# Flat conformal structures on 3-manifolds, II. Uniformization of hyperbolic and Seifert manifolds with boundary

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**Abstract**

This is the second in a series of papers proving prove an existence theorem for flat conformal structures on finite-sheeted coverings over a certain class of Haken manifolds.

## 1 Introduction

This is the second in the series of papers whose goal is to prove an existence theorem for uniformizable flat conformal structures on finite-sheeted coverings $N$ over Haken manifolds $M$. In the present paper we prove the existence of flat conformal structures on hyperbolic and Seifert manifolds $N_j$ with toral boundary, satisfying certain "boundary conditions." These structures appear as the "building blocks" for a flat conformal structure on $N$ that will be obtained via conformal gluing of the structures on the manifolds $N_j$. This "assembly" will be done in the subsequent paper [6]. In the paper we will use extensively definitions from theory of Kleinian groups and flat conformal structures which the reader can find for instance in [9, 4].

### 1.1 Outline

The rough idea of the conformal gluing is the following:

Suppose that $M_1, M_2$ are two conformally flat manifolds with boundary and $f : \partial M_1 \to \partial M_2$ is a homeomorphism, then the manifolds $M_1, M_2$ can be "conformally glued via $f$" iff there exists a Moebius homeomorphism $h : \partial M_1 \to \partial M_2$ which is isotopic to $f$. We refer the reader to §12 for the precise definitions. At the moment we only note that if the conformal structures on $M_1, M_2$ are *uniformizable* then under certain conditions on the gluing map (Maskit Combination Theorem) the structure on $M_1 \cup_f M_2$ is uniformizable as well. The reader interested only in the existence theorem can safely ignore the issues related to Maskit Combination.

In this paper we will use only very special class of conformally flat manifolds with boundary $(M, C)$:

(1) We will always assume that the structure $C$ is uniformizable.

(2) $M$ is compact, 3-dimensional and all boundary components of $M$ are homeomorphic to the 2-dimensional torus $T^2$.

(3) The "developing image" (the image under the developing map of $M$) of each boundary component of $M$ is either a *torus of revolution* or a *cone of revolution* (images of standard tori and cones of revolution in $\mathbb{R}^3$ under Moebius transformations, see sections 2.2 and 8 for details).

(4) The holonomy of each boundary component of $M$ is either trivial (in the case of a torus of revolution) or an infinite cyclic *loxodromic* group (in the case of a cone of revolution)

**Definition 1.1.** *A flat conformal manifold* $(M, C)$ *satisfying the above conditions will be called* toroidal.

We observe that we have the canonical coorientation (directed away from $M$) on each boundary component of $M$ and on its developing image.

Suppose now that we are given a compact oriented 3-manifold $M$ with boundary such that the interior of $M$ admits a complete hyperbolic structure of finite volume. For each boundary torus $\partial_j$ of $M$ we pick a basis $u_j, v_j$ of $\pi_1(\partial_j)$. Given each (sufficiently large) prime number $p$ let $M_p \to M$ denote the finite covering given by Theorem 1.3 in [4] (recall that according to this theorem, the restriction of the covering $M_p \to M$ to each boundary torus is the characteristic $p^2$-fold covering).

In Section 3 we prove that for all but finitely many prime numbers $p$ there is a conformally flat structure $C_p$ on $M_p$ such that:

(a1) The manifold with boundary $(M_p, C_p)$ is *toroidal*, moreover the developing image of each boundary torus is a cone of revolution with the vertex angle $\pi/4$ ($\pi/4$-cone for short).

(b1) The kernels of the holonomy representations of boundary tori of $M_p$ are generated by the elements $u_j^p$.

The structures $C_p$ are constructed by applying Thurston's "cusp closing" theorem.

Now suppose that we are given a Seifert manifold $Z = \Sigma \times S^1$ where $\Sigma := \Sigma_{g,k}$ is the compact oriented surface of genus $g$ with $k$ boundary components. Let $\sigma : \partial\Sigma \to Z$ be a section of the trivial bundle $Z \to \Sigma$. In §4 we define the Euler number $e(Z, \sigma) \in \mathbb{Z}$ of $Z$ relative to $\sigma$. We suppose that each boundary component of $Z$ is assigned either "hyperbolic" label (such component is to be denoted $T_j^H$) or "Seifert" label (in which case we will use the notation $T_i^Z$ for the component).

**Remark 1.2.** *At this point, the assignment of "hyperbolic" and "Seifert" labels is arbitrary. However if $Z$ is a maximal Seifert component of a Haken manifold $W$ then the* hyperbolic *tori $T_j^H$ are adjacent to the hyperbolic components of $W$, and* Seifert *tori $T_i^Z$ are adjacent to the Seifert components of $W$.*

In §10 we prove a "relative uniformization theorem" for $Z$. Namely, suppose we are given a collection of $\gamma_j$ of loxodromic elements[1] in $\mathbf{Mob}_3$. Each $\gamma_j$ preserves a

[1] Which will appear as the holonomies along the loops $v_j^p$ discused above.

certain $\pi/4$-cone $C_j$; assume also that for each $j$ we are given a $\gamma_j$-invariant "spiral" $s_j$ in $C_j$.

Then for each sufficiently large $g$ there exists a toroidal flat conformal structure on $Z$ such that:

(a2) The developing images of Seifert boundary components $T_i^Z$ are tori of revolution which bound solid tori $\widehat{T}_i^Z \subset S^3$, so that the canonical coorientation is directed "inward" $\widehat{T}_i^Z \subset S^3$.

(b2) The developing images of hyperbolic boundary components $T_i^H$ are $3\pi/4$-cones, the holonomy of each loop $\delta_i = \sigma(\partial\Sigma) \cap T_i^H$ is a loxodromic element which is conjugate to $\gamma_i$ in $\mathbf{Mob}_3$.

(c2) The developing image of each loop $\delta_i$ is a spiral on a $3\pi/4$-cone as above which has the same slope as the corresponding spiral $s_j$.

(d2) For each loop $\delta_i \in T_i^Z$ the developing image is null-homotopic in the solid torus $S^3 - \hat{T}_i^Z$, the developing image of each fiber $f_i \subset T_i^Z$ of Seifert fibration is null-homotopic in $\hat{T}_i^Z$.

The proof of this theorem is the most technically difficult part of the proof of the existence theorem of flat conformal structures (Theorem 5.1 of [4]). The proof consists of two steps:

**Step 1.** We first find a flat conformal structure on $Z$ with the required properties provided that $e(Z, \sigma) = 0$.

**Step 2.** We modify the construction so that the prescribed relative Euler number is realized.

We deal with the **Step 1** in Section 5. The main tool in this section is a theorem of A. Weil about the restriction map from the representation variety of the fundamental group of a surface $\Sigma$ with boundary to the product of the representation varieties of its boundary components. This map is open for a Zariski open and dense subset of the representation variety $\mathrm{Hom}(\pi_1(\Sigma), \mathbf{Mob}_3)$. As the result we construct flat conformal structures on Seifert manifolds with boundary such that the conditions (a2—d2) are satisfied for the zero value of the relative Euler number.

On the other hand, there are *closed* Seifert manifolds $\hat{Z}'$ with an arbitrary Euler number provided that the genus of the base is sufficiently large (Theorem 2.1 of [4]). We then cut out a solid torus from $\hat{Z}'$ to get a compact *toroidal* flat conformal manifold $Z'$ with boundary. To finish the **Step 2** we organize a conformal gluing of these two types of manifolds: the manifold $Z$ constructed in the Step 1 and of the manifold $Z'$. This gluing is performed in Section 10, proof of Theorem 11.2.

We use the above *relative uniformization* theorem, in the following paper [6] as follows. Suppose that we have a collection of hyperbolic and Seifert manifolds with flat conformal structures satisfying the conditions (a1), (b1), (a2), (b2), (d2), then we can glue them conformally so that:

(a) The loops $u_i^p$ on $D_i \subset M_p$ are identified with fibers $f_j$ of the corresponding Seifert components.

(b) The loops $v_i^p$ on $D_i$ are identified with loops $\delta_i$ on $T_i^H$.

(c) The loops $\delta_i \subset T_i^Z$ are identified with the fibers $f_j \subset T_j^Z$ of the adjacent Seifert manifolds.

This means that we can construct a flat conformal structure on any Haken manifold with the special properties (a-c) of boundary identifications, provided that the relative Euler numbers are sufficiently small. In [6] we will prove that for any Haken manifold there exists a finite-sheeted covering such that the gluing maps satisfy all properties above.

# 2 Definition and notation

Let $(X, d_X)$ be a metric space and $C \subset X$ a subset. Then we set $d(x, C) := \inf\{d_X(x, c) : c \in C\}$. For subsets $A, B$ of $(X, d_X)$ we will use the notation $\mathrm{dist}(A, B)$ for the *nonsymmetric distance*:

$$\mathrm{dist}(A, B) = \sup\{d_X(a, B) : a \in A\}.$$

For a connected compact hypersurface $S \subset \mathbb{R}^n$, we let $\mathrm{ext}(S)$ denote the unbounded component of $\mathbb{R}^n \setminus S$.

## 2.1 Deformation theory

We will need the following result of A. Weil, [12]. Let $\mathbf{G}$ be a reductive real Lie group with the Lie algebra $\mathfrak{g}$; let $\Gamma = \langle \gamma_1, ..., \gamma_n \rangle$ be a free group of rank $n$ and $\rho : \Gamma \to \mathbf{G}$ a representation. We will denote by $ad(\rho) : \Gamma \to Aut(\mathfrak{g})$ the composition of $\rho$ with the adjoint representation. Set $\gamma_0 := \gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_n$. We will denote by $Ad$ the action of $G$ on itself by conjugation.

Define $PZ_0^1(\Gamma, ad(\rho))$ as the space of those cocycles in $Z^1(\Gamma, ad(\rho))$ whose restriction to each $\langle \gamma_j \rangle$ is a coboundary $(j = 1, ..., n)$. Let

$$PZ^1(\Gamma, ad(\rho)) := \{z \in PZ_0^1(\Gamma, ad(\rho)) : z \mid_{\langle \gamma_0 \rangle} \text{ is a coboundary}\}$$

Let $Z_{\mathbf{G}}(A)$ denote the centralizer of a subgroup $A$ in $\mathbf{G}$.

