Characterization of covering maps via path-lifting property

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A continuous map between topological $f : X \to Y$ is said to satisfy *the path-lifting property* if for any path $p : [0, 1] \to Y$ and any $x \in f^{-1}(p(0))$ there exists a *lifting* $\tilde{p}$ *of the path* p *with the intitial value* x, i.e. there exists a path $\tilde{p}$ such that $f \circ \tilde{p} = p$ and $\tilde{p}(0) = x$.

Similarly, a smooth map between Riemannian manifolds $f : X \to Y$ is said to satisfy *the rectifiable path-lifting property* if the above definition holds for the rectifiable paths $p(t)$.

Suppose that $f : X \to Y$ is a local homeomorphism (resp. diffeomorphism) between topological spaces X and Y (resp. Riemannian manifolds X and Y).

Lemma 0.1. *f satisfies tha path-lifting (resp. rectifiable path-lifting) property if and only if the following holds: For each continuous (resp. rectifiable) path $q : [0, T] \to Y$ and each partial lift $\tilde{q} : [0, T) \to Y$ extends continuously to the point $t = T$.*

Proof: The implication $\Rightarrow$ is clear, we will prove the other implication. We will use the standard arguments of the covering theory: Let $A \subset [0, 1]$ denote the largest subinterval on which a lift $\tilde{p}$ of the path p (with the initial value x) exists. This subset is nonempty (since $0 \in A$). Suppose that A is a half-open interval $[0, T), T \leq 1$. Then, by our assumption the lift $\tilde{p}$ existens continuously to the point T. Thus $A = [0, T]$ is a closed interval, it remains to show that $T = 1$. Suppose that $T < 1$. Let U denote a neighborhood of $x := \tilde{p}(T)$ which maps homeomorphically (by f) onto a neighborhood V of the point $y := p(T)$. Then there exists $0 < \epsilon < 1 - T$ such that $p([T, T + \epsilon)) \subset V$ and we define the lift $\tilde{p}$ on $[T, T + \epsilon)$ by

$$f^{-1} \circ p : [T, T + \epsilon) \to U.$$

This contradicts maximality of A. □

It is a standard fact of the covering theory that if f is a covering map then f satisfies the path-lifting property.

Theorem 0.2. *Suppose that X and Y are connected, semilocally simply-connected (e.g. are manifolds or cell-complexes), resp. Riemannian manifolds and $f : X \to Y$ is a local homeomorphism (resp. diffeomorphism) which satisfies the path-lifting (resp. rectifiable path-lifting) property. Then f is a covering map.*

Proof: Let $\tilde{X}$ denote the universal cover of X and let $g : \tilde{X} \to \tilde{Y}$ denote a lift of f. It suffices to show that g is a homeomorphism (resp. diffeomorphism).

Lemma 0.3. *g satisfies the path-lifting (resp. rectifiable path-lifting) property.*

Proof: Let $q : [0,1] \to \tilde{Y}$ be a (rectificable) path in $\tilde{Y}$, p be its projection to X and $\tilde{x} \in \tilde{X}$ be such that $g(\tilde{x}) = q(0)$. Let x denote the projection of $\tilde{x}$ to X, then $f(x) = p(0)$. Thus there exists a lift $\tilde{p} : [0,1] \to X$ of the path p with the initial value x. Then, since $\tilde{X} \to X$ is a covering, the path p lifts to a path $\tilde{q} : [0,1] \to \tilde{X}$ such that $\tilde{q}(0) = \tilde{x}$. it is clear from the construction that $\tilde{q}$ is the required lift of the path q. □

Lemma 0.4. *The mapping g is onto.*

Proof: Suppose that g is not onto. Then, since $\tilde{Y}$ is connected, there exists a (rectifiable) path $p : [0,1] \to \tilde{Y}$ so that $p(0) = g(\tilde{x}) \in g(\tilde{X})$ and $p(1) \notin g(\tilde{X})$. Then the path p does not admit a lift with the initial value $\tilde{x}$, which is a contradiction. □

