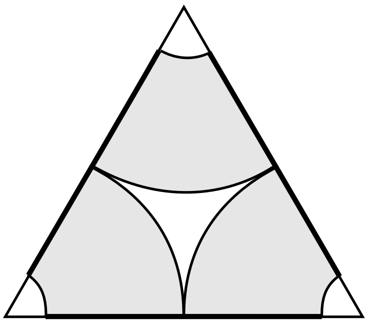

Figure 6: A band complex structure for a simplex. The bands are in grey, Γ is in bold and the 2-cells are unshaded.

Theorem 2.9 *If a finitely-presentable group G acts on an $\mathbb{R}$-tree then a resolution exists.*

Proof: Let X be a simplicial 2-complex with fundamental group G, which exists since G is finitely-presentable. Construct $f : \tilde{X} \to T$ and a band-complex-structure for X as follows. Define f on the 0-skeleton of $\tilde{X}$ to be G-equivariant.

Extend it to the 1-skeleton by requiring that it is constant on small neighbourhoods of vertices and then monotonic on the rest of the edges. Finally extend it to the interiors of the 2-simplices in the natural manner. This is illustrated in the case where the three vertices have distinct images in T in figure 6. The case where two vertices are collapsed is similar, and if all three vertices have the same image the whole simplex becomes a 2-cell. *QED*

2.5 A Graph of Spaces Decomposition

Let Ω be a graph. For each vertex v let X_v be a connected space, and for each edge e let X_e be a connected space. Orient the edges arbitrarily, assigning to each edge e a terminal vertex $t(e)$ and an initial vertex $i(e)$. For each edge and each terminal (respectively initial) vertex let $f_e : X_e \to X_{t(e)}$ (respectively $g_e : X_e \to X_{i(e)}$) be an embedding. It is also required that f_e and g_e are π_1-injective. Then define X_Ω to be the space

$$\coprod X_v$$

quotiented by equating the images of f_e and g_e in the natural manner for each e.

Definition 2.10 *The graph Ω, the spaces X_v and X_e, and the embeddings f_e and g_e constitute a* ***graph of spaces decomposition*** *for X_Ω. X_Ω is called a* ***graph of spaces****.*

By van Kampen's theorem, $\pi_1(X_\Omega)$ is a graph of groups with graph Ω, vertex groups $\pi_1(X_v)$ and edge groups $\pi_1(X_e)$.

A stable G-tree is *pure* if it admits a minimal resolving band complex X such that underlying union of bands Y is connected and $\pi_1(Y) \to \pi_1(X)$ is surjective.

Theorem 2.11 *Let G be finitely presented and T a stable G-tree. Suppose that X is a band complex which resolves the action of G. Let X have underlying union of bands Y. Assume that no leaf of Y has a fault. Then G has a graph of groups decomposition such that:*

1. *the action on T of a vertex group is either trivial or pure;*
2. *the action on T of an edge group of the decomposition is trivial.*

Proof: Let X be a resolving band complex. Subdivide the boundary of each 2-cell of X so that if the interior of an edge of the subdivision meets Y then the edge is contained in Y. Then the closure of $X - Y$ has an obvious CW-structure. By adding (possibly infinitely many) 2-cells to the edge spaces it can be ensured that the edge embeddings are π_1-injective. Now X has a graph of spaces decomposition where the vertex spaces are the components of Y together with the closure of $X - Y$, and an edge space is a component of the intersection of the closure of $X - Y$ with Y.

Each component of Y is either minimal or simplicial. If the component is simplicial then it an interval-bundle over a leaf, so any loop is homotopic to a loop in that leaf and fixes a corresponding point in T; otherwise, the component is minimal and the action is pure.

An edge space is a component of the intersection of the closure of $X - Y$ with Y, so is contained in a leaf; in particular, the action of its fundamental group on T is trivial. *QED*

2.6 The Kazhdan-Margulis Lemma for resolutions of $\mathbb{R}$-trees

Let c be a non-degenerate subarc of an edge of Γ with a base-point z_0 in the interior of c. A loop α in X based at z_0 is *c-short* if it is a composition $p_1 \star \lambda \star p_2$ of paths where p_1, p_2 are paths in c and λ is a path in a non-singular leaf.

Assume that X is a band complex with no faults, let $C(Y)$ be a minimal component of Y and write $C(\Gamma)$ for $\Gamma \cap C(Y)$.

Proposition 2.12 *Let c be a non-degenerate subarc of an edge of $C(\Gamma)$ and let z_0 be a base-point in the interior of . Then the image of $\pi_1(C(Y))$ in $\pi_1(X)$ is generated by c-short loops.*

Proof: Since there are no faults, Y has only finitely many singular leaves. Let α be a loop based at z_0. Then α is homotopy equivalent to a loop of the form

$$p_0 \star \lambda_1 \star p_1 \star \dots p_{n-1} \star \lambda_n \star p_n$$

where p_i is horizontal for all $0 \leq i \leq n$ and λ_i is a vertical path in a non-singular leaf for all $1 \leq i \leq n$. By the minimality of $C(Y)$ each p_i can be subdivided into segments $p_{i,j}$, each of which pushes into c. Further, it can be ensured that the end-points of $p_{i,j}$ lie in non-singular leafs. Realizing these pushes as homotopies, $p_{i,j}$ is homotopic to a path of the form $\mu^+_{i,j} \star q_{i,j} \star \mu^-_{i,j}$, where $\mu^{+,-}_{i,j}$ are vertical paths in non-singular leafs, and $q_{i,j}$ is a path in c. If z_0 does not lie in the interior of $q_{i,j}$ it may be necessary to append a null-homotopic loop in c to $q_{i,j}$, containing z_0. This gives the required decomposition. *QED*

It is now easy to prove the Kazhdan-Margulis lemma for resolutions of actions on $\mathbb{R}$-trees.

