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# SOME PROPERTIES OF DEVELOPMENTS OF CONFORMAL STRUCTURES ON THREE-DIMENSIONAL MANIFOLDS

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**1.** The study of the properties of developments of conformal structures begun in [1] is continued in this paper. Definitions of the conformal structure, development, holonomy homomorphism and holonomy group can be found, for example, in [1] or [2]. In what follows the development of a conformal structure on $M$ will always be denoted by $d\colon \tilde{M} \to D = d(\tilde{M}) \subset S^n$; $p\colon \tilde{M} \to M$ is the universal covering; also, $G$ will be the group of covering transformations, $d_*\colon G \to H = d_*(G)$ the holonomy homomorphism, $H$ the holonomy group, and $D$ the domain of the development. The structure $K$ is said to be *relatively complete* if $d\colon \tilde{M} \to D$ is a covering. If $M$ is a compact three-dimensional manifold with $|\pi_1(M)| = \infty$, then from [1] it follows that a structure being relatively complete is equivalent to $D$ being distinct from $S^3$ and equivalent (excluding a certain narrow class of structures) to the action of the group $H$ on $D$ being discontinuous. Our aim is to characterize relatively complete conformal structures on certain classes of three-dimensional manifolds in terms of the holonomy group.

*A Schottky manifold of genus* $(r,p)$ is defined to be the connected sum of $r$ manifolds homeomorphic to $S^2 \times S^1$ and of $p$ manifolds homeomorphic to $S^1 \times S^1 \times S^1$ (here $r + p > 0$). An orientable closed three-dimensional manifold will be called an *almost trivial Seifert fibration* (ATSF) if it is finitely covered by $S_g \times S^1$, where $S_g$ is a surface of genus $g > 1$. It is known (see [3]) that $M$ is an ATSF if and only if it is closed, orientable and admits an $(\mathbf{H}^2 \times \mathbf{R}, \mathrm{Isom}(\mathbf{H}^2 \times \mathbf{R}))$-structure. Let $\mathrm{M}(3)$ be the group of all orientation-preserving Möbius transformations of $S^3$. If $\Gamma$ is a discrete group, its discontinuity set will be denoted by $R(\Gamma)$, and the limit set $L(\Gamma) = S^3 \backslash R(\Gamma)$. A group $G \subset \mathrm{M}(3)$ is called a *Schottky group of genus* $(r,p)$ if it is obtained by a Klein combination of $r$ cyclic loxodromic groups (with the spherical fibers as the fundamental domains) and $p$ parabolical free Abelian groups of rank 3 (with fundamental domains homeomorphic to a parallelepiped). Let $L = \{x \in R^3\colon x_2 = x_3 = 0\} \cup \{\infty\}$ and $\mathrm{M}(L) = \{\gamma \in \mathrm{M}(3); \gamma(L) = L\}$. A group $H \subset \mathrm{M}(L)$ is said to be *almost discrete* if $H|_L$ is a discrete group and the subgroup of $H$ consisting of rotations around $L$ is isomorphic to $\mathbf{Z}$. If $H \subset \mathrm{M}(L)$ and $L(H) = L$, then $H$ is called a *Fuchsian* group. If $G \subset \mathrm{M}(3)$ is conjugate to the Fuchsian group by a homeomorphism, then $G$ will be called a *quasi-Fuchsian group.*

**2.** In the formulations of the theorems it will be assumed everywhere that $(M,K)$ is a closed three-dimensional conformal manifold.

THEOREM 1. (a) *Let the holonomy group $H$ of the manifold $(M,K)$ be a Schottky group of genus $(r,p)$. Then the domain of the development is $R(H)$, $d\colon \tilde{M} \to D$ is a homeomorphism, and $M$ is a Schottky manifold of genus $(r,p)$.*

(b) *Let $M$ be a Schottky manifold and let $K$ be a relatively complete conformal structure on $M$. The holonomy group $H$ is a Schottky group.*

THEOREM 2. *If $M$ is an almost trivial Seifert fibration, then there exists a Fuchsian group $H$ acting freely in $S^2 \backslash L$ such that $M$ is homeomorphic to $R(H)/H$.*

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THEOREM 3. (a) *If the holonomy group $H$ of the manifold $(M, K)$ is either quasi-Fuchsian or conjugate in* $\mathrm{M}(3)$ *to the subgroup* $\mathrm{M}(L)$, *then $K$ is a relatively complete conformal structure and either $M$ is an* ATSF *or is finitely covered by a Schottky manifold of genus $(r, 0)$, or $M$ is a lens space.*