**Theorem 2.1** (A. Weil, [12])**.** *The dimension of $PZ^1(\Gamma, ad_G \circ \rho)$ equals*

$$-\dim(\mathbf{G}) + e_1 + \ldots + e_n + \zeta,$$

*where*

$$e_j := \dim(\mathbf{G}) - \dim Z_{\mathbf{G}}(\langle \rho(\gamma_j) \rangle) = \dim B^1(\langle \gamma_j \rangle, ad_{\mathbf{G}} \circ \rho),$$
$$\zeta := \dim H^0(\Gamma, ad(\rho)) = \dim(Z_{\mathbf{G}}(\rho(\Gamma))).$$

**Corollary 2.2.** *Suppose that $H^0(\Gamma, ad(\rho)) = 0$. Then the natural restriction map*

$$res_0 : PZ_0^1(\Gamma, ad(\rho)) \to Z^1(\langle \gamma_0 \rangle, ad(\rho))$$

*is onto.*

*Proof.* We have $e_\Gamma := \dim PZ_0^1(\Gamma, ad(\rho)) = e_1 + \ldots + e_n$ ,

$$\dim \ker(res_0) = PZ^1(\Gamma, ad(\rho)) = e_\Gamma + e_0 - \dim(\mathbf{G})$$

and

$$\dim Z^1(\langle\gamma_0\rangle, ad(\rho)) = \dim(\mathbf{G}) - e_0,$$

which implies the assertion. □

Consider a free group $\Gamma$ on $r + s$ generators

$$h_1, h_2, \ldots, h_r, c_1, \ldots, c_s,$$

and set

$$h_{r+1} := h_1 \cdot h_2 \cdot \ldots \cdot h_r \cdot \ c_1 \cdot \ldots \cdot c_s.$$

Then the representation variety $\mathrm{Hom}(\Gamma, \mathbf{G})$ is canonically identified with $\mathbf{G}^{r+s}$. Define the following smooth subvariety in $\mathrm{Hom}(\Gamma, \mathbf{G})$:

$$W := \mathrm{Hom}(\Gamma, \mathbf{G}) \cap \left( Ad(\mathbf{G})(\rho(h_1)) \times \ldots \times Ad(\mathbf{G})(\rho(h_r)) \times \underbrace{\mathbf{G} \times \ldots \times \mathbf{G}}_{s \text{ times}} \right).$$

Here $Ad(\Gamma)(\gamma)$ is the $G$-orbit through $\gamma \in G$ under the adjoint action of $G$ on itself.

Let $res_{r+1} : W \to \mathrm{Hom}(\langle h_{r+1}\rangle, \mathbf{G})$, $p_j : W \to \mathrm{Hom}(\langle c_j\rangle, \mathbf{G})$ denote the restriction maps. Corollary 2.2 implies the following

**Theorem 2.3.** *Suppose that $H^0(\Gamma, ad(\rho)) = 0$, i.e. $\rho(\Gamma)$ has finite centralizer in $\mathbf{G}$. Then the map*

$$(res_{r+1}, p_1, \ldots, p_s) : W \to \mathrm{Hom}(\langle h_{r+1}\rangle, \mathbf{G}) \times \prod_{i=1}^{s} \mathrm{Hom}(\langle c_i\rangle, \mathbf{G}) = \mathbf{G}^{s+1}$$

*is a submersion at the point $\rho \in W$.*

## 2.2 Cones of revolution in $S^3$

Suppose that $x, y$ are distinct points in $S^3$ and $\mathcal{J} \subset S^3$ is a circular arc connecting $x$ and $y$. Then there exists a Moebius transformation $\gamma \in \mathbf{Mob}_3$ such that $\gamma(x) = 0$ and $\gamma(y) = \infty$. Therefore $\rho = \gamma(\mathcal{J})$ is a Euclidean ray emanating from the origin.

Let $K(\rho, \theta)$ be denote the solid Euclidean cone of revolution with the axis $\rho$ and the angle $\theta$ between the axis and the surface of the cone. We let $K(\mathcal{J}, \theta) := \gamma^{-1}(K(\rho, \theta))$ denote the *cone with the axis $\mathcal{J}$ and the angle $\theta$*, a *$\theta$-cone* for short. Observe that $K(\rho, \theta)$ is invariant under the stabilizer of $\{0, \infty\}$ in $\mathbf{Mob}_3$, thus our definition is independent of the choice of $\gamma$. We provide the boundary of $K = K(\mathcal{J}, \theta)$ with a flat Riemannian metric $d_{\partial K}$ invariant under the stabilizer of $K$ in $\mathbf{Mob}_3$, normalized so that in the case $K = K(\rho, \theta)$, the metric is the restriction of the metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{x^2 + y^2 + z^2}$$

on $\mathbb{R}^3 - \{0\}$. We will refer to non-periodic geodesics on $\partial K$ as *spirals*. The *slope* of a spiral $\sigma$ is the angle between $\sigma$ and any periodic geodesic on $\partial K$. Observe that if $g \in \mathbf{Mob}_3$ is a loxodromic element preserving $K$ then each axis (i.e. invariant geodesic) of the induced isometry $g : \partial K \to \partial K$ is a spiral $\sigma$. Then slope of $\sigma$ and the choice of $g$ uniquely determine the free homotopy class of the projection of $\sigma$ to the torus $T^2 = \partial K/\langle g\rangle$. Thus, given a loxodromic element $g$ as above and a slope $s$ of a $g$-invariant spiral, we get a basis $\langle u, v\rangle$ for $H_1(T^2)$, where $u$ is represented by a simple loop on $T^2$ which lifts to a periodic geodesic in $\partial K$ and $v$ is represented by a loop $\sigma/\langle g\rangle$. Here $\sigma$ is a $g$-invariant spiral of the slope $s$ oriented consistently with the action of $g$ on $\sigma$ and $u$ is oriented so that the orientation on $T^2$ given by the pair $(u, v)$ is the same as the orientation induced from the solid torus $K/\langle g\rangle$.

# 3 Cusp Closing

Let $N$ be a compact connected orientable manifold whose boundary is a union of tori $\partial N = T_1 \cup ... \cup T_k$. Assume that $\mathrm{int}(N) = \mathbb{H}^3/\Gamma$ for some torsion-free discrete group $\Gamma \subset PSL(2,\mathbb{C})$. We will identify $\pi_1(N)$ with the group $\Gamma$; let $\rho_0 : \pi_1(N) \to \Gamma \subset PSL(2,\mathbb{C})$ denote this isomorphism. Set

$$X(\Gamma) = \mathrm{Hom}(\pi_1(N), PSL(2,\mathbb{C}))//PSL(2,\mathbb{C})$$

be *character variety* of the group $\Gamma$ (see for instance [2] for the precise definition). For each boundary torus $T_j \subset \partial M$ pick generators $u_j, v_j$ of $\pi_1(T_j)$. Given $\rho \in \mathrm{Hom}(\Gamma, PSL(2,\mathbb{C}))$ let $[\rho]$ denote the projection of $\rho$ to the character variety. The following theorem is mostly due to W. Thurston [11, Chapter 5, Theorem 5.6 ] (see also [2], [1]) and [5, Theorem 8.44], where smoothness is proven.

**Theorem 3.1.** *(a) There exists a neighborhood $U$ of $[\rho_0]$ in $X(\Gamma)$ which is a smooth complex manifold of the complex dimension $k$.*

*(b) Furthermore, the projection*

$$p : U \to \mathbb{C}^k/(\mathbb{Z}/2)^k,$$

*given by*

$$p([\rho]) = (tr(u_1) \mod \mathbb{Z}/2, ..., tr(u_k) \mod \mathbb{Z}/2)$$

*is a homeomorphism onto a neighborhood of the point $(\pm 2, ...., \pm 2) \in \mathbb{C}^k/(\mathbb{Z}/2)^k$. The image of $[\rho_0]$ under $p$ is the point $(\pm 2, ...., \pm 2)$.*

Here the generator 1 of the group $\mathbb{Z}/2$ acts on $\mathbb{C}$ by sending each $z \in \mathbb{C}$ to $-z$.

**Corollary 3.2.** *There exists an $\epsilon > 0$ such that for each vector $\overrightarrow{\tau} = (\tau_1, ..., \tau_k) \in [2-\epsilon, 2]^k$, there exists a representation $\rho_\tau : \Gamma \to PSL(2,\mathbb{C})$ (depending continuously on $\tau$), such that*

$$\rho_0 = \rho_{(2,..,2)} \text{ and } \pm tr\rho_\tau(u_i) = \pm\tau_i.$$

As another corollary one gets (see [1]):

**Theorem 3.3.** *For each positive integer $n$ set*

$$\overrightarrow{\tau}(n) := (2\cos(\frac{2\pi}{n}), ..., 2\cos(\frac{2\pi}{n})).$$

*Then for each sufficiently large $n$, the representation $\rho_{\tau(n)}$ is the holonomy of a closed hyperbolic orbifold $\mathcal{O}_n$ obtained by performing a "generalized Dehn filling" on $\partial N$.*

We will use the notation $\Gamma(n)$ for the image of $\rho_n := \rho_{\tau(n)}$. Observe that the group $\Gamma(n)$ is discrete since it contains a finite-index subgroup which is a holonomy group of a closed hyperbolic manifold, see for instance [4, Section 1.4]. The singular locus of the orbifold $\mathcal{O}_n = \mathbb{H}^3/\Gamma(n)$ is a totally-geodesic link $L_n \subset \mathcal{O}_n$. The link $L_n$ is the projection to $\mathcal{O}_n$ of the union of axes $\mathcal{J}_i(n)$ of the hyperbolic elements $\rho_n(v_i), i = 1, ..., k$. Note that since the length of $L_n$ converges to zero as $n \to \infty$, the normal injectivity radius of $L_n$ diverges to $\infty$, by the Kazhdan–Margulis Lemma. In particular, for all sufficiently large $n$'s the following holds:

If $K_i := K(\mathcal{J}_i(n), \pi/4)$, then each cone $K_i$ is precisely invariant under its stabilizer $\rho_n(\pi_1(T_i)) = \langle \rho_n(v_i), \rho_n(u_i)\rangle$, and the projections of the cones $K_i$ to $\mathcal{O}_n$ are pairwise disjoint, $i = 1, ..., k$.

An effective bound on $n$ can be derived from an estimate on the Margulis' constant for $\mathbb{H}^3$ (compare [3]) however we are will not pursue this issue here.

# 4 Relative Euler Class

Let $\Sigma := \Sigma_{g,r}$ be a compact connected oriented surface with $r$ boundary circles. We will regard the manifold $M = \Sigma \times S^1$ as a trivial circle bundle over $\Sigma$; let $\sigma : \partial\Sigma \to \partial M$ is a section of the restriction of this bundle to $\partial\Sigma$. By abusing notation we will retain the name $\sigma$ for the image of the section $\sigma$.