Theorem 2.13 (Kazhdan-Margulis Lemma) *Let X be a band complex resolving a stable G-tree T. Let $C(Y)$ be a minimal component of the underlying union of bands Y, and let H be the image of $\pi_1(C(Y))$ in $G = \pi_1(X)$. Then T_H is stable. It follows immediately that if $h \in H$ fixes an arc of T then h lies in the kernel of the action of H on T_H.*

Proof: Let c be an arc of $C(\Gamma)$ mapping via the resolution to a non-degenerate arc of a maximal stable sub-tree T' in T_H. Let S_c be the set of generators of H consisting of c-short loops. Whenever $g \in S_c$, it follows by construction that $g(c) \cap c$ is a non-degenerate arc, and so is $g(\rho(c)) \cap \rho(c)$ by G-equivariance.

The intersection of maximal stable subtrees must be a point by remark 1.5. Since $g(T')$ is a maximal stable subtree, it follows that $g(T') = T'$, so T' is H-invariant. Therefore $T' = T_H$. *QED*

Corollary 2.14 *If $\alpha, \beta \in H$ act as hyperbolic isometries of T and the length of $L = \mathrm{Axis}(\alpha) \cap \mathrm{Axis}(\beta)$ is greater than the sum of the translation lengths of α and β then the two axes coincide.*

Proof: Let $\gamma = [\alpha, \beta]$ and let J be a segment in L adjacent to one of the end-points with

$$\mathrm{length}(J) = \mathrm{length}(L) - l(\alpha) - l(\beta).$$

Then γ fixes J, so by the theorem γ acts trivially on T_H and the actions of α and β on T_H commute. Therefore their axes coincide. *QED*

3 The Machine

3.1 Moves

There are 6 basic moves which, given a band complex resolving a G-tree, transform this complex into another one resolving the same action.

M0 **Add a 2-cell.** Attach a 2-cell along a vertical loop in $Y \cup X^{(1)}$ which is null-homotopic in X.

M1 **Add an annulus.** For c any subarc of an edge of Γ, attach to c an annulus B; then attach a 2-cell along one of the vertical fibres of B.

M2 **Subdivide a band.** Let $B = b \times I$ be a band, and let x lie in the interior of b. Then split B down the fibre x. There are now two fibres over x; attach a 2-cell along them. See figure 7

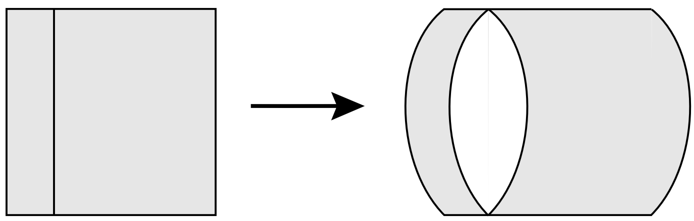

Figure 7: Subdividing a band along the indicated fibre. The added 2-cell is unshaded.

M3 **Split a point.** Let $z \in \Gamma$ be a point which does not lie in the interior of any bases, but lies in at least one block and locally separates Γ. Split Γ at z, giving the graph Γ'. Let Z be the subset of Γ' which lies over z. Attach to Γ' the cone over Z, as a collection of 1-cells. The move is completed by attaching bands and cells to the resulting graph as before, to get a band complex.

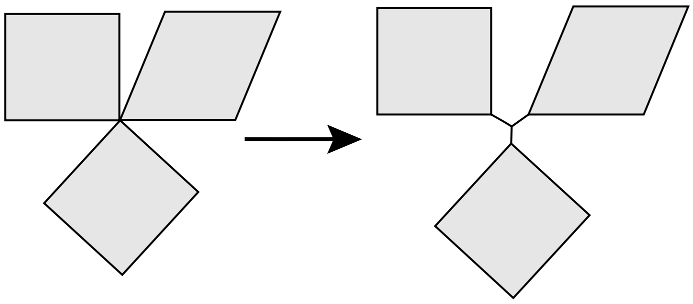

Figure 8: Splitting a point. Note that the cone is added as 1-cells.

Lemma 3.1 *Applying a finite sequence of moves M2 and M3 it may be ensured that a band complex contains no faults.*

Proof: All dead ends can be eliminated by subdividing bands. Note that this may create cut points (for example in the situation of figure 3). Then all cut points can be removed by splitting. This will not create dead ends. *QED*

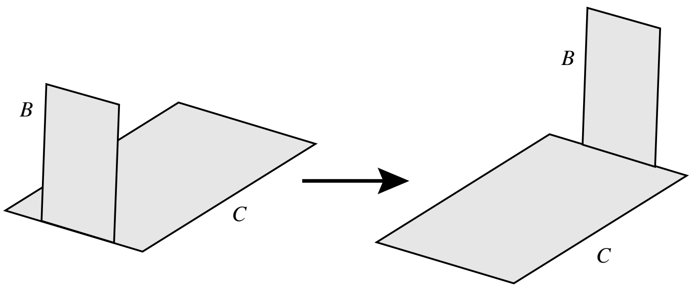

Figure 9: A slide.