(b) *If $M$ is an* ATSF *and $K$ is a relatively complete conformal structure on $M$, then the holonomy group $H$ is either quasi-Fuchsian or almost discrete.*

**3.** For the proof of Theorems 1 and 3 we need the following lemma.

LEMMA. *Let $(M, K)$ be a compact conformal $n$-dimensional manifold, and let $N$ be a closed proper subset of $S^n$ containing more than one point and invariant under the holonomy group $H$. Let $d^{-1}(D\backslash N) = \bigcup_{i\in I} \tilde{M}_i$ be the decomposition into connected components and $d_i\colon \tilde{M}_i \to D_i = d_i(\tilde{M}_i)$ the restriction of $d$ to $\tilde{M}_i$.*

*Then, for any $i \in I$, $d_i\colon \tilde{M}_i \to D_i$ is a covering.*

PROOF OF THEOREM 1. (a) It is not hard to see that the limit set of the Schottky group $H$ is a discontinuum with simply-connected complement and $R(H)/H$ is a Schottky manifold of the same genus as the group $H$. Assertion (a) is implied by these two facts and the lemma.

(b) Let $M$ be a Schottky manifold and $K$ a relatively complete conformal structure. In this case the domain of the development $D$ is an invariant component of $H$ (see [1]). We choose in $H$, if necessary, a subgroup $H_0$ of finite index without torsion, and we consider the subgroup $G_0 = d_*^{-1}(H_0)$ which has finite index in $G$. It is not hard to see that $M_0 = \tilde{M}/G_0$ is a finite-sheeted covering of $M$ and $D/H_0$. Kurosh's theorem on the subgroup of a free product, Kneser's theorem, and the fact that $M$ is a Poincaré manifold easily imply that $M_0$ is again a Schottky manifold. We prove that $H_0$ is a finite extension of a Schottky group (in this case $\pi_1(R(H)) = \{1\}$ and $d\colon \tilde{M} \to D$ is a homeomorphism).

Let $M_0 = R(\Gamma)/\Gamma$, where $\Gamma$ is a Schottky group; then $D/H_0 = X = R(\Gamma)/F$, where $F$ is the group of homeomorphisms containing $\Gamma$ as a subgroup of finite index (here we consider $M_0$ and $X$ as topological manifolds without conformal structure). Using the results of [4] we obtain $X = Y \# A_1 \# \cdots \# A_q$, where $\#$ is the symbol for the connected sum, $|\pi_1(A_i)| < \infty$, and $\pi_1(Y)$ is a torsion-free group. Since $R(\Gamma)$ is simply-connected, then $F = E_1 * \cdots * E_q * F_1$, where $E_i \simeq \pi_1(A_i)$, $F_1 \simeq \pi_1(Y)$, and $L(F_1) \subset L(F) = L(\Gamma)$ and is also a discontinuum. The results in [5] imply that $F_1 \simeq R_1 * \cdots * R_s$, where $L(R_j)$ is a singleton or a two-point set and each of the groups $R_j$ contains a Möbius Abelian subgroup of finite index. Since $X$ is covered by a domain in $S^3$, then $X$ is a Poincaré manifold and again applying Kneser's theorem it is not hard to see that

$$Y = T_1 \# \cdots \# T_k \# (S^2 \times S^1) \# \cdots \# (S^2 \times S^1),$$

where $T_i$ has $S^1 \times S^1 \times S^1$ as a finite-sheeted covering. Lifting the spheres (partitioning into a connected sum) to the domain $D$ and passing, if necessary, to a subgroup of finite index in $H_0$, we ascertain that $H_0$ is actually a finite extension of a Schottky group (see [6]).

PROOF OF THEOREM 2. Following [1], we identify $\mathbf{H}^2 \times \mathbf{R}$ with the space $Y = \{(x, r, \varphi) \in \mathbf{R}^3; r > 0\}$ on which we introduce the metric $ds^2 = (dx^2 + dr^2)/r^2 + d\varphi^2$. Let $q\colon Y \to S^3\backslash L$, $q(x, r, \varphi) = (x, r\cos\varphi, r\sin\varphi)$. If $G \subset \mathrm{Isom}(Y, ds^2)$, then there is a natural homomorphism $d_*\colon G \to \mathrm{M}(L)$. Since $M$ is an almost trivial Seifert fibration, there exists a group $G \subset \mathrm{Isom}(Y, ds^2)$, acting freely and disconnectedly on $Y$, such that $Y/G$ is homeomorphic to $M$. It is easy to see that $G$ can be chosen so that the maximal normal cyclic subgroup in $G$ is generated by the shift $k\colon (x, r, \varphi) \to (x, r, \varphi + 2\pi)$. Then $q_*(G) = H$ is a Fuchsian group acting freely on $S^3\backslash L$. Obviously $M = Y/G = (S^3\backslash L)/H$.