Choose an orientation on the fiber $S^1$. Recall the definition of the *relative Euler class* in the above context (see for instance [10] for the general discussion):

**Definition 4.1.** *The Euler class $e(M, \sigma)$ of the bundle $M \to \Sigma$, relative to $\sigma$, equals to $-o_1(\sigma)$, where $o_1(\sigma)$ is the first obstruction $o_1(\sigma) \in H^2(\Sigma, \partial\Sigma; \pi_1(S^1)) \cong \mathbb{Z}$ to the extension of $\sigma$ to a section $\Sigma \to M$. The image of $-o_1(\sigma)$ in $\mathbb{Z}$ will be denoted $e(M, \sigma)$.*

The number $e = e(M, \sigma)$ can be explicitly computed as follows: Project the homology class $[\sigma]$ to $H_1(M)$ and then to $H_1(S^1, \mathbb{Z}) \otimes H_0(\Sigma, \mathbb{Z}) = \mathbb{Z}$. The number $e$ is the image of $[\sigma]$ in $\mathbb{Z}$.

It is easy to see that if $\sigma, \sigma'$ are sections $\partial\Sigma \to M$ with the same relative Euler class then there exists an automorphism $f$ of the bundle $M \to \Sigma$ such that $f \circ \sigma = \sigma'$.

Suppose now that the number $r$ is even. Let $p :\to M$ be *a standard $n^2$-fold covering* over $M$ (see [4, §1.3]); there exists a unique (up to homotopy) section

$$\tilde{\sigma} : \partial\tilde{\Sigma} \to \tilde{M}$$

which is a lift of $\sigma : \partial\Sigma \to M$. Then the above homological description of the Euler number implies that

$$e(\tilde{M}, \tilde{\sigma}) = e(M, \sigma).$$

Let $M_1 \to \Sigma_1, M_2 \to \Sigma_2$ be two trivial circle bundles and $\sigma_i : \partial\Sigma_i \to M_i$ be their sections. Suppose that the manifold $M$ is obtained by gluing $M_1$ and $M_2$ along some of their boundary components as follows: The fiber is glued to fiber (preserving the orientation) and a section is glued to a section with the change of the induced orientation. The sections $\sigma_1, \sigma_2$ define a section $\sigma : \partial\Sigma_1 \cup \Sigma_2 \to \partial M$. Then

$$e(M, \sigma) = e(M_1, \sigma_1) + e(M_2, \sigma_2).$$

## 4.1 Relative Euler Class of a Kleinian Group

Let $\Gamma \subset \mathbf{Mob}_3$ be a free finitely generated Kleinian group. We assume that $\mathcal{K} \subset \Omega(\Gamma)$ is a union of pairwise disjoint cones $K_i$ each of which is stabilized by an infinite cyclic (loxodromic) subgroup $\langle h_i \rangle$ of $\Gamma$. We assume that the manifold $M^*(\Gamma) := \Omega(\Gamma) \setminus \mathcal{K}$ is the product $\Sigma \times S^1$, where $\Sigma$ is a compact surface with boundary. Suppose that we are given a $\Gamma$-invariant collection $\tilde{\sigma}$ contained in $\partial K$, one spiral for each cone $K_i$. Then the projection $\sigma$ of $\tilde{\sigma}$ to $M^*(\Gamma)$ defines a section $\sigma : \partial\Sigma \to M^*(\Gamma)$ of the trivial bundle $\Sigma \times S^1 \to \Sigma$. Choice of one of the generators $h_i$ (which we will suppress henceforth to simplify the notation) determines orientation on $\Sigma$ and hence on the fiber $S^1$.

**Definition 4.2.** $e(\Gamma, \mathcal{K}, \tilde{\sigma}) := e(M^*(\Gamma), \sigma))$ *is the relative Euler number of the Kleinian group* $\Gamma$.

Below is an example of the relative Euler number for a Kleinian group that we will use in what follows. Let $\Gamma := G(10, 1)$, $\Gamma' := H(12, 1)$ be the Kleinian groups constructed in [4, Section 2.6]. Recall that $\Gamma'$ is isomorphic to a surface group of genus 12; the quotient $\Omega(\Gamma')/\Gamma' = S(12, 1)$ is homeomorphic to the total space of the circle bundle over a closed surface of genus 12 and the Euler number of this bundle equals 1. The group $\Gamma'$ splits as $\Gamma' = \Gamma *_{\langle h \rangle} \Gamma''$, where $h$ is a hyperbolic element, $\Gamma''$ is a free group of rank 2 which preserves a round circle in $S^3$. This amalgam is realized as Maskit decomposition of the group $\Gamma'$: There is a sphere $\Pi$ in $S^3$ which is precisely invariant under the subgroup $\langle h \rangle$, the twice punctured sphere $\Pi \setminus \Lambda(\langle h \rangle)$ (which we regard as a $\pi$-cone) projects to a torus $T^2 \subset M(\Gamma') = \Omega(\Gamma')/\Gamma'$. This torus splits the manifold $M(\Gamma)$ into two components: $M^*(\Gamma), M^*(\Gamma'')$. The above $\pi$-cone contains a Euclidean segment $\tilde{s}$ (an $h$-invariant spiral of slope $\pi/2$). Projection of this segment to $T^2$ is a simple loop $\sigma$ which is a section of the trivial circle bundles $M^*(\Gamma) \to \Sigma_{10,1}$ and $M^*(\Gamma'') \to \Sigma_{2,1}$. Since the group $\Gamma''$ is Fuchsian and $s$ is a straight line segment in $\mathbb{R}^3$ it follows that $e(M^*(\Gamma''), \sigma) = 0$. Hence additivity of the relative Euler number implies that $e(M^*(\Gamma), \sigma) = 1$ and thus the corresponding relative Euler number of the group $\Gamma$ equals to 1 as well.

# 5 Regular collars in hyperbolic surfaces

**Definition 5.1.** *Let $S$ be a compact hyperbolic surface with geodesic boundary. A $w$-*collar *of a boundary circle $\gamma \subset \partial S$ is a $w$-neighborhood of $\gamma$ in $S$. A collar is said to be* regular *if it is disjoint from other boundary components of $S$ and is homeomorphic to the annulus.*

**Lemma 5.2.** *For any finite $w, \ell \in (\mathbb{R}_+)^2$ there is an integer $g_0 = g_0(w, \ell)$ such that, for all $g \geq g_0$ there exists a compact surface $\Sigma_{g,1}$ of genus $g$ with one (geodesic) boundary loop which has length $\ell$ and a regular $w$-collar.*

*Proof.* Start with a closed hyperbolic surface $S$ of genus 2 which contains a simple loop $\gamma$ of length $\ell$; let $\Gamma := \pi_1(S) \subset PSL(2, \mathbb{R})$. Then, since $\Gamma$ has the LERF property with respect to the maximal abelian subgroup $\langle \gamma \rangle$ (see for instance [8]), there exists a finite covering $S' \to S$ such that $\gamma$ lifts to $S'$ homeomorphically to a loop $\gamma'$ whose normal injectivity radius in $S'$ is $\geq w$. Moreover, without loss of generality we can assume that $\gamma'$ does not separate $S'$. Now, cut $S'$ open along $\gamma'$: The resulting surface $\Sigma = \Sigma_{g_0-1,2}$ has two geodesic boundary components $\gamma'$ and $\gamma''$ and $\gamma'$ possesses a regular $w$-collar. Suppose that $g \geq g_0$. Set $h := g - g_0 + 1$. Glue to $\Sigma$ a hyperbolic surface $\Sigma_{h,1}$ (with one geodesic boundary component of the length $\ell$) along the loop $\gamma''$. The resulting surface $\Sigma_{g,1}$ has single geodesic boundary component of the □

Recall that a compact hyperbolic surface $S$ with geodesic boundary is called *pair of pants* if it is homeomorphic to the 2-sphere with three holes. It is known that $S$ is uniquely determined (up to an isometry) by the three numbers $\ell_1, \ell_2, \ell_3$: the lengths of the boundary loops $\alpha_1, \alpha_2, \alpha_3 \subset \partial S$.

Furthermore, for each triple $(\ell_1, \ell_2, \ell_3) \in (\mathbb{R}_+)^3$ there exist pair of pants $S = S(\ell_1, \ell_2, \ell_3)$, such that where $\ell_j$ is the length of $\alpha_j$ $(j = 1, 2, 3)$.

**Remark 5.3.** *As $\ell_1 \to 0$ the component $\alpha_1 \subset \partial\ S$ degenerates to a puncture (which is infinitely far from any finite point of $S$). Hence for any fixed $0 < \ell_2, \ell_3, w < \infty$ there exists $\lambda > 0$ such that for all $0 < \ell_1 < \lambda$ the curve $\alpha_1 \subset \partial S(\ell_1, \ell_2, \ell_3)$ has a regular $w$-collar.*

**Lemma 5.4.** *(a) For any finite $w, \ell \geq 0$ there is an integer $g_0 = g_0(w, \ell)$ such that, for all $g \geq g_0$ there exists a compact surface $\Sigma_{g,1}$ of genus $g$ with one (geodesic) boundary loop which has length $\ell$ and a regular $w$-collar.*

*(b) If $w = 0$ then we can take $g_0(w, \ell) = 1$.*

*Proof.* First we prove Part (b). Let $S = S(\ell, \ell, \ell)$ be a *pair of pants*, $\alpha_1 \subset \partial S$. The required surface of genus 1 is obtained from $S$ by gluing its boundary curves $\alpha_2, \alpha_3$ . A surface of arbitrary genus can be obtained from $S$ by consecutive gluing along $\alpha_2$ and $\alpha_3$ of $2g$ copies of $S(\ell, \ell, \ell)$ and then gluing pairwise the $2g$ boundary curves. Part (b) follows.

Consider the general case: $w \geq 0$ . By the Remark 5.3, for some $n \in \mathbb{N}$ there exist a "pair of pants" $S(\ell/n, \ell, \ell)$ such that $\alpha_1 \subset \partial S(\ell/n, \ell, \ell)$ has a regular $w$-collar. For this surface we construct a $n$-sheeted covering $S \to S(\ell/n, \ell, \ell)$ such that the loop $\alpha_1$ is $n$ times covered by a component $\alpha \subset \partial S$. Then the length of $\alpha$ is equal to $\ell$ and the loop $\alpha$ has a regular $w$-collar in $S$ . Denote the genus of $S$ by $h$ and the number of its boundary components by $m$ . We can glue to each component $\beta$ of $\partial S - \alpha$ a copy of the surface of genus 1 whose unique (geodesic) boundary component has the same length as $\beta$ . The resulting surface $\Sigma_{g_0,1}$ has the genus $g_0 = h + m - 1$ and precisely one boundary curve $\alpha$ which has a regular $w$-collar.

