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ON CONFORMAL STRUCTURES WITH FUCHSIAN HOLONOMY

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1. W. M. Goldman, in [1], has described a "grafting" procedure for a flat conformal structure, generalizing to higher dimensions Maskit's construction [2] of a surjective development. He proved that if a representation of the holonomy of a complex projective structure (on a closed surface) is faithful and has a Fuchsian image H, then the structure is obtained by such a grafting operation from the standard structure on Δ/H, where Δ is the unit disk. He also conjectured that a similar statement can be made for flat conformal structures in higher dimensions. The present note proves this conjecture for three-dimensional manifolds.

2. The definitions of flat conformal structure (here simply to be called conformal), development, holonomy homomorphism, and holonomy group can all be found in [3]–[5]. Let $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$, $\Delta = \{x \in \mathbf{R}^n : |x| < 1\}$, $\Sigma = \{x \in \mathbf{R}^n : |x| = 1\}$, and $\Delta^* = \overline{\mathbf{R}}^n \backslash (\Delta \cup \Sigma)$, and let $\mathscr{M}_n$ be the group of orientation-preserving Möbius automorphisms of $\overline{\mathbf{R}}^n$. If $G \in \mathscr{M}_n$, we denote by $R(G)$ its set of discontinuity; and we put $L(G) = \overline{\mathbf{R}}^n \backslash R(G)$.

Let G be a subgroup of $\mathscr{M}_n$ such that $G(\Delta) = \Delta$, G acts on Δ freely and discontinuously, and $\Delta/G = M(G)$ is compact (i.e., G is a Fuchsian group). Throughout, $M(G)$ will denote the conformal manifold with fixed conformal structure K_0 dropped down from Δ. Let $S = \bigcup_1^m S_i$ be a family of incompressible aspherical connected closed $(n-1)$-submanifolds in $M(G)$, $\bigcup_1^m \widetilde{S}_i$ the lift of S in Δ, and Γ_i the stabilizer of $\widetilde{S}_i$ in G. Suppose the conformal manifold $(M_i, K_i) = R(\Gamma_i)/\Gamma_i$ is homeomorphic either to $S_i \times S^1$, where S^1 is a circle, or to a fiber bundle over the nonorientable hypersurface $S_i' = \widetilde{S}_i/\Gamma_i$ with fiber S^1. It is easily seen that the hypersurfaces S_i and S_i' have conformally equivalent neighborhoods in $M(G)$ and M_i, respectively. Slit the manifolds $M(G)$ and (M_i, K_i) along S_i and S_i', and paste corresponding "edges" of the slits conformally, for each $i = 1, \dots, m$ (see Figure 1, where the pasting is pictured along an orientable surface). The conformal manifold $M[S]$ so obtained is then homeomorphic to $M(G)$, and the image of the development $d: \widetilde{M}[S] \to \overline{\mathbf{R}}^n$ is all of $\overline{\mathbf{R}}^n$ (if $S \neq \varnothing$).

DEFINITION. The conformal manifold $M[S]$ is obtained by "grafting" along S on $M(G)$.

THEOREM. *Let M be a closed three-dimensional manifold, $M = \Delta/G$, where G is a Fuchsian group; and let K be an arbitrary conformal structure on M, the representation of whose holonomy $d_*: G \to \mathscr{M}_3$ is the identity. Then (M, K) is obtained by a grafting operation on $M(G)$. If the family S' is homotopic in M to the family S, then $M[S]$ is conformally equivalent to $M[S']$.*

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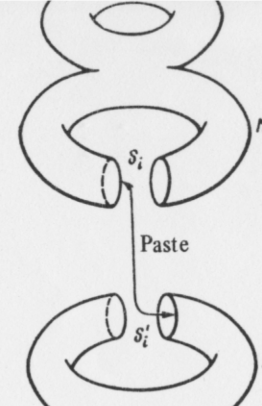

FIGURE 1

3. For the proof, we need three auxiliary lemmas.

LEMMA 1. *Let X be a manifold with boundary $\partial X = Y_1 \cup Y_2$, where the imbedding $Y_1 \to X$ is a homotopy equivalence, while Y_2 is a nonvoid compact orientable manifold. Then X is compact.*

Let us introduce the following *standard notation.* $p: \widetilde{M} \to M$ is the universal covering; G is the corresponding group of transformations of the covering space; $d: \widetilde{M} \to \overline{\mathbf{R}}^n$ is the development of a conformal structure K on M; and $d_*(G) = H$ is the holonomy group.