PROOF OF THEOREM 3. (a) Let $N = L$ if $H \subset \mathrm{M}(L)$, and $N = L(H)$ if $H$ is a quasi-Fuchsian group. Without loss of generality we may suppose that $|\pi_1(M)| = \infty$, as otherwise $(M, K) = S^3/G$, where $G$ is a finite Möbius group leaving $L$ invariant (it is easy to see that in this case $M$ is a lens space). We assume that $D = S^3$. Then $d^{-1}(S^3\backslash N)$ is a connected set and the lemma implies that $d_1: \tilde{M}_1 \to S^3\backslash N$ is a covering (here $\tilde{M}_1 = d^{-1}(S^3\backslash N)$). If $d_1$ is a homeomorphism, then arguments analogous to the proof of assertion (a) of Theorem 1 lead immediately to a contradiction. Let $d_1: \tilde{M}_1 \to S^3\backslash N$ be a nontrivial covering, $x \in N$, $y \in d^{-1}(x)$, and let $U$ and $V$ be neighborhoods of $x$ and $y$ respectively such that $d|_V: V \to U$ is a homeomorphism. We choose a loop $\gamma$ in $U\backslash N$ such that $\langle[\gamma]\rangle$ is a subgroup determining the covering $d_1$. Let the left of $\gamma$ be the path $\tilde{\gamma}$ in $V$. Obviously $d_1: \tilde{\gamma} \to \gamma$ is not a homeomorphism. The contradiction thus obtained proves that $D \neq S^3$ and $K$ is a relatively complete structure. If $H$ is an almost discrete group, then $M$ is an ATSF (see [1]). If $H$ is quasi-Fuchsian, then $M$ is finitely covered by the manifold $M_0 = \tilde{M}/G_0$, where $G_0 = d_*^{-1}(H_0)$ is a torsion-free subgroup of finite index in $H$. Obviously $R(H)/H_0$ is an ATSF, $M_0$ covers $R(H)/H_0$, and consequently $M_0$ and $M$ are also ATSF. As $M$ is compact, only one possibility remains: $L(H)$ is a discontinuum lying on $L$ and $R(H)/H$ is a compact manifold. It is not hard to see that such a group $H$ is a finite extension of a Schottky group of genus $(r, 0)$ and $M = R(H)/H$.

(b) Let $M$ be an ATSF, and $K$ a relatively complete conformal structure on $M$. If the holonomy group $H$ is not discrete, then it is almost discrete (see [1]); consequently we need only consider the case of a discontinuous action $H$ on $D$. Let $H_0$ be a torsion-free subgroup of finite index in $H$, $G_0 = d_*^{-1}(H_0)$, $M_0 = \tilde{M}/G_0$, and $R = D/H_0$. It is easy to see that $R$ is an ATSF. Therefore $R$ admits an $S^1$-action (see [3] or [7]). Since $H$ is a discrete group, then $K \subset \ker d_*$, where $K$ is a maximal normal cyclic subgroup of $G_0$. It is not hard to prove that in this case the $S^1$-action on $R$ lifts to an $S^1$-action on $D$ inducing the identity automorphism of $H_0$. Arguing analogously to [8], we can extend this $S^1$-action to the whole sphere $S^3$, and $S^1$ will act on $S^3\backslash D$ as the identity. Using the results of [9] it is easy to prove that $S^3\backslash D$ is an unknotted topological circle in $S^3$ and $R(H_0) = R(H) = D$. Since $R = D/H_0$ is an ATSF, application of Theorem 2 immediately gives us that $H_0$ is a quasi-Fuchsian group. One may suppose that the branched covering $R \to D/H$ is regular and has the covering group $\Gamma$. The results of [10] imply that the action of the finite group $\Gamma$ is equivalent to the action of the finite group of automorphisms of the conformal structure $K'$ introduced on $R$ by the Fuchsian group $F$ $((R, K') = (S^3\backslash L)/F)$. This now implies assertion (b) in the theorem.

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