If instead of the surface of genus 1 we will take a surface of genus $(g - g_0 + 1)$ then the constructed surface $\Sigma_g$ will satisfy the assertion (a) of Lemma. □

# 6 Deformations of representations of free Kleinian groups

We identify the Euclidean space $\mathbb{R}^3$ with the product $\mathbb{C} \times \mathbb{R}$. We introduce the following notations: the disk

$$\Delta = \Delta_R = \{z \in \mathbb{C} : \mid z \mid < R\} \subset \mathbb{R}^3$$

is the Poincare model for the hyperbolic plane; $d_\Delta(,)$ is the hyperbolic distance between points in $\Delta_R$; $O_R = \partial\Delta_R$ is the circle of the Euclidean radius $R$. We fix the counterclockwise orientation on $O_R$. The sphere $S^3 = \overline{\mathbb{R}^3}$ has a metric $d_S$ of constant sectional curvature $+1$ such that $O_R$ is a geodesic in this metric. (This metric depends on $\mathbb{R}$.)

Recall that $d_{\mathbb{E}}(,)$ and $\mathrm{dist}_{\mathbb{E}}(,)$ are the distances between sets with respect to the Euclidean metric in $\mathbb{E} = \mathbb{R}^3$; $\mathrm{dist}_{\mathbb{H}}(,)$ is the distance between sets in the hyperbolic plane $\Delta_R$ (see [4, §1.1] and §2 of this paper). Recall also that for $\alpha \in \mathbf{Mob}_3$ such that $\alpha(\infty) \neq \infty$, $I(\alpha)$ denotes the *isometric sphere* of $\alpha$ (see [4]).

Fix nonnegative integers $r \geq 3, m, s$. Set $R = 10m + 6$. We construct a Kleinian group $\tilde{H} \subset \mathbf{Mob}_3$ which is freely generated by $r$ hyperbolic elements and $s$ parabolic elements so that:

(1a) The disk $\Delta_R$ is invariant under $\tilde{H}$.

(2a) The hyperbolic generators $\tilde{h}_i$ $(i = 1, ..., r)$ are conjugate in $\mathbf{Mob}_3$ to the hyperbolic element $h \in G(10, 1)$ (see [4, §2.6]).

(3a) The Euclidean radii of the isometric spheres of $\tilde{h}_i$ and parabolic generators $\tilde{h}_i, \tilde{c}_j$ $(i \leq r, j \leq s)$ are equal to $1/8$ .

(4a) These isometric spheres are pairwise disjoint except for the pairs

$$I(\tilde{c}_j), I(\tilde{c}_j^{-1})$$

which are tangent $(j = 1, \ldots, s)$.

(5a) There are no isometric spheres in this set of generators between $I(\tilde{h}_i)$ and $I(\tilde{h}_i)^{-1}$ $(i = 1, ..., r)$ (see Figure 1).

Define the element $h_{r+1}^{-1} = \tilde{c}_s \cdot ... \cdot \tilde{c}_1 \cdot \tilde{h}_r \cdot ... \cdot \tilde{h}_1$ .

(6a) The axis $A(\tilde{h}_i) \subset \mathbb{H}^2 = \Delta_R$ of $\tilde{h}_i$ is contained in a Euclidean circle of radius $< 1/8$, $i = 1, ..., r$.

(7a) Suppose that $D, D'$ be a pair of consecutive (on $O_R$) isometric spheres as above, which is different from the pair $I(\tilde{c}_s^{-1})$, $I(\tilde{h}_1)$. Then we need the Euclidean distance $(d_{\mathbb{E}})$ between $D$ and $D'$ to be $< 1$ (Figure 1).

The group $\tilde{H}$ may be easily constructed via the first Maskit Combination Theorem (see [4, Section 1.2]). Thus the domain

$$\tilde{\mathcal{P}} = \bigcap_{i=1}^{r} \mathrm{ext}(I(\tilde{h}_i) \cup I(\tilde{h}_i^{-1})) \cap \bigcap_{j=1}^{s} \mathrm{ext}(I(\tilde{c}_j) \cup I(\tilde{c}_j^{-1}))$$

is a fundamental domain for the action of $H'$ in $S^3$. Denote by $\mathcal{J}(h_{r+1})$ a component of $O_R \backslash Fix(h_{r+1})$ such that

$$\mathcal{J}(h_{r+1}) \cap (I(\tilde{h}_1^{-1}) \cup I(\tilde{c}_1)) = \emptyset$$

It is easy to see that $\mathcal{J}(h_{r+1})$ is precisely invariant under $\langle h_{r+1} \rangle$ in $\tilde{H}$. This implies that the axis $A(h_{r+1}) \subset \Delta_R$ of the element $h_{r+1}$ and the strip in $\Delta_R$ between $A(h_{r+1})$ and $\mathcal{J}(h_{r+1})$ are precisely invariant as well.

Let $K(h_{r+1}) = K(\mathcal{J}(h_{r+1}), \vartheta)$ be the cone with the axis

$$\mathcal{J}(h_{r+1})$$

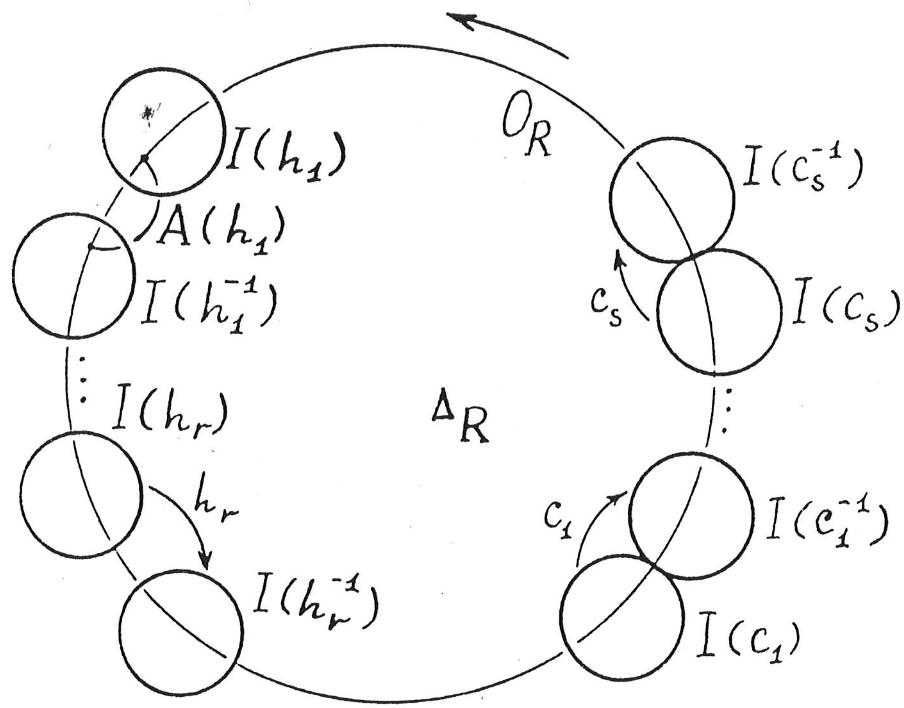


Figure 1: Group $\tilde{H}$.

and sufficiently small vertex angle so that

$$\vartheta < \pi/2, \tag{1}$$

$$\mathrm{dist}_{\mathbb{E}}(K(h_{r+1}, O_R) < 1/4. \tag{2}$$

3.6.2. The hyperbolic distance $\mathrm{dist}_{\mathbb{H}}(z, A(h_{r+1}))$ is equal to

$$w = \operatorname{arccosh}(1/\sin(\vartheta))$$

for any $z \in \partial K(h_{r+1}) \cap \Delta_R$. Pick a point $a \in A(h_{r+1})$ and denote the distance $d_\Delta(h_{r+1}(a), a)$ by $\ell$. It is easy to see that the elements $\tilde{h}_i$ have precisely invariant cones $K(\tilde{h}_i)$ with the vertex angles $\pi/2$ and axes lying in $O_R$ $(i = 1, ..., r)$.

3.6.3. First, we apply Theorem 2.3 in the case $\mathbf{G} = PSL(2, \mathbb{C})$, $\mathbf{G}_{\mathbb{R}} = PSL(2, \mathbb{R})$, $\tilde{H}$ is the free group constructed in §3.5.1, and $\rho_0 : \tilde{H} \to \mathbf{G}$ is the natural inclusion. Set $C = \langle \tilde{c}_1, ..., \tilde{c}_s, h_{r+1} \rangle$; let

$$\mathrm{Hom}_C(\tilde{H}, \mathbf{G}) = \{\beta : \tilde{H} \to \mathbf{G} : \beta|_C = \rho_0|_C\}$$

Set

$$\mathcal{D}_{\mathbb{R}}(\tilde{H}) = W \cap (\mathrm{Hom}_C(\tilde{H}, \mathbf{G}_{\mathbb{R}}) \times Z_{\mathbf{G}}(C)/Z_{\mathbf{G}_{\mathbb{R}}}(C))$$
$$\mathcal{D}_{\mathbb{C}}(\tilde{H}) = W \cap \mathrm{Hom}_C(\tilde{H}, \mathbf{G})$$

Then we have:

**Lemma 6.1.** *There exists a representation $\rho : \tilde{H} \to \mathbf{G}$ such that:*
*(1b) $\rho(\tilde{c}_j) = \tilde{c}_j$ , $\rho(h_{r+1}) = h_{r+1}$ $(j = 1, ..., s)$;*
*(2b) the elements $\tilde{h}_i$ and $\rho(\tilde{h}_i)$ are conjugate in $\mathbf{G}$;*
*(3b) the group $\rho(\tilde{H})$ has no invariant Euclidean circle;*
*(4b) $\mathrm{dist}_{\mathbb{E}}(I(\rho(\tilde{h}_i^{\pm 1})), O_R) < 1/4$;*
*(5b) Choose the shortest geodesic $\ell_i$ in $S^3$ between fixed points of $\rho(\tilde{h}_i)$ as the axis of the cone $K(\rho(\tilde{h}_i)) = K(\ell_i, \pi/2)$. Then $K(\rho(\tilde{h}_i))$ are precisely invariant under $\langle \rho\tilde{h}_i \rangle$ in $\rho(\tilde{H})$, their orbits are disjoint and $\mathrm{dist}_{\mathbb{E}}(K(\rho\tilde{h}_i), O_R) < 1/8, i = 1, ..., r$.*

*Proof.* First, the condition $r > 2$, Theorems 2.1 and Theorem 2.3 imply that

$$\dim_{\mathbb{R}}(\mathcal{D}_{\mathbb{R}}(\tilde{H})) < \dim_{\mathbb{R}}(\mathcal{D}_{\mathbb{C}}(\tilde{H}))$$

near the point $\rho_0$ where these varieties are smooth. Thus, we have at least a (real) curve of representations $\rho : \tilde{H} \to \mathbf{G}$ containing $\rho_0$ such that the condition (3b) is satisfied for all $\rho \neq \rho_0$ and the conditions (1b), (2b), (4b), (5b) are satisfied for all $\rho$ sufficiently close to $\rho_0$. □

Denote the group $\rho\tilde{H}$ by $H$ and for elements $\tilde{g} \in \tilde{H}$ we set $\rho(\tilde{g}) = g$. Accordingly, the isometric fundamental domain for $H$ will be denoted $\mathcal{P}$; this domain is bounded by the isometric spheres of $h_j^{\pm 1}, c_i^{\pm 1}$ $(1 \le j \le r \; ; 1 \le i \le s)$.