LEMMA 2 (see [3] and [6]). *Let (M, K) be a compact conformal manifold; E a closed subset of $\overline{\mathbf{R}}^n$ invariant with respect to H and consisting of more than one point; $\widetilde{M}_0 = \widetilde{M}\backslash d^{-1}(E)$; and $d_0 = d\,|_{\widetilde{M}_0}$. Then $d_0: \widetilde{M}_0 \to d(\widetilde{M})\backslash E$ is a covering.*

LEMMA 3. *Suppose $H(\Delta) = \Delta$, and (M, K) is compact and aspherical. Then $F = p(d^{-1}(\Sigma))$ is a finite union of incompressible aspherical closed hypersurfaces in M.*

PROOF. Obviously $d^{-1}(\Sigma)$ is closed in $\widetilde{M}$ and invariant with respect to G. Therefore F is compact and is a submanifold of codimension 1 in M (since p is a covering); and consequently F has only finitely many connected components. We assert that for any component $\widetilde{F}_1 \subset p^{-1}(F)$ we have $\pi_1(\widetilde{F}_1) = \{1\}$ and $\pi_k(\widetilde{F}_1) = 0$ for $k > 1$ (which implies incompressibility and asphericity for F). Indeed, let $\widetilde{M}_1^+$ and $\widetilde{M}_1^-$ be the components of $\widetilde{M}_0 = \widetilde{M}\backslash d^{-1}(\Sigma)$ adjoining $\widetilde{F}_1$. By Lemma 2, $d_0\,|_{\widetilde{M}_1^+}$ and $d_0\,|_{\widetilde{M}_1^-}$ are coverings; and since $d(\widetilde{M}_2^+) = \Delta$ and $d(\widetilde{M}_1^-) = \Delta^*$, these coverings are homeomorphisms. Let K^+ be the cone over $d(\widetilde{F}_1)$ with vertex at zero, and K^- its image under inversion with respect to Σ. The complex $Q = (\widetilde{M}\backslash \partial d^{-1}(K^+ \cup K^-)) \cup d^{-1}\{0, \infty\}$ is homotopically equivalent to $\widetilde{M}$. If $\pi_1(\widetilde{F}_1) \neq \{1\}$ or $\pi_k(\widetilde{F}_1) \neq 0$, $k > 1$, then the exactness of the homotopy sequence of the pair $(K^+, \widetilde{F}_1)$ implies that $T = \pi_{k+1}(K^+ \cup K^-) \neq 0$. But T occurs as a summand in $\pi_{k+1}(Q)$, and this contradicts the contractibility of Q.

4. **Proof of the Theorem.** Without loss of generality we can suppose that $d\colon \widetilde{M} = \Delta \to \overline{\mathbf{R}}^n$ preserves the orientation induced on Δ by its imbedding in $\mathbf{R}^n$ (otherwise we take the composite of d with inversion with respect to Σ), and that $d(\Delta) = \overline{\mathbf{R}}^n$ (otherwise (M, K) is uniformized by the group G; see [5]). Let

$$d^{-1}(\Delta) = \widetilde{M}^+ = \bigcup_{j=1}^{\infty} \widetilde{M}_j^+, \qquad d^{-1}(\Delta^*) = \bigcup_{j=1}^{\infty} \widetilde{M}_j^- = \widetilde{M}^-,$$

let G_j^+ and G_j^- be the stabilizers of $\widetilde{M}_j^+$ and $\widetilde{M}_j^-$ respectively, and $\Omega_j^{\pm} = R(G_j^{\pm}) \cap \Sigma$ the sets of discontinuity of the groups G_j^+ and G_j^- restricted to the sphere Σ.