M4 Slide. Let $B = b \times I$ and $C = c \times I$ be distinct bands, and suppose $f_b(b) \subset f_c(c)$. Then a new band complex can be created by replacing f_b by $f_{c*} \circ \delta_C \circ f_c^{-1} \circ f_b$. Note that there is an obvious homotopy between the two complexes, so the fundamental group has not been changed.

Lemma 3.2 *If $s, s' \subset \Gamma$ are non-degenerate arcs and s pushes onto s' then there exists a finite sequence of moves M1 and M4 converting the band complex X with underlying union of bands Y into a band complex with underlying union of bands $Y \cup B$, where B is a band with bases s and s'.*

Proof: Attach an annulus to s using move M1. Then realize the push of s onto s' as a sequence of moves of type M4. *QED*

The next lemma provides a way to simplify Möbius bands.

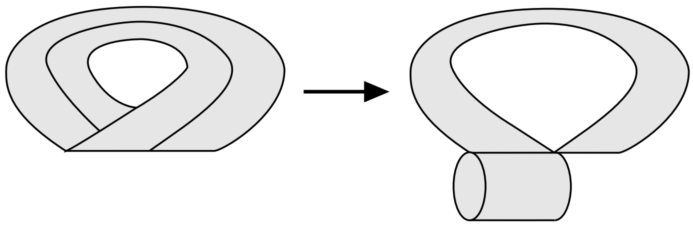

Figure 10: Simplifying a Möbius band in case 1 of lemma 3.3.

Lemma 3.3 *Let $B = b \times I$ be a Möbius band in X. Let $x \in f_b(b)$.*

1. *If x is the midpoint of the base then B can be split over x to give an annulus and band of weight 1.*
2. *If x lies in the interior of a half-base then B can be split over x to give an annulus, a Möbius band and a band of weight 1.*

Proof: In the first case there is a single fibre over x. Otherwise there are two fibres over x. Subdivide B down the fibre(s) over x, and slide one of the resulting Möbius bands over the resulting band of weight 1, creating an annulus. *QED*

Definition 3.4 *A subarc c of a base b is **free** if either:*

1. *b is of weight 1 and the interior of c meets no other bands of positive weight; or*
2. *b is of weight $\frac{1}{2}$, the interior of c does not contain the midpoint of b, and also meets no positive weight bases other than b and b^*.*

The final move can now be defined.

M5 Collapsing from a free subarc. Let $B = b \times I$ be a band, and let $c \subset b$ be a free subarc of b. First use M2 to subdivide all annuluses meeting c, until a base of any such band is contained in c. Then use M4 twice on each such annulus to slide it over B, until it is attached instead to $\delta_B(c)$. If b is of weight 1, then $c \times I$ can now be replaced by $(\partial c \times I) \cup \delta_B(c)$. In the case where an endpoint x of c is also an endpoint of b, and $x \times (0, 1)$ does not meet any 2-cells, also eliminate $x \times (0, 1)$.

If b is of weight $\frac{1}{2}$, use lemma 3.3 to subdivide B over the end-point of c nearest the midpoint of b. Now c is contained in a band of weight 1, which can be collapsed from c as before.

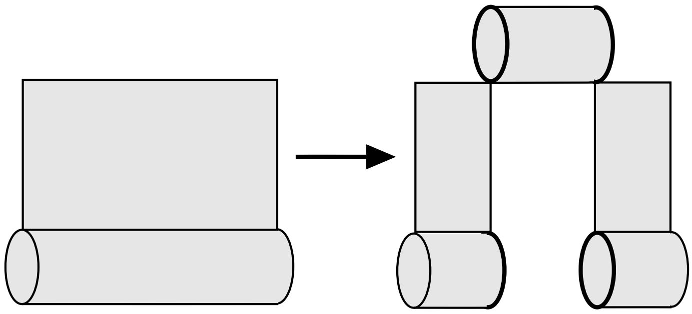

Figure 11: Collapsing a weight 1 band from a free subarc. The subdivision annuluses are attached to the bold circles.

3.2 Preliminary assumptions and outline

Let X be a band complex with underlying union of bands Y and graph Γ. Suppose also that X resolves a G-action on an $\mathbb{R}$-tree T. The following assumptions may be made.

A1 The graph Γ is the disjoint union of its edges. Each edge is a block. This can be achieved by splitting the vertices of Γ and points in the intersection of blocks.

A2 The union of bands Y is a disjoint union of components each of which is either minimal or simplicial, by proposition 2.6.

A3 If $A = a \times I$ and $B = b \times I$ are Möbius bands whose bases share midpoints then $B = C$. For if so, simply slide one over the other to form an annulus.

A4 The resolution embeds lifts of bases into T.

A5 X has no faults, by lemma 3.1.

Once these have been arranged, the machine consists of two processes, I and II. Process I is applied first, repeatedly. If after some repetitions it can no longer be applied, pass to process II. Again, if after some repetitions it can no longer be applied, pass back to process I. Going from I to II and back reduces complexity, so eventually either process I or process II is applied. Information about the resolution, and so the group action, can be deduced from this eventual data. Note that the process never stops, since the complexity is at least 0.

3.3 Process I

Choose a minimal component $C(Y)$ of the union of bands Y. To explain the process properly, the following definitions are needed.

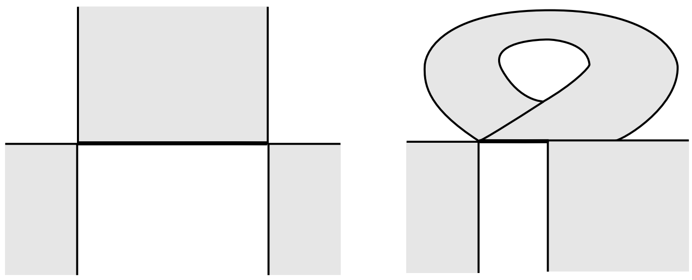

Figure 12: An isolated base and an isolated half-base, marked by the bold lines.