3.6.4. Now we set $\mathbf{G} = \mathbf{Mob}_3$; then the group $H$ constructed in 3.6.3 has centralizer in $\mathbf{G}$. Thus we can again apply Theorem 2.3.

Denote by $\mu$ the minimum of the following numbers:

1. the radius of $I(c_i^{\pm 1})$,
2. the radius of $I(h_j^{\pm 1})$, (where $1 \le j \le r, 1 \le i \le s$),
3. $\min\{d_{\mathbb{E}}(D, D') : D, D'$ are mutually disjoint components of $\partial\mathcal{P}\}$.

Let $\{h_j(\epsilon), c_i(\epsilon), 1 \le j \le r, 1 \le i \le s\}$ be a set of elements of $\mathbf{Mob}_3$ such that:

(1c) $\mathrm{dist}_{\mathbb{E}}(I(c_i^{\pm 1}(\epsilon)), I(c_i^{\pm 1}))$ and $\mathrm{dist}_{\mathbb{E}}(I(h_j^{\pm 1}(\epsilon)), I(h_j^{\pm 1}))$ are less than $\frac{1}{8}\mu$.

(2c) Moreover, for each $j$ the Euclidean distance between the fixed points of $h_j(\epsilon)$ is less than 1/4 and $\mathrm{dist}_{\mathbb{E}}(Fix(h_j(\epsilon)), O_R) \le 1/4$ , $j = 1, ..., r$ (see (7a)).

(3c) $I(c_i(\epsilon)) \cap I(c_i^{-1}(\epsilon)) = \emptyset$ ( in particular, all elements $c_i(\epsilon)$ are loxodromic). Let $\mathcal{J}(c_i(\epsilon))$ be the shortest among circular arcs in $S^3$ connecting the fixed points of $c_i(\epsilon)$ and invariant under $c_i(\epsilon)$. Then we require:

(4c) $\mathrm{dist}_{\mathbb{E}}(\mathcal{J}(c_i(\epsilon), O_R) \le 1/8$.

(5c) Suppose also that $\mathrm{dist}_{\mathbb{E}}(K(\mathcal{J}(c_i(\epsilon)), 3\pi/4), O_R) < 1/2$

(6c) $h_j(\epsilon)$ is conjugate to $h_j$ for each $j = 1, ..., r$.

Denote by $\overrightarrow{c_i(\epsilon)}$ the pair $(c_i(\epsilon), \mathcal{J}_i(\epsilon))$ (see [4, Section 1.1, Definition 1]).

**Definition 6.2.** *A collection of elements*

$$(h_1(\epsilon), \dots, h_r(\epsilon), c_1(\epsilon), \dots, c_s(\epsilon))$$

*with the properties (1c – 6c) above is called* admissible. *A collection*

$$(c_1(\epsilon), \dots, c_s(\epsilon))\ )$$

*is called admissible c–collection if it satisfies the properties (1c –5c).*

Thus according to Theorem 3.1, the exists a neighborhood $U$ of $(c_1, \dots, c_s)$ in $\mathbf{Mob}_3^s$ such that for every $c$-admissible collection of elements $(c_1(\epsilon), \dots, c_s(\epsilon)) \in U$ there is a representation $\rho_\epsilon : H \to \mathbf{Mob}_3$ such that:

$$\rho_\epsilon(h_{r+1}) = h_{r+1}, \tag{3}$$

$$\rho_\epsilon(c_i) = c_i(\epsilon), i = 1, .., s; \tag{4}$$

$$(\rho_\epsilon h_1 = h_1(\epsilon), .., \rho_\epsilon h_r = h_r(\epsilon), \rho_\epsilon(c_1) = c_1(\epsilon), .., \rho_\epsilon(c_s) = c_s(\epsilon)) \text{ is admissible} \tag{5}$$

**Definition 6.3.** *Representations $\rho_\epsilon$ satisfying the properties* (3) — (5)*, will be called* admissible.

As the image of a continuous family of admissible representations we obtain the family $H_\epsilon = \rho_\epsilon(H)$ of free Kleinian groups of the rank $r + s$ which have fundamental polyhedra $\mathcal{P}(\epsilon)$ bounded by the isometric spheres of $c_i^{\pm 1}(\epsilon), h_j^{\pm 1}(\epsilon), 1 \le i \le s, 1 \le j \le r$. The domain $\mathcal{P}(\epsilon)$ has the following properties:

(1e) $\mathrm{dist}_{\mathbb{E}}(\partial \mathcal{P}(\epsilon), O_R) < 1/2$;

(2e) Define $\mathcal{J}(h_j(\epsilon))$ as the shortest geodesics segments between the fixed points of $(h_j(\epsilon))$. Then the cones $K(\mathcal{J}(h_j(\epsilon), \pi/2)$ , $K(\mathcal{J}(c_i(\epsilon)), 3\pi/4)$ , $K(h_{r+1})$ are precisely invariant under corresponding cyclic subgroups in $H(\epsilon)$ , $i = 1, ..., s, j = 1, ..., r$.

(3e) For each cone $K$ (except for $K(h_{r+1})$) as in (2e), we have $\mathrm{dist}_{\mathbb{E}}(K, O_R) < \frac{1}{2}$ and $K \cap \mathcal{P}(\epsilon)$ is a fundamental domain for the action in $K$ of its stabilizer.

In what follows we will denote the element $h_{r+1}$ by $h_{r+1}(\epsilon)$. Denote by $\mathcal{K}_\epsilon$ the union of the cones

$$K(h_j(\epsilon)) = K(\mathcal{J}(h_j(\epsilon), \pi/2)\ ,\ K(h_{r+1})$$

$$\text{and}\quad K(c_i(\epsilon)) = K(\mathcal{J}(c_i(\epsilon)), 3\pi/4)\quad (i = 1, ..., s, j = 1, ..., r)$$

Let

$$p_\epsilon : R(H_\epsilon) \to R(H_\epsilon)/H_\epsilon = M(H_\epsilon)$$

be the natural projection. Then the manifold

$$M^*(H_\epsilon) = M(H_\epsilon) - p_\epsilon(\mathcal{K}_\epsilon)$$

is homeomorphic to $S^1 \times \Sigma$, where $\Sigma$ is a compact surface with $r + s + 1$ boundary circles and zero genus.

**Remark 6.4.** *As the result of this long and technical discussion we have a family of conformally flat manifolds with boundary $M^*(H_\epsilon)$. These manifolds have prescribed topological type and prescribed flat conformal structures near the boundary.*

## 7 The shortest arcs on boundaries of invariant cones.

Let $K$ be a cone in $S^3$ with the vertices $p_1, p_2$. Recall that in section 3 we have defined a flat metric $d_{\partial K}$ on the boundary of $K$.

Let $g$ be a loxodromic transformation preserving $K$ such that $g(p_i) = p_i$ $(i = 1, 2)$, complex coefficient $k(g)$ is not a negative real number (see [4, Section 1.1]). Choose a point $x \in \partial K - \{p_1, p_2\}$ and the shortest geodesic segment $\mu$ (with respect to the metric $d_{\partial K}$) between points $x$ and $g(x)$. Under our assumptions this segment is unique. Denote by $p_g : K \to K/\langle g\rangle$ the projection.

**Definition 7.1.** *The infinite arc $\nu =$ a connected component of $p_g^{-1} p_g(\mu)$ which contains $\mu$. is called a* shortest directed arc corresponding to $(K, g)$. *The orientation on $\nu$ is given by the action of $g$.*

**Remark 7.2.** *The homotopy class of the projection of $\nu$ to $\partial K/\langle g\rangle$ does not depend on the choice of the point $x$. We needed a metric on $\partial K$ to make the choice of this homotopy class unique.*

Given points $x \in \partial K(h_j(\epsilon))$ and $y \in \partial K(c_i(\epsilon))$ we will construct the shortest directed arcs $\tilde{\gamma}_j(\epsilon)$ and $\tilde{\beta}_i$ $(\epsilon)$ corresponding to

$$(K(h_j(\epsilon)), h_j(\epsilon)) \text{ and } (K(c_i(\epsilon)), c_i(\epsilon))$$

respectively. Because $k(h_j(\epsilon)) > 0$ and $k(c_i(\epsilon)) \to 1$ under $\epsilon \to 0$ the arcs $\tilde{\gamma}_j(\epsilon)$ and $\tilde{\beta}_i$ $(\epsilon)$ are defined correctly and depends continuously on $\epsilon$.

## 8 Certain Toric Constructions.

Let $O(P, \mathfrak{r})$ be a circle with the center $P$ and radius $\mathfrak{r}$ lying in the plane $\pi \subset \mathbb{R}^3$. Let $\mathcal{J} \subset \pi$ be a straight line such that $\text{dist}_{\mathbb{E}}(P, \mathcal{J}) = R$, where $0 < \mathfrak{r} < R$. Denote by $T(R, \mathfrak{r})$ the torus obtained by rotating $O(P, \mathfrak{r})$ around the axis $\mathcal{J}$ (see Figure 2). Then $\mathcal{J}$ is called the *axis* of $T(R, \mathfrak{r})$. Let $O_R$ be a circle of the radius $R = 10m + 6$ (as in Section 6), $O_R$ is the boundary of a disc $\Delta_R$. Choose the axis $\mathcal{J}$ which passes through the point $O$ orthogonally to the disc $\Delta_R$. Let $Q$ be any point on $O_R$ and denote by $D(Q, 1/2)$ the Euclidean disc in $\mathbb{R}^3$ with center $Q$ and radius $1/2$, which is orthogonal to the circle $O_R$.

The solid torus $\mathcal{T}(m)$ is the solid of revolution of the disc $D(Q, 1/2)$ around the axis $\mathcal{J}$. Then $S^3 - \mathcal{P}(\epsilon)$ and orbit of $\mathcal{K}(\epsilon)$ under the action of $H(\epsilon)$ lay in $\mathcal{T}(m)$. Furthermore, the circle $O_R$ is so large that it is possible to arrange mutually disjoint Euclidean balls $B(P_i, 8)$ with centers at $P_i \in O_{R+2}$ and radii 8 $(i = 1, ..., m)$.