It is easily seen that

$$d(\mathrm{cl}_\Delta \widetilde{M}_j^+) = \Delta \cup \Omega_j^+, \qquad d(\mathrm{cl}_\Delta \widetilde{M}_j^-) = \Delta^* \cup \Omega_j^-,$$

and the manifolds $d(\mathrm{cl}_\Delta \widetilde{M}_j^{\pm})/G_j^{\pm}$ are compact. It follows from Lemma 3 that all the connected components of the sets $\Omega_j^{\pm}$ are contractible.

Lemma 4. *For every j, the set Ω_j^- has exactly two connected components.*

Proof. Let $\widetilde{M}_1^-$ be a component of $\widetilde{M}^-$, $\widetilde{F}_1$ a component of $\partial \widetilde{M}_1^-$, $\widetilde{F}_1 \subset \partial \widetilde{M}_1^- \cap \partial \widetilde{M}_1^+$, and $W = d(\widetilde{F}_1) \subset \Omega_1^+ \cap \Omega_1^-$. Since the holonomy homomorphism d_* is the identity and $d\,|_{\widetilde{M}_1^+ \cup \widetilde{M}_1^-}$ is a homeomorphism (see Lemma 3), the stabilizer Γ_1 of $\widetilde{F}_1$ in the group G is the stabilizer of W in the groups G_1^+ and G_1^-. The hypersurface $\widetilde{F}_1$ separates Δ into two components U^+ and U^-, with $\widetilde{M}_1^{\pm} \subset U^{\pm}$. Since $G_1^{\pm}(\widetilde{M}_1^{\pm}) = \widetilde{M}_1^{\pm}$, we have $L(G_1^{\pm}) \subset \mathrm{cl}(U^{\pm}) \cap \Sigma$, $\mathrm{cl}\, U^+ \cap \mathrm{cl}\, U^- = \mathrm{cl}\, \widetilde{F}_1$, and $W \cap \mathrm{cl}\, \widetilde{F}_1 = \varnothing$ (all closures taken in $\overline{\mathbf{R}}^n$). Suppose $W \subset \mathrm{cl}\, U^+$. Then since d preserves orientation, so does $\varphi = d|_{\widetilde{F}_1}\colon \widetilde{F}_1 \to W$ (the orientation being induced from U^+). On the other hand, for every $\gamma \in \Gamma_1$ we have $\varphi \circ \gamma = \gamma \circ \varphi$, and therefore φ drops to a homeomorphism $f\colon \widetilde{F}_1/\Gamma_1 \to W/\Gamma_1$ of the boundary of the manifold $N = (U^+ \cup \widetilde{F}_1 \cup W)/\Gamma_1$. Since all transformations in Γ_1 are orientation-preserving, it follows that N is orientable and f preserves orientation of the boundary ∂N. Let $N' = N/f$ be the manifold obtained by identifying the points x and $f(x)$ in ∂N. In view of the properties of φ and f just described, the manifold N' is nonorientable, aspherical, closed (since by Lemma 1 N is compact), and has fundamental group $\pi_1(N') \simeq \mathbf{Z} \times \Gamma_1$. Therefore N' is homotopically equivalent to the manifold $S^1 \times (\widetilde{F}_1/\Gamma_1)$, which is orientable. The contradiction proves that $W \subset \mathrm{cl}\, U^- \cap \Sigma$.

Applying Lemma 1 to the manifold $(U^- \cup \widetilde{F}_1 \cup W)/\Gamma_1$, we see that $R(\Gamma_1) \cap \mathrm{cl}\, U^- \cap \Sigma$ consists of just the one component $W = (\mathrm{cl}\, U^- \setminus \mathrm{cl}\, \widetilde{F}_1) \cap \Sigma$. But $W \subset \Omega_1^-$ (since $d|_{\widetilde{M}_1^-}$ is a homeomorphism), and $L(G_1^-) \subset \mathrm{cl}\, U^- \cap \Sigma$. Therefore $L(G_1^-) = \Sigma \cap \mathrm{cl}\, \widetilde{F}_1 = L(\Gamma_1)$; and G_1^- equals either Γ_1 or its $\mathbf{Z}_2$-extension. If the dimension n is 3, the lemma now follows immediately (since Γ_1 is geometrically finite and isomorphic to a Fuchsian group, and therefore quasi-Fuchsian [7]). But in the given situation we can give a simple proof for arbitrary dimension.