Definition 3.5 *A weight 1 base b is called* ***isolated*** *if it is a free subarc of itself; that is, its interior meets no other positive weight bases. Half of a weight* $\frac{1}{2}$ *base b is* ***isolated*** *if it is free; that is, its interior meets no bases other than b and* b^**.*

A weight 1 base b is ***semi-isolated*** *if one of its endpoints is contained in a free subarc of b. Likewise, half of a weight* $\frac{1}{2}$ *base b is* ***semi-isolated*** *if one of the endpoints of the half-base is contained in a free subarc.*

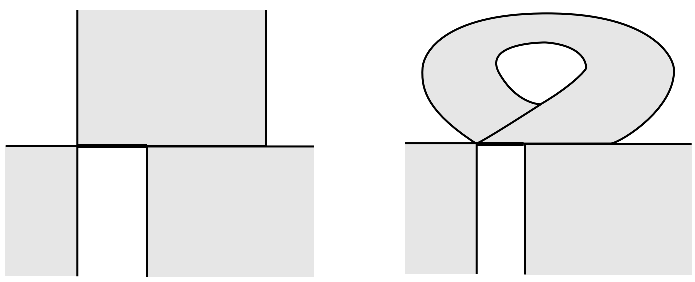

Figure 13: A semi-isolated base and a semi-isolated half-base, marked by the bold lines.

Note that collapsing from a (semi-)isolated (half-)base does not increase the number of positive-weight bands.

Definition 3.6 *A* ***long band*** *is a sequence of weight 1 bands* $B_1, \ldots, B_n$*, with for each j a choice of 'top' and 'bottom' base, such that for* $j = 1, \ldots, n-1$ *the top of* B_j *is identified with the bottom of* B_{j+1} *and does not meet any other positive weight base. It is also required that the sequence is maximal with respect to these properties.*

The process takes a band complex X and creates a new one, X', as follows. If possible, let c be a maximal free subarc in $C(Y)$. If not, set $X' = X$ and pass to process II. Choose c according to the following criteria.

I1 If there exists an isolated (half-)base choose that for c.

I2 If there does not exist an isolated (half-)base, but there does exist a semi-isolated (half-)base, choose that for c.

I3 If there exist no (semi-)isolated (half-)bases, choose any maximal free subarc for c.

Now collapse from c. If c lay at the end of a long band, repeat the collapse along the whole length of the long band. Continue by applying process I to X'.

Excercise 3.7 *Process I does not increase complexity.*

The next proposition gives some of the principal features of infinite sequences of band complexes obtained in this way.

Proposition 3.8 *Let $X_0, X_1, \ldots$ be an infinite sequence of band complexes created by applying process I recursively.*

1. *The number of weight $\frac{1}{2}$ bases does not increase.*
2. *The number of long bands is uniformly bounded.*
3. *Choice I1 occurs only finitely many times.*
4. *For any integer n there exists a segment of length n contained in some leaf, which is eventually collapsed.*
5. *Choice I3 occurs infinitely often.*

Proof: 1 is trivial. Without loss of generality it may be assumed that the complexity is a constant, σ. Now suppose that the number of long bands in X_{i+1} is greater than the number in X_i. This can only occur if choice I3 has been made. So there are no (semi-)isolated (half-)bases, and each block of complexity 0 must contain either two bases of weight one, or one of weight one and two of weight half. Therefore each long band has one end in a block of positive weight. The number of long bands originating in such a block is its complexity, plus two. It follows that the number of long bands in X_i is no greater than 3σ, and the number in X_{i+1} is no greater than $3\sigma + 1$. This proves 2.

Choice I1 leads to a reduction in complexity. So it can only be chosen finitely many times. This proves 3.

Define $N(k)$ inductively for $k \in \mathbb{N}$. Set $N(0) = 0$, and let $N(k+1)$ be the least integer such that all points on free subarcs of $\Gamma \cap X_{N(k)}$ that are eventually collapsed are collapsed by stage $N(k+1)$. Note that at any stage there are only finitely many maximal free subarcs, so this is well-defined. Now any point collapsed at stage $N(n)$ belonged to a segment of length at least n which has been collapsed. This proves 4.

Suppose that eventually only collapses of type I2 occur. Then the number of positive weight bands remains constant. So it can be assumed that the number of positive weight bands is some constant η. By 4 there exists some λ of length $n > 4\eta$ which is eventually collapsed, at stage $N(n)$. Then λ must intersect some band $B = b \times I$ in at least 5 vertical segments. Orient λ, and choose vertical segments $\lambda_1, \lambda_2, \lambda_3$ of λ in B with the following properties. According to the orientation of λ, λ_1 must be the first segment and λ_3 the last. Let x_i be the first point of λ_i in b. Then the segments $[x_1, x_2]$ and $[x_2, x_3]$ must be orientation-preserving.