Let $\pi_i$ be the Euclidean plane that contains $\mathcal{J}$ and the point $P_j$; the line $\mathcal{J}_i \subset \pi_i$ is parallel to $\mathcal{J}$ and $\text{dist}_{\mathbb{E}}(P_j, \mathcal{J}_i) = 2$ , $\text{dist}_{\mathbb{E}}(\mathcal{J}, \mathcal{J}_i) = R$.

Denote by $T_i'(1, 1)$ the torus with the axis $\mathcal{J}_i$; let $\mathcal{J}_i^{\perp}$ be the perpendicular from $P_i$ to the line $\mathcal{J}$. Then $T_i(1, 1)$ is the torus, which is obtained of $T_i(1, 1)$ by the Euclidean rotation around the axis $\mathcal{J}_i^{\perp}$ to the angle $\pi/2$. It is easy to see that:

1) $\text{int}(T_i(1, 1)) \cap \mathcal{T}(m) = \emptyset$ and these solid tori form a solid Hopf link in $S^3$ (see Figure 3);

2) $T_i(1, 1)$ is contained in $B(P_i, 8)$.

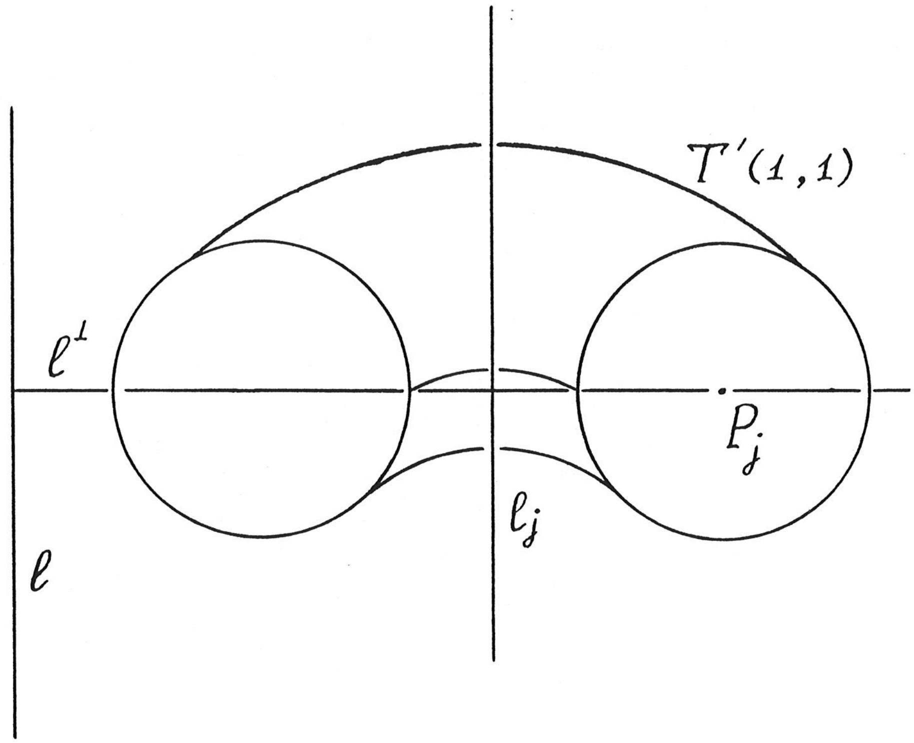


Figure 2: Torus of revolution.

Denote by $S(P_j, 1)$ the unit sphere, which is tangent to $T'_j(1,1)$ along a great circle; let $\omega_j$ be the inversion in the sphere $S(P_j, 1)$. Denote by $T^*_j(1,1)$ the image of $\omega_j(T'_j(1,1))$ under the homothety with the center $P_j$ and coefficient 7.5. An easy calculation shows that $T^*_j(1,1)$ is contained in $B(P_j, 8) - \mathcal{T}(m)$ and the topological solid tori $\mathrm{int}(T^*_j(1,1))$ and $\mathcal{T}(m)$ form a solid Hopf in $S^3$ (see Figure 4).

The manifold

$$L = S^3 - \mathcal{T}(m) - \cup_{j=1}^{m} \mathrm{int}(T^*_j(1,1))$$

is homeomorphic to $\Sigma_{0,m+1} \times S^1$. There is a canonical flat conformal structure on $L$ induced from $S^3$, this flat conformal structure is "toric" (in the sense of paragraph 3.1). If we remove from the manifold $M^*(H_\epsilon)$ the projection of the solid torus $\mathcal{T}(m)$, then there exists natural conformal gluing of the resulting manifold $M^{**}(H_\epsilon)$ with the manifold $L$. The manifold $M^{**}(H_\epsilon) \cup L$ is one of two building blocks for construction of flat conformal structures on Seifert manifolds which will be done in Section 10.

Finally we introduce the following notations: the clockwise oriented loop $\Delta_R \cap \partial\mathcal{T}(m)$ will be denoted by $\tilde{\delta}$. The oriented loop $\tilde{\theta} = \partial D(Q, 1/2)$ is oriented so that the pair $(\tilde{\delta}, \tilde{\theta})$ provides $\partial\mathcal{T}(m)$ with the orientation induced from $\mathrm{ext}(\mathcal{T}(m))$ (see Figure 5).

# 9 Cusp Closing and Shortest Arcs.

Let $N$ and $\Gamma$ be the hyperbolic manifold and discrete group from the section 3. Let $B_i$ be open horoballs in $\mathbb{H}^3$ which are precisely invariant under $\langle u_i \rangle \oplus \langle v_i \rangle \subset \Gamma$ and have

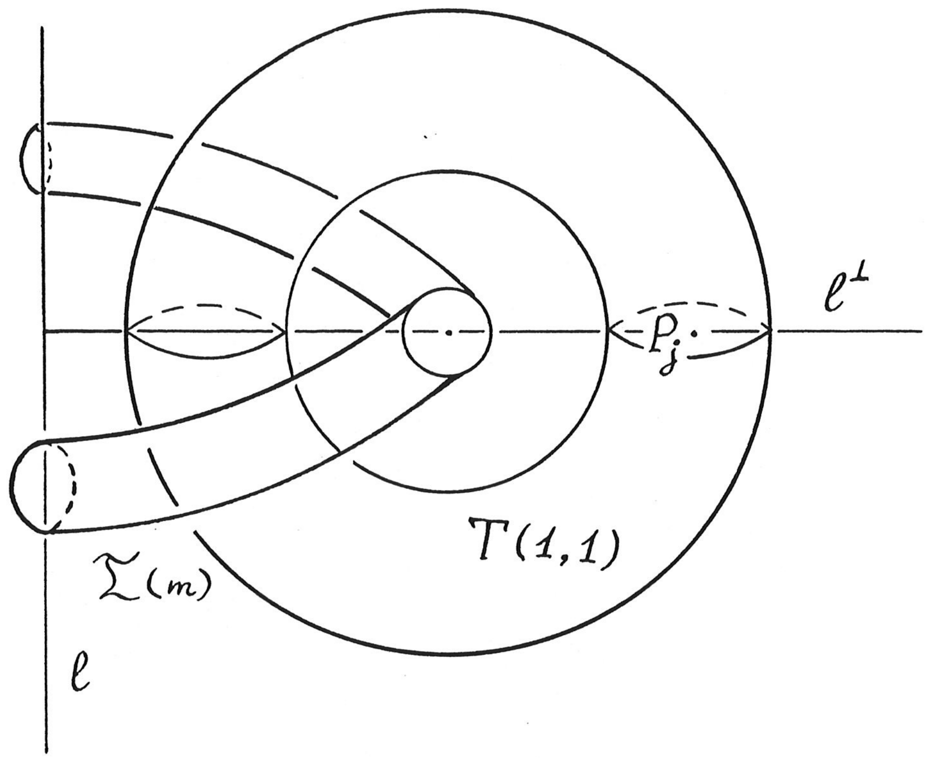


Figure 3: Solid Hopf link.

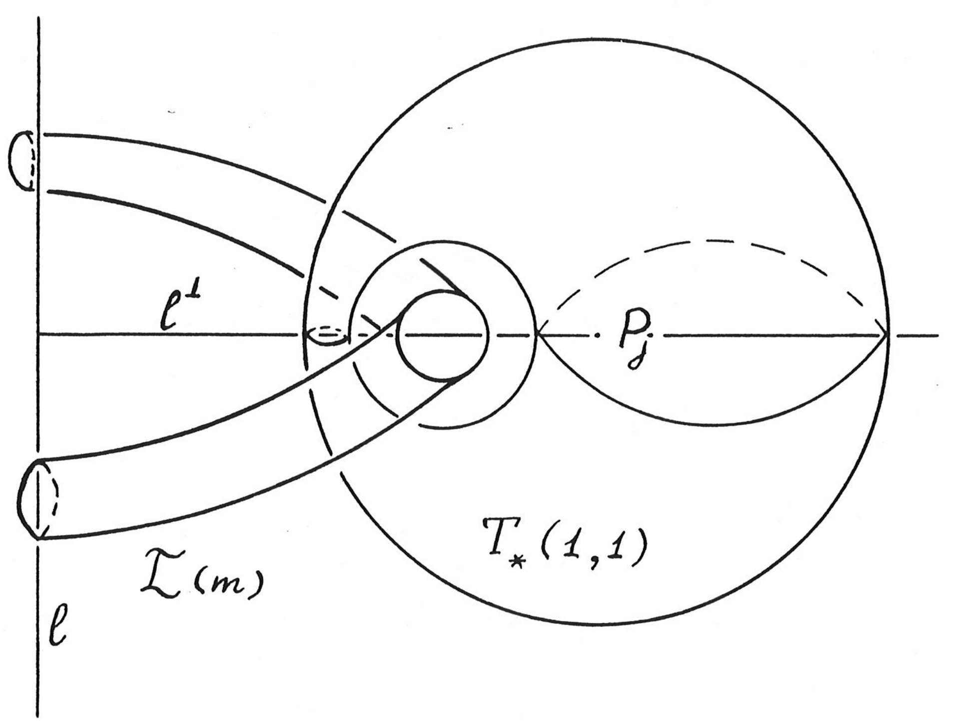


Figure 4: Solid tori $\mathrm{int}(T_j^*(1,1))$ and $\mathcal{T}(m)$.