Thus it suffices to show that g is 1-1. We first consider the easier topological setting:

Lemma 0.5. *In case g satisfies the path-lifting property, the map g is 1-1.*

Proof: We imitate the usual arguments of the covering theory. Suppose that $x, x' \in \tilde{X}$ be distinct points such that $y = g(x) = g(x')$. Let $\alpha : [0,1] \to \tilde{X}$ be a path connecting x to x'. The composition $\beta := g \circ \alpha$ is a loop in $\tilde{Y}$. Hence, since $\tilde{Y}$ is simply-connected, there exists a continuous map

$$H : [0,1] \times [0,1] \to \tilde{Y}$$

so that $H(1,s) = y = H(t,0) = H(t,1)$ for all $s,t \in [0,1]$ and $H(t,0) = \beta(t)$. Our goal is to show that the homotopy H admits a lift $\tilde{H}$ to $\tilde{X}$, which again satisfies:

$x = \tilde{H}(t,0), x' = \tilde{H}(t,1)$ for all $t \in [0,1]$ and $H(t,0) = \alpha(t)$.

This would yield a contradiction since $x \neq x'$. Let $A \subset [0,1] \times [0,1]$ be a maximal rectangle on which the lift $\tilde{H}$ exists, this rectangle contains the segment $[0,1] \times \{0\}$ (use α as the lift of β). By the same covering theory arguments (as in the proof of Lemma 0.1), if the maximal rectangle A is closed then it coinsides with $[0,1] \times [0,1]$ and we are done. Suppose that A is a half-open rectangle: $A = [0,1] \times [0,S)$. Let $\tilde{H} : A \to \tilde{X}$ denote the required lift of H. Suppose that H does not admit a continuous extension to a point $u := (t,S)$, for some $0 \le t \le 1$. This means that there are sequences $z_i, w_i \in A$ convergent to u such that

$$\lim_i \tilde{H}(z_i) = a \neq b = \lim_i \tilde{H}(w_i).$$

Let $\gamma : [0,1) \to A$ denote the piecewise-linear path in A which connects z_1 to w_1, w_1 to z_2, z_2 to w_2, etc. Since $\lim_i z_i = u = \lim_i w_i$, the path γ extends continuously to the point 1, $\gamma(1) = u$. Thus the composition $H \circ \gamma : [0,1] \to \tilde{Y}$ is a continuous path which has the partial lift

$$\tilde{\gamma} := \tilde{H} \circ \gamma : [0,1) \to \tilde{X}.$$

However, since $a \neq b$, the path $\tilde{\gamma}$ does not extend continuously to the point 1. This contradicts the path-lifting property of g. $\square$

We now modify the above arguments in the setting of Riemannian manifolds:

Lemma 0.6. *In case g satisfies the rectifiable path-lifting property, the map g is 1-1.*

Proof: We follow the proof of Lemma 0.5, modifying it when necessary. We will take α a smooth curve in $\tilde{X}$, then β is smooth as well and hence there exists a smooth homotopy H. We again argue that the maximal rectangle A is closed. Note that if the path $\gamma : [0,1] \to [0,1] \times [0,1]$ in the proof of Lemma 0.5 was rectifiable, its image $H \circ \gamma$ would be rectifiable as well and we would get a contradiction as before. Apriori however γ has infinite length. Note that instead of the original sequences z_i and w_i we can freely choose their subsequences: the limits a and b would be still different.

We therefore choose subsequences (again denoted $z_i, w_i \in A$) such that

$$d(z_i, u) < 2^{-i-1}, d(w_i, u) < 2^{-i-2}, \forall i.$$

Then

$$d(z_i, w_i) + d(w_i, z_{i+1}) < 2^{-i}, \forall i,$$

and hence the curve γ is rectifiable. $\square$

This also concludes the proof of Theorem 0.2. $\square$