Let $\theta > 0$ be the minimum of all possible non-zero translation lengths for orientation-preserving holonomy transformations determined by paths of length no more than n. The segments $\lambda_1, \lambda_2, \lambda_3$ are separated by horizontal distances at least θ which depends only on the original band complex X_0. One of the segments (say λ_2) lies between the other two, and so lies at least θ from each boundary. At stage $N(n)$, λ is collapsed, using choice I2 by assumption. It follows inductively that between stages $N(kn)$ and stages $N((k+1)n))$, at least one base is reduced in length by at least θ. Therefore after finitely many steps all bases have been reduced to length 0, and the process stops. This is a contradiction, proving 5. *QED*

3.4 Process II

Once again let $C(Y)$ be a minimal component of the union of bands Y underlying the band complex X. Process II is applied after process I, so it may be assumed that for any point z in a base in $C(Y)$:

- $\star$ the sum of the weights of the bases containing z is at least 2.

Identify the components of $C(\Gamma) = \Gamma \cap C(Y)$ with disjoint closed intervals in $\mathbb{R}$, giving $C(\Gamma)$ a total order. Let x be the first point of $C(\Gamma)$. Let b be the longest positive-weight base containing x. If possible, choose b to have weight 1. Now produce the new band complex X' as follows.

1. Slide over $B = b \times I$ all positive-weight bases c whose midpoint is moved away from x as a result, apart from b and b^*.

2. Note that there is now necessarily a non-degenerate maximal free initial segment c of b. Collapse c.

Note that each base in X naturally corresponds to a base in X', with the possible exception of b which may have been collapsed entirely. In particular, Γ_i the underlying graph of X_i embeds in Γ.

If complexity is reduced by process II, go back to process I. Otherwise, apply process II again.

Proposition 3.9 *Suppose X' results from X via process II.*

1. *The complexity of X' is no greater than the complexity of X.*

2. *If the complexity of X' equals the complexity of X then $\star$ holds for X'.*

Proof: To prove 1 the effect of slides on complexity needs to be understood. The final collapse will certainly not increase complexity, exactly as in process I. A slide can only increase complexity in the following cases:

1. the carried base has weight 0 or $\frac{1}{2}$, in which case after it is carried it might have weight 1;

2. the carried base has weight 1 and, at first, is contained in a block of zero complexity.

The first case is not a problem, because if a weight $\frac{1}{2}$ base is carried so is its dual, and the result is still two weight $\frac{1}{2}$ bases. The second case can only occur at the last slide. If complexity is increased then after the slide the base b is isolated, and so collapsing reduces complexity again. This proves 1.

Condition $\star$ might fail for two different reasons. First, sliding a base might reduce its weight to 0 or $\frac{1}{2}$; in this case, complexity is clearly reduced. Alternatively, after all the slides, the weight of a particular point in the dual of the maximal free initial segment of b might be less than 3, so collapsing would bring its weight below 2. This can only occur if the dual of b contains a free segment, in which case it must have done so before the slides, contradicting condition $\star$ initially. *QED*

This ensures that the machine works as required. Complexity is not increased, and passing from process I to process II and back again reduces complexity. It follows that eventually only one is applied.

Remark 3.10 *Process II only moves bases in the positive direction in $\mathbb{R}$, or deletes a neighbourhood of the lower boundary point. Since the positions of bases in $\mathbb{R}$ are bounded above, bases have a limiting position in Γ under this process.*

Proposition 3.11 *Let $X_0, X_1, \ldots$ be a sequence of band complexes formed by applying process II recursively. Then one of the following holds.*

1. *The sequence terminates after finitely many steps.*

2. *The **surface** case. For all sufficiently large i and each point z of $\Gamma \cap X_i$ which is not the endpoint of a base, $w(z) = 2$.*

3. *The **toral** case. Some base b is a carrier and is carried infinitely often, and has length that does not converge to 0. The bases b and b^* have the same limiting position.*

Without loss of generality, it may be assumed that if a band participates, it does so infinitely often. It was seen in the proof of proposition 3.9 that unless the process terminates no bands decrease in weight, so it may also be assumed that all bands' weights stay the same.

Let D_i denote the total measure of the subset of points of Γ_i between the initial point and the minimal point of Γ that lies in a band that never participates. Also let

$$\tau_i = \sum_b \text{length}(b) w(b)$$

where the sum is over all bases b that eventually participate. Now set

$$\text{excess}(X_i) = \tau_i - 2D_i.$$

The $\text{excess}(X_i)$ measures the extent to which X_i fails to be of the surface case.

Lemma 3.12 $\text{excess}(X_i) > 0$ *and* $\text{excess}(X_i) = \text{excess}(X_{i+1})$.

The proof is left as an exercise.

Proof of proposition 3.11: Assume that neither 1 nor 2 occur. Let b be a base which carries infinitely often. By lemma 3.12 there exists a band which carries infinitely often whose length does not converge to 0. Suppose that b and b^* are carried only finitely many times; without loss of generality they are carried zero times. Then if b and b^* have disjoint interiors for all i, any base can only be carried by b finitely many times, a contradiction. Likewise if $B = b \times I$ is orientation reversing, any band can only be carried by b once, also a contradiction. Therefore suppose B has weight 1 and is orientation preserving. Under the assumption that b and b^* remain fixed, the translation length η of B is constant, and some base gets translated by η infinitely often. This contradicts remark 3.10.

Now suppose b has weight 1. Let η_i be the distance between the mid-points of b and b^*. Then any base carried by b moves a distance η_i. So $\eta_i \to 0$ as $i \to \infty$. *QED*

4 Machine Output

Let T be an $\mathbb{R}$-tree, G a finitely-generated group and X a resolving band complex with underlying union of bands Y and graph G. Applying the machine to a component $C(Y)$ of Y, there are four possibilities.