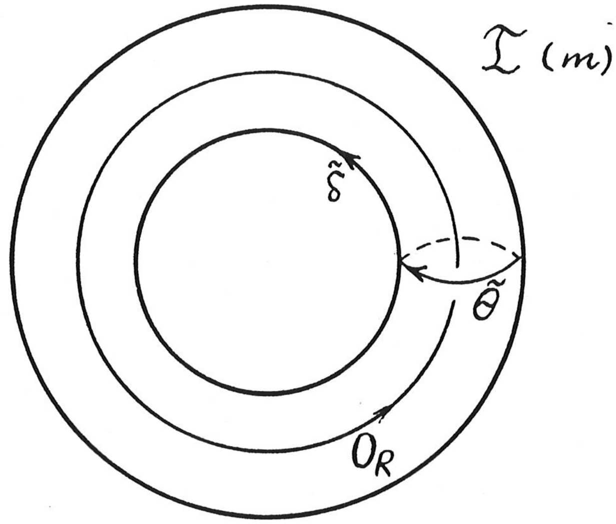


Figure 5: Orientation on $\tilde{\theta}$.

disjoint orbits (see Section 3). Let $\mathbb{H}^3_*$ denote the complement in $\mathbb{H}^3$ to the $\Gamma$-orbit of the union $\mathcal{B} = B_1 \cup \ldots \cup B_k$).

After some change of coordinates in $\mathbb{H}^3$ and choice of generator $v_1$ we can assume that

$$B_1 = \{(x_1, x_2, x_3) : x_3 > 1, x_1 + ix_2 \in \mathbb{C}\}$$

$$u_1 : z \mapsto z + 1, v_1 : z \mapsto z + w_1$$

$$\text{where} \quad -\frac{1}{2} < Re(w_1) \leq \frac{1}{2}$$

Let $\rho_n : \Gamma \to \Gamma(n)$ be a small deformation of $\Gamma$ (see Section 3), normalized so that

$$\rho_n(u_1) = u_1(n) : z \mapsto e^{2\pi i/n} z + \lambda_n$$

The homomorphisms $\rho_n$ are holonomy representations of certain hyperbolic structures on $N$ with the development maps $d_n : \tilde{N} \to \mathbb{H}^3$, where $\tilde{N} = \mathbb{H}^3_*$ is the universal cover of $N$; $d_0 = id$ and

$$\lim_{n \to \infty} d_n = d_0$$

The development maps $d_n$ were chosen so that $d_n(B_i)$ is the cone

$$K(i, n) = K(\mathcal{J}(i, n), \theta_n)$$

and

$$\lim_{n \to \infty} \theta_n = \frac{\pi}{2} \quad (i = 1, ..., k) \tag{6}$$

Let $V_0$ be the Euclidean segment between $z \in \partial B_1$ and $v_1(z)$; set $\bar{V}_n = d_n(V)$. Denote by $\bar{\nu}_n$ the *shortest geodesic segment* on $\partial K(\mathcal{J}(1, n)$ between the points $d_n(z)$, $d_n(v_1(z))$ (see Section 6). Let $\hat{V}_n$ be the orbit of $V_n$ under $\langle v_1(n) \rangle$.

**Lemma 9.1.** *For all but finitely many $n \in \mathbb{N}$, the arcs $\bar{\nu}_n$ and $\bar{V}_n$ are homotopic on $\partial K(\mathcal{J}(1,n)$ (rel $\{x, v_1(n)(x)\}$).*

*Proof.* It follows from (6) that

$$\lim_{n\to\infty} \bar{V}_n = V$$

Consider the line $\mathcal{J}(1,n)$ as the axis of the cylindrical coordinates $(r, \mathfrak{h}, \vartheta)$ in

$$\mathbb{H}^3 - \mathcal{J}(1,n)$$

Then, in the for all but finite $n$ the total variation of the angle $\vartheta$ along the arc $\bar{V}_n$ is less than $\pi$. Now, the direct calculation in the cylindrical coordinates implies that the same is true for $\bar{\nu}_n$ . □

Let $\hat{\mu}(n)$ be the shortest infinite arc corresponding to $(K(\mathcal{J}(1,n), \pi/4), v_1(n))$. Given any prime $n$ we can construct a normal subgroup of finite index $\Gamma_0 \subset \Gamma$ according to Theorem 1.3 [4, §1.3]. Define $\Gamma_0(n) := \rho_{\tau(n)}(\Gamma_0)$. Note that

$$\ker(\rho_\tau) = \langle\langle u_1^n, ..., u_k^n \rangle\rangle < \Gamma_0,$$

where the double brackets refer to the normal closure. Therefore, the group $\Gamma_0(n)$ is torsion-free and the covering

$$\mathbb{H}^3_{*,n}/\Gamma(n) \to \mathbb{H}^3_{*,n}/\Gamma_0(n)$$

is equivalent to the covering

$$\mathbb{H}^3/\Gamma = N \to \mathbb{H}^3/\Gamma_0.$$

Denote by $p_{0,n}$ the projection $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma_0(n)$ ($p_{0,0}$ is the projection from $\mathbb{H}^3_*$ to $\mathbb{H}^3_*/\Gamma_0$).

**Corollary 9.2.** *For all but finitely many primes $n \in \mathbb{N}$, the projection*

$$p_{0,n}(\hat{\mu}(n)) \subset \partial K(\mathcal{J}(1,n), \pi/4)/\langle v_1^n(n)\rangle$$

*of $\hat{\mu}(n)$ is freely homotopic to $p_{0,n}(\hat{V}_n)$.*

# 10 Uniformization of Seifert manifolds with boundary

# 11 Outline

In this section we will construct groups $G$ which uniformize Seifert components of Haken manifolds. The groups $G$ arise as Maskit combination of two types of Kleinian groups:

$G(10,1)$ which has the relative Euler number 1 (see section 4) and subgroups of $H_\epsilon$ which have zero relative Euler number (see sections 5, 6).

**Notation 11.1.** *Suppose that $\overrightarrow{T} = (T, \mathcal{L})$ is a directed loxodromic transformation in* **Mob**$_3$*. Then we set $\overrightarrow{T}^p = (T^p, \mathcal{L})$. Define the axis $Ax(\overrightarrow{T})$ of $\overrightarrow{T}^p$ to be $\mathcal{L}$.*

**Theorem 11.2.** *Let $e \in \mathbb{Z}, g, m, s/2 \in \mathbb{N}$ be integers such that $2g+m-|e| > 0$, $\{\overrightarrow{V_j(p)}\}$ be be a sequence of directed loxodromic transformations indexed by the system of all but finitely many primes $p \in \mathbb{N}$, and $1 \leq j \leq s$. Suppose that*

$$\lim_{p \to \infty} V_j(p) = 1$$

*Then for all but finitely many $p$ there exists a free Kleinian group*

$$G = G(e, m, s, p) \subset \mathbf{Mob}_3$$

*and a set of directed loxodromic elements $\overrightarrow{Y_j} \in \mathbf{Mob}_3$ conjugate to $\overrightarrow{V_j(p)}$ such that:*

*(1) The group $G$ contains $s$ directed loxodromic elements $\overrightarrow{Y_j}^p$ $(1 \leq j \leq s)$;*

*(2) Let $K_j$ be the cone $K(Ax(\overrightarrow{Y_j}), 3\pi/4)$,*

$$\mathcal{K} = \cup_{j=1}^{s} K_j$$

$$\mathcal{Y} = \cup_{j=1}^{s} Y_j^p$$

*Then the pair $(\mathcal{K}, \mathcal{Y})$ satisfies the conditions (1e—3e) of Section 3.6.4.*

*(3)*

$$\Sigma = \Sigma_{\tilde{g},s}, \tilde{g} = (p-1)g + (p-1)(m+s)/2 - p + 1$$

*(4) The group $G$ has a fundamental set $\Phi$ which contains the topological solid torus $S^3 \setminus \mathcal{T}(m)$, where $\mathcal{T}(m)$ was constructed in Section 8. Moreover, the intersection of $\Phi$ with each $K_j$ is a fundamental domain for the action of $\langle Y_j^p \rangle$ in $K_j$;*

*(5) Denote by $\mathfrak{p}: R(G) \to M(G) = R(G)/G$ the quotient map, then*

$$M_*(G) = M(G) - (\mathcal{K} \cup \mathfrak{p}(\mathcal{T}(m)))$$

*is homeomorphic to*

$$\Sigma_{\tilde{g},s+1} \times S^1$$

*(6) Let $\tilde{\delta}$ be the oriented loop on $\partial\mathcal{T}(m)$ as in Section 7; $\tilde{\beta}_j \subset \partial F_j$ be the shortest directed arc which corresponds to $(K_j, Y_j)$. Define $\beta_j = \mathfrak{p}(\tilde{\beta}_j)$, $\delta = \mathfrak{p}(\tilde{\delta})$*

$$\beta = \cup_{j=1}^{s} \beta_j$$

*Then:*

$$e(M_*(G), \beta \cup \delta) = e$$

*(see Sections 3, 4 for the definitions).*

**Remark 11.3.** *We will explain the importance of the condition (3) in [6].*

*Proof of Theorem 11.2.*

Denote the number $\max\{2, |e|\}$ by $r$. Choose a (large) prime number $p_0$ so that:

$$\xi(p_0) = p_0(g + \frac{m}{2} - r) - \frac{m}{2} - 9r + 1 \geq g_0(w, \ell) \tag{7}$$

Here $g_0(w, \ell)$ is the function defined in Lemma 5.4, $\ell$ and $w$ are defined according to Subsection 3.6.2, using the parameters $(r, s, m)$ of the current section.

### 11.1 Case 1: $s$ is positive

Let $\mathcal{S}$ be a complete hyperbolic surface of finite area of the genus 0 with $s$ punctures and $r+1$ geodesic boundary components; thus i.e. $\mathrm{int}(\mathcal{S}) = \Delta_R/\tilde{H}$ where the group $\tilde{H}$ was constructed in Section 5. Because the number $s$ is even, there exists a regular $p$-fold covering $\tilde{\mathcal{S}} \to \mathcal{S}$ such that $\tilde{\mathcal{S}}$ has $s$ punctures and $p\cdot(r+1)$ boundary loops (see [4]). Then the genus of $\tilde{\mathcal{S}}$ equals $g' = 1-p+s(p-1)/2$. We will assume that $p \geq p_0$.

Denote by $b_j$ the components of $\partial\tilde{\mathcal{S}}, 1 \leq j \leq (r+1)p$. Next we glue the following compact hyperbolic surfaces[2] to $\tilde{\mathcal{S}}$:

(a) To each component $b_j$ of $\partial\tilde{\mathcal{S}} \setminus (b_1 \cup \ldots \cup b_{r+1})$ we glue a hyperbolic surface $\Sigma_{1,1}$ with the geodesic boundary of the same length as $b_j$.

(b) To each loop $b_j, 1 \leq j \leq r$ we glue an isometric copy of the hyperbolic surface $\Sigma_{10,1}$ with the geodesic boundary of the same length as $b_j$.