Suppose Ω_1^- consists of just one component (i.e., $\Omega_1^- = W$). Since $\widehat{M}_1^- = (\Delta \cup \Omega_1^- \cup \Delta^*)/G_1^-$ is closed and aspherical, and $\pi_1(\widehat{M}_1^-) \simeq G_1^-$, it follows that $\widehat{M}_1^-$ is homotopically equivalent to W/Γ_1. But this is impossible, since $H_n(\widehat{M}_1^-, \mathbf{Z}) \neq 0 = H_n(W/\Gamma_1, \mathbf{Z})$. Thus, $\Omega_1^- \setminus W \neq \varnothing$. The manifold $X = (\Delta^* \cup \Omega_1^-)/\Gamma_1$ has exactly two boundary components $Y_1 = W/\Gamma_1$ and Y_2 (by Lemma 1), while by Lemma 3

Y_1 and Y_2 are aspherical and the maps $i_{k*}: \pi_1(Y_k) \to \pi_1(X)$ are monomorphic (i_k is the imbedding $Y_k \to X$, $k = 1, 2$). Since $\Gamma_1(W) = W$, i_1 is an isomorphism. If $(i_{1*})^{-1} \circ i_{2*}(\pi_1(Y_2))$ has infinite index j in $\pi_1(Y_1)$, then the covering $\widetilde{Y}_1$ over Y_1 corresponding to this subgroup in noncompact and homotopically equivalent to the orientable manifold Y_2 of the same dimension; which is impossible. If $1 < j < \infty$, then there exists a finite-sheeted covering over X having more than two boundary components, which is also impossible, by Lemma 2. Thus, Ω_1^- consists of exactly two contractible components, and G_1^- either equals Γ_1 or is a $\mathbf{Z}_2$-extension of Γ_1. This proves the lemma.

We restrict ourselves now to the case $G_i^- = \Gamma_i$ (the argument for $\mathbf{Z}_2$-extensions is similar). In each component $\widetilde{M}_i^-$ pick a surface $\widetilde{S}_i$ invariant with respect to Γ_i. Let $\widetilde{M}_j^+$ and $\widetilde{M}_s^+$ be the components adjoining $\widetilde{M}_i^-$; $\widetilde{P}_j$ and $\widetilde{P}_s$ the inverse images of $\widetilde{S}_i$ in $\widetilde{M}_j^+$ and $\widetilde{M}_s^+$ with respect to the mapping d; $\widetilde{N}_j^+$ and $\widetilde{N}_s^+$ the components of $\widetilde{M}_j^+ \backslash \widetilde{P}_j$ and $\widetilde{M}_s^+ \backslash \widetilde{P}_s$ whose stabilizers are G_j^+ and G_s^+; and $\widetilde{N}_i^-$ the union of $\mathrm{cl}\, \widetilde{M}_i^-$ and the two components of $\widetilde{M}_j^+ \backslash \widetilde{P}_j$ and $\widetilde{M}_s^+ \backslash \widetilde{P}_s$ adjoining $\partial \widetilde{M}_i^-$. Then $(N^+, K^+) = (\bigcup_{i=1}^\infty \widetilde{N}_i^+ / G_i^+)/G$ is conformally equivalent to $(\Delta \backslash \bigcup_1^\infty \widetilde{S}_i)/G$. At the same time, $(N^-, K^-) = (\bigcup_{i=1}^\infty N_i^- / G_i^-)/G$ is conformally equivalent to $(\bigcup_1^\infty (R(\Gamma_i) \backslash S_i)/G_i^-)/G$. If $n = 3$, then $R(\Gamma_i)/\Gamma_i$ is homeomorphic to $S_i \times S^1$ (where $S_i = \widetilde{S}_i/\Gamma_i$) (see [7]) and, as is easily seen, (M, K) is conformally equivalent to $M[S]$, where $M(G) \supset S = (\bigcup_1^\infty \widetilde{S}_i)/G$. This proves the theorem.

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