Definition 4.1

1. *The **simplicial** case, if $C(Y)$ is simplicial.*
2. *The **thin** case, in which eventually only process I is applied.*
3. *The **surface** case, in which eventually only process II is applied and* $\text{excess} = 0$.
4. *The **toral** case, in which eventually only process II is applied and* $\text{excess} > 0$.

Let $\overline{Y}$ denote the union of bands Y with the annuluses removed. For a component $C(Y)$ of Y, write $\overline{C(Y)}$ for $\overline{Y} \cap C(Y)$, and let H be the image of $\pi_1(C(Y))$ in Y. Let N be the normal subgroup normally generated by the images of the fundamental groups of the annuli in the closure of $C(Y) - \overline{C(Y)}$, and let $\overline{H} = H/N$.

4.1 The Surface Case

Proposition 4.2 *Suppose $C(Y)$ is of the surface type. Then the following holds.*

1. *Let L be a leaf of $C(Y)$. The image of $\pi_1(L)$ admits a map onto a cyclic group such that the kernel is contained in the kernel of the action of H on T_H.*
2. *Suppose X is pure. Then there is a short exact sequence*

$$1 \to \ker(G \curvearrowright T) \to G \to \pi_1(O) \to 1$$

where O is a 2-dimensional cone-type orbifold.

Proof: Assume that the machine has been iterated enough so that only process II will be applied to $C(Y)$, and excess $= 0$. Applying lemma 3.3 it may also be assumed that there are no Möbius bands. Therefore $\overline{C(Y)}$ is a surface with boundary.

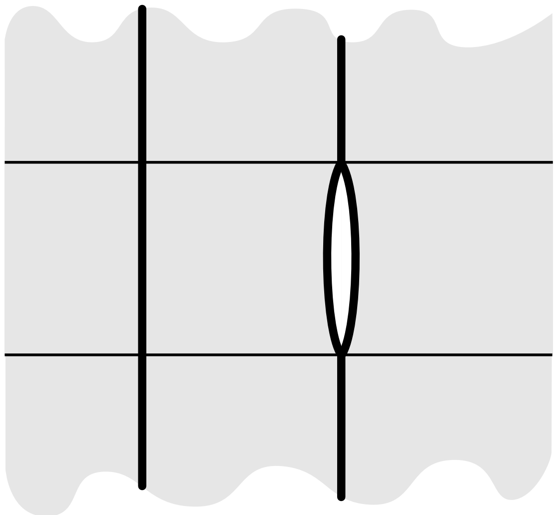

Figure 14: Leaves in the surface case. A non-singular leaf on the left and a singular leaf on the right are in bold.

Let L be a leaf in $\overline{C(Y)}$. Note that any point in L which separates it into more than 2 components must lie on the boundary of $C(Y)$. So if L is non-singular it is homeomorphic to $\mathbb{R}$, and if L is singular it is a boundary circle with finitely many rays attached – if it contained two or more boundary components then an interval between them would be a fault. So all leaves in $\overline{C(Y)}$ have cyclic fundamental group.

Now let L be a closed vertical loop in an annulus $C(Y)$. Then the image of $\pi_1(L)$ in G is a cyclic subgroup which fixes an arc in T, and so by the Kazhdan-Margulis theorem lies in the kernel of the action of H on T_H.

Finally let L be a general leaf in $C(Y)$. Then $\pi_1(L)$ has a normal subgroup $N \cap \pi_1(L)$, corresponding to conjugates of vertical loops in closed annuluses, which fixes T_H. The quotient group corresponds to the fundamental group of a leaf in $\overline{C(Y)}$, and so is cyclic. This proves the first assertion.

Now suppose that X is pure, so $Y = C(Y)$ and $H = G$. There is the short exact sequence

$$1 \to N \to G \to \overline{H} \to 1.$$

N certainly lies in the kernel of the action of H on T_H, and since annuluses are the only way in which trivial actions can arise, in fact N is that kernel. Now $\overline{H}$ is the fundamental group of $\overline{Y}$ with some 2-cells attached. This is naturally the fundamental group of a 2-dimensional cone-type orbifold, with singular locus where the attaching maps have degree greater than 1. *QED*

4.2 The Toral Case

Let $C(Y)$ be a toral component of the union of bands Y underlying the band complex X, and let $X_1, X_2, \ldots$ be a sequence of band complexes obtained by applying process II to the component $C(Y)$. By proposition 3.11 it may be assumed that $C(Y_i)$ has only one block, J. Each base b of $C(Y)$ determines a partial isometry $h(b)$ of J, which can be extended to an isometry of $\mathbb{R}$. Thus $C(Y_i)$ determines a (finitely generated) group $\mathcal{B}_i$ of isometries of $\mathbb{R}$. Since slides correspond to compositions of elements and collapses of proper subarcs do not change the group, it follows that $\mathcal{B}_i = \mathcal{B}_j$ for all i, j. Denote this constant group $\mathcal{B}$, the *Bass group*. Of course each element of $\mathcal{B}_i$ is either a translation or a reflection.

Lemma 4.3 *Let $\mathcal{B}$ be a finitely generated group of isometries of $\mathbb{R}$.*

1. *Let τ_i be a sequence of translations in $\mathcal{B} - \{1\}$ converging to the identity. Then there exist i, j such that the ratio of the translation lengths of τ_i and τ_j is irrational.*
2. *Let σ_i be a sequence of reflections in $\mathcal{B} - \{1\}$ converging to a reflection in $\mathcal{B}$. Then there exist i, j, k such that the ratio of the translation lengths of $\sigma_i\sigma_j$ and $\sigma_i\sigma_k$ is irrational.*

Proposition 4.4 *Moves can be applied to transform X to a band complex X' having two orientation-preserving bands A, A' determining partial isometries α, α' such that the domain of the composition $\alpha\alpha'\alpha^{-1}\alpha'^{-1}$ is a non-degenerate segment and the ratio of the translation lengths of α and α' is irrational.*

Proof: By proposition 3.11 there is a base b which carries and is carried infinitely often. Let b_i be the base corresponding to b in X_i, the complex obtained by applying process II i times. Then the corresponding partial isometries $h(b_i)$ form a convergent sequence, and are either all orientation-preserving or orientation-reversing. If they are all orientation-preserving then the limit is the identity. In either case, apply lemma 4.3.

Suppose first that the $h(b_i)$ are orientation-preserving. Then there are i, j such that the ratio of the translation lengths of $h(b_i), h(b_j)$ are irrational. Use lemma 3.2 to attach bands $A = a \times I, A' = a' \times I$ to $J \subset X$ by identifying a and a^* with b_i and b_i^* respectively, and identifying a' and a'^* with b_j and b_j^* respectively. Since the limiting position of b_i and b_i^* is the same, it follows that for sufficiently large i, j the result of this construction is gives a non-degenerate domain for $\alpha\alpha'\alpha^{-1}\alpha'^{-1}$.

Now suppose the $h(b_i)$ are orientation-reversing. Then there are i, j, k such that the ratio of the translation lengths of $h(b_i)h(b_j)$ and $h(b_i)h(b_k)$ is irrational. Choose i, j, k sufficiently large that $b_i \cap b_j$ and $b_i \cap b_k$ are non-degenerate segments. Attach bands $B_i = b_i \times I, B_j = b_j \times I, B_k = b_k \times I$ to J in X using lemma 3.2, and subdivide B_j and B_k so that $B'_j = b_i \cap b_j \times I, B'_k = b_i \cap b_k \times I$ are bands. Then slide B'_j and B'_k over B_i to get orientation-preserving bands A, A' with corresponding translation lengths equal to those of $h(b_i)h(b_j)$ and $h(b_i)h(b_k)$ respectively. Again, the domain of $\alpha\alpha'\alpha^{-1}\alpha'^{-1}$ is a non-degenerate segment for i, j, k sufficiently large. *QED*

Proposition 4.5 *The group H has an invariant line in T.*

This has an immediate corollary. The second assertion follows from the observation that π fixes a point in the invariant line.

Corollary 4.6 *1. There is a short exact sequence*

$$1 \to \ker(H \curvearrowright T_H) \to H \to \mathcal{A} \to 1$$

where $\mathcal{A}$ contains a free abelian subgroup of index 2.

2. Let L be a leaf in $C(Y)$, and let π be the image of $\pi_1(L)$ in H. Then the following short exact sequence holds.

$$1 \to \kappa \to \pi \to \Phi \to 1$$

Here κ acts trivially on the invariant line in T and Φ is a group of order at most 2.

Proof of proposition 4.5: Let A, A' be as in the previous lemma, and let α, α' be the corresponding elements of H. Without loss of generality suppose the base-point of X lies in J. Then α is obtained by concatenating a vertical path in A and a horizontal path in J to form a loop; α' is obtained from A' similarly. Let $\rho(J)$ be a segment in T, so α, α' are partial orientation-preserving isometries of J. Then the commutator $[\alpha, \alpha']$ fixes a non-degenerate subarc of $\rho(J)$, and hence by the Kazhdan-Margulis theorem fixes T_H. It follows that α, α' are hyperbolic isometries with the same axis, L. Let F be the group of isometries of L generated by α and α'. Since the ratio of the translation lengths of α and

α' is irrational, there are arbitrarily small translation in F. Recall that H is generated by the set S_J, the set of J-short loops. It has to be shown that $\gamma \in S_J$ fixes L. Let $s \subset \rho(J)$ be a subarc such that $\gamma(s) \subset \rho(J)$.

Suppose first that the partial isometry of $\rho(J)$ induced by γ is orientation-preserving. Then it follows that γ is hyperbolic, and its axis contains $s \cup \gamma(s)$. Let ϵ be the length of the smallest connected segment containing $s \cup \gamma(s)$. Choose $w \in F$ with translation length less than the length of s, so that $l(w) + l(\gamma) < \epsilon$. Then it follows from corollary 2.14 that the axis of γ is L.

Now suppose that the partial isometry of $\rho(J)$ induced by γ is orientation-reversing. Let m, m' be the midpoints of $s, \gamma(s)$ respectively, and let t' be the interval centered at m' with length half the length of s. Choose $w \in F$ with translation length less than the length of t'. Now $w' = \gamma w \gamma^{-1}$ is a partial orientation-preserving isometry of $\rho(J)$ mapping t' into s'. It follows that the axis of w' is L, and thence that γ preserves L. *QED*

4.3 The Thin case

Proposition 4.7 *Let $C(Y)$ be of thin type, so there is an infinite sequence of band complexes $X_0, X_1, \ldots$ obtained by applying process I recursively to $C(Y)$. Then G splits over a subgroup that fixes an arc in T. If $C(Y)$ has no zero-weight bands then the decomposition is a free product.*

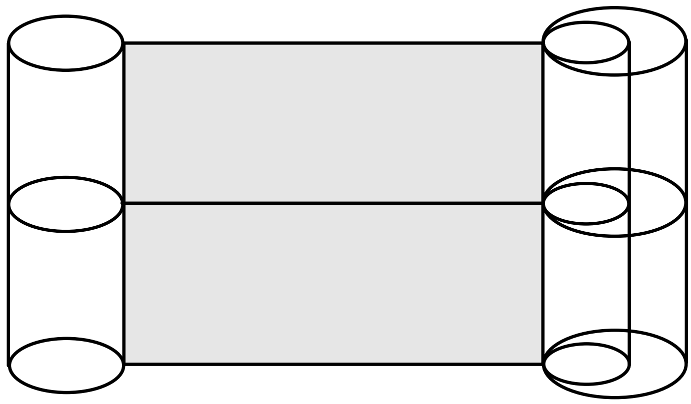

Figure 15: The band B_n with subdivision annuli, and the graph Λ_n marked in the middle.

Proof: Let $\mathcal{A}_n$ denote the set of subdivision annuli in X_n. By proposition 3.8 eventually there is a band $B_n = b_n \times I$ of weight 1 that meets the closure of $X_n - (Y_n \cup \mathcal{A}_n)$ only in a subset of $b_n \cup b_n^*$. This band is formed by a collapse under choice I3. For any integer $t > 0$ there is a $k \geq 1$ such that $B_n \cap Y_{n+k}$ has at least t components.

Let $h_n = b_n \times \{\frac{1}{2}\}$, and let $A_{1,n} \ldots, A_{m,n} \in \mathcal{A}_n$ be the subdivision annuli intersecting B_n in the endpoints of h_n. Each is of the form

$$\alpha_{i,j} : S^1 \times I \to X_n$$

for $\alpha_{i,j}$ a continuous map. Let $\pi : S^1 \times I \to I$ be projection onto the right factor. So for each i there is a collection of horizontal circles $C_{i,n}$ given by $\alpha_{i,n}(\pi^{-1}(\pi(\alpha_{i,n}^{-1}(\partial h_n))))$ in X_n. Then $\Lambda_n = h_n \cup C_{1,n} \cup \ldots \cup C_{m,n}$ is a finite graph consisting of two bouquets of circles connected by an edge, and locally separates X_n. So X_n can be thought of as a graph of spaces with edge space Λ_n and vertex spaces the closures of the components of $X_n - \Lambda_n$.

Note that the generators of the fundamental groups of the circles $C_{i,n}$ act trivially on T_H (since they are homotopic to leaves of 0-weight bands). So the edge group does fix an arc in T. It remains to show that the splitting is non-trivial.

If $X_n - \Lambda_n$ has only one component then G splits as an HNN-extension. So it may be assumed that $X_n - \Lambda_n$ has two components, X'_n containing b and X''_n containing b^*. Applying process I further would eventually divide B_n into arbitrarily many bands. Recall that the number of long bands is bounded above. Therefore it follows that there exist paths $p' : I \to X'_n, p'' : I \to X''_n$ and long bands B'_n, B''_n such that:

1. $p'(I), p''(I)$ are contained respectively in non-singular leaves L', L'';
2. $L' \subset B'_n$ and $L'' \subset B''_n$;
3. $p'(I) \cap B_n \subset b, p''(I) \cap B_n \subset b^*$ each consist of two distinct points.

In other words, both paths have non-trivial holonomy, and so each vertex group acts non-trivially on T. Since the edge group fixes T, it is a proper subgroup of each of the vertex subgroups, so the splitting is non-trivial. *QED*

5 The Decomposition Theorem

Let G be a finitely presentable group with a stable action on a tree T.

Theorem 5.1 *If the action of G on T is not pure then G splits (possibly trivially) over a subgroup E which fits into a short exact sequence*

$$1 \to K_E \to E \to C \to 1$$

where K_E fixes an arc in T and C is finite or cyclic. E fixes a point in T.

If the action of G on T is pure, one of the following holds.

1. ***Surface type.*** *G satisfies*

$$1 \to K \to G \to \pi_1(O) \to 1$$

where K is the kernel of the action of G on T and O is a 2-dimensional cone-type orbifold.

2. ***Toral type.*** *Without loss of generality T is a line and*

$$1 \to K \to G \to A \to 1$$

where K is the kernel of the action of G on T and A is a subgroup of $\mathrm{Isom}(\mathbb{R})$.

3. ***Thin type.*** *G splits over a subgroup that fixes an arc of T.*

Proof: By theorem 2.9 there exists a resolving band complex X. By lemma 3.1, applying a finite number of moves to X, it may be assumed that X has no faults. Now the Rips Machine is applied to each component $C(Y)$ of the union of bands Y.

If the action is pure, the result follows immediately from propositions 4.2 and 4.7 and corollary 4.6. Further, if any component of Y is of thin type the result follows from proposition 4.7. Therefore it may be assumed that the action is not pure and no component of Y is of thin type.

By theorem 2.11 X has a graph-of-spaces decomposition, with the action of the fundamental group of each vertex space on T either pure or trivial, and the action of the fundamental group of each edge space trivial. Therefore the assumption that the action is not pure implies that this decomposition is non-trivial. Let E be an edge group of the corresponding graph-of-groups decomposition of G. Then G splits over E, and E fixes a point in T. Then E is contained in the image of the fundamental group of some leaf in some component $C(Y)$. If $C(Y)$ is minimal then E satisfies the theorem's assertion by proposition 4.2 and corollary 4.6. Therefore suppose $C(Y)$ is simplicial. Then $C(Y)$ is an I-bundle over a leaf, so a finite covering $\widetilde{C(Y)}$ of $C(Y)$ is a trivial I-bundle, having fundamental group which fixes an arc in T. So the theorem holds. *QED*

References

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