(c) Finally, along the component $b_{r+1}$ we glue a hyperbolic surface $\Sigma_{\xi(p),1}$ whose geodesic boundary loop has the length $\ell$ and admits a regular $w$-collar. (Such hyperbolic surface exists due to (7).)

As the result of this gluing we obtain a hyperbolic surface $\tilde{S}^*$ of finite area which has $s$ punctures and the genus

$$(9+p)r - p + s(p-1)/2 + \xi(p) = \tilde{g}$$

If we remove $m+1$ disjoint closed discs from $\tilde{S}^*$ then we get a surface homeomorphic to $\mathrm{int}(\Sigma_{\tilde{g},s+1})$ (see Part (5) of Theorem 11.2).

### 11.2 Case 2: $s = 0$

We have $r \geq 2, p \geq 2$, therefore (*) implies the inequality

$$\xi'(p_0) = p_0(2g+m-2)/2 + 1 - m/2 - 10r > g_0(\ell, w)$$

Then we set $\tilde{S} = S \to S$ be the trivial covering. Let $p \geq p_0$. In the same way as in the Case 1 we glue surfaces $\Sigma_{10,1}$ to $b_1, \ldots, b_r \subset \partial S$ and the surface $\Sigma_{\xi'(p),1}$ (such that $\partial\Sigma_{\xi'(p),1}$ has a regular $w$-collar) is glued to $b_{r+1}$. As the result we obtain a surface $\tilde{S}^*$ of the same genus as $\mathrm{int}(\Sigma_{\tilde{g},s+1})$.

### 11.3 Construction of a finite-index subgroup in $H(\epsilon_p)$.

Set $\hat{p} = p$ in the Case 1 and $\hat{p} = 1$ in the Case 2. In the following sections 11.4—11.7 we will construct the required group $G$ via the first Maskit Combination. The combination process corresponds to gluing of $\tilde{S}$ from pieces, that was described in Sections 11.1,11.2.

Recall that

$$\lim_{p\to\infty} V_j(p) = 1$$

Then for all $p \geq p_1 \geq p_0$ there is $\epsilon = \epsilon_p$ and a $c$-admissible collection

$$(c_1(\epsilon), \ldots, c_s(\epsilon))$$

[2] Each of which has a single geodesic boundary component.

such that the directed loxodromic elements $\overrightarrow{c_j} = (c_j, \mathcal{J}(c_i(\epsilon))$ are conjugate to $\overrightarrow{V_j(p)}$.

The fundamental set $\mathcal{P}(\epsilon_p)$ is not good enough for the Maskit Combination (along $\langle h_i \rangle$) even in the Case 2. The reason is that

$$\mathcal{K}(h_{r+1}(\epsilon_p)) \cap \mathcal{P}(\epsilon_p)$$

is not a fundamental set for action of $\langle h_{r+1} \rangle$ in this cone. Hence we have to change $\mathcal{P}(\epsilon_p)$.

Set $\mathcal{S}_{r+1} := I(h_1(\epsilon_p))$ and $\mathcal{S}'_{r+1} := h_{r+1}(I(h_1(\epsilon_p)))$.

Since $\mathcal{P}(\epsilon_p)$ is a fundamental domain for $H(\epsilon_p)$ then

$$\mathcal{S}'_{r+1} \cap \mathcal{S}_{r+1} = \emptyset$$

thus the exterior of $\mathcal{S}'_{r+1} \cup \mathcal{S}_{r+1}$ is a fundamental set for the group $\langle h_{r+1} \rangle$. Attach the intersection

$$\nabla := K(\mathcal{J}(h_{r+1}), \pi/2) \cap \text{int}(I(c^{-1}(\epsilon_p)) - \text{int}(\mathcal{S}'_{r+1})$$

to the set $\mathcal{P}(\epsilon_p)$. See Figure 6.

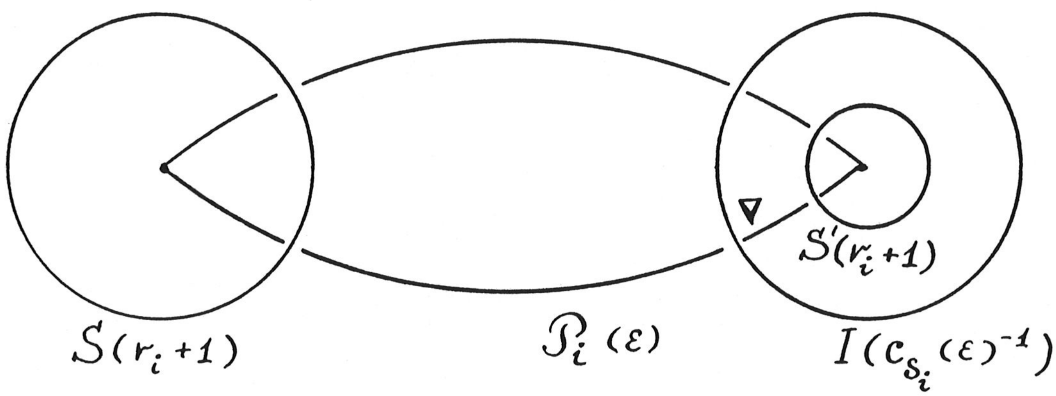


Figure 6: Modification of the fundamental domain

To preserve the fundamentality of $\mathcal{P}(\epsilon_p)$ after this attachment, we have to remove the orbit

$$(H(\epsilon_p) - 1)(\nabla)$$

from the set $\mathcal{P}(\epsilon_p) \cup \nabla$. Then

$$\mathcal{P}^0(\epsilon_p) := (\mathcal{P}(\epsilon_p) \cup \nabla) - (H(\epsilon_p) - 1)(\nabla)$$

is a fundamental domain.

It follows from the properties of the group $H(\epsilon_p)$ that $S^3 \backslash \mathcal{P}^0(\epsilon_p)$ is contained in $\mathcal{T}(m)$, where $\mathcal{T}(m)$ is the solid torus defined in Section 7.

Recall that the surface int$(S)$ is homeomorphic to $\Delta_R/H$. Then the regular covering $\tilde{S} \to S$ corresponds to a normal subgroup $H_p \subset H$ of the index $\hat{p}$.

Let $H(p)$ denote the group $\rho_\epsilon(H_p)$. Then we have the coset decomposition:

$$H(\epsilon_p) = 1 \cdot H(p) + c_1(\epsilon_p) \cdot H(p) + ... + (c_1(\epsilon_p))^{p-1} \cdot H(p)$$

Denote by $\varphi_{j'}$ the element $h_{j'}(\epsilon)$ of the group $H_\epsilon$. Then

$$\varphi_j := \varphi_{j'+q(r+1)} := (c_1(\epsilon_p))^q \cdot \varphi_{j'} \cdot (c_1(\epsilon_p))^{-q}$$

$(0 \leq q < \hat{p})$ are the elements generating the fundamental groups of components of $\partial\tilde{S}$. The cone $K(\varphi_j) = c(\epsilon_p)^q K(\varphi_{j'})$ is precisely invariant under $\varphi_j$ $(1 \leq j' \leq r)$.

For all values $j' = r+1 \neq j$ we choose a precisely invariant cone $\mathcal{K}(\varphi_j)$ which has the vertex angle $\pi/2$ and the common axis with $(c_1(\epsilon_p))^q(\mathcal{K}(\varphi_{j'}))$. All these cones lie in $S^3 \backslash P(\epsilon_p)$ and, hence, the complement to the set

$$P^-(\epsilon_p) = P^0(\epsilon_p) \backslash (\bigcup_{j=1}^{\hat{p}(r+1)} \mathcal{K}(\varphi_j) \cup \bigcup_{l=1}^{s} \mathcal{K}(c_l(\epsilon_p^{\hat{p}}))$$

is inside of the solid torus $\mathfrak{T}(m)$ as well.

## 11.4 Fundamental domains for the groups $\langle\varphi_j\rangle$ and $H(p)$.

Let $j' < r+1$. Then

$$P\langle\varphi_{j'}\rangle := \mathrm{ext}(I(\varphi_{j'})) \cap \mathrm{ext}(I(\varphi_{j'}^{-1}))$$

is the isometric fundamental domain for the group $\langle\varphi_{j'}\rangle$. Thus

$$P\langle\varphi_j\rangle = c_1(\epsilon_p)^q(P\langle\varphi_{j'}\rangle)$$

Similarly, for $j' = r+1$ we let

$$P\langle\varphi_{j'}\rangle := \mathrm{ext}(S(r+1) \cup S'(r+1))$$

and for $j = j' + q(r+1)$ we let

$$P\langle\varphi_{j'}\rangle := c_1(\epsilon_p)^q(P\langle\varphi_{j'}\rangle)$$

The fundamental set $P^0(\epsilon_p)$ has a defect: the intersection $P^0(\epsilon_p) \cap \mathcal{K}(c_l(\epsilon_p))$ is not a fundamental set for the action of $\langle c_l(\epsilon_p)^{\hat{p}}\rangle$ in this cone, $2 \leq l \leq s$. For this reason we perform the following modification of $P^0(\epsilon_p)$. We let

$$P\langle c_l(\epsilon_p)^{\hat{p}}\rangle := \bigcup_{q=1}^{\hat{p}-1} c_l^{-q}(\epsilon_p)(\mathcal{K}(c_l(\epsilon_p) \cap P^0(\epsilon_p))$$

It is easy to see that this set is a fundamental domain for the action of the group $\langle c_l(\epsilon_p)^{\hat{p}}\rangle$ in the cone $\mathcal{K}(c_l(\epsilon_p)$ Furthermore, the set

$$P(p) := P^-(\epsilon_p) \cup \cup_{l=1}^{s} P\langle c_l(\epsilon_p)^{\hat{p}}\rangle \cup \bigcup_{j=1}^{\hat{p}(r+1)} P\langle\varphi_j\rangle$$

is a fundamental domain for the group $H(p)$.

In what follows we will denote the manifold

$$(R^*(H(p)) := R(H(p) \backslash H(p) \cdot \cup_j \mathcal{K}(\varphi_j))/H(p)$$

$M^*(H(p))$. Also we denote the shortest infinite arc corresponding to $(K(\varphi_j), \varphi_j)$ by $\tilde{\gamma}_j$.

## 11.5 Construction of Kleinian groups corresponding to surfaces which are glued to $\tilde{S}$.

Given an element $\varphi_j \in \mathbf{Mob}_3$ we let $\mathfrak{l}_j$ denote the translation length of the induced isometry $\varphi_j : \mathbb{H}^4 \to \mathbb{H}^4$.

Consider first the generic case: $|e| \geq 2$, i.e. $|e| = r$. For the elements $\varphi_j$ $(j > r+1)$ we choose a hyperbolic surface$S(j)$ of the genus 1 with the unique geodesic boundary loop of the length $\mathfrak{l}_j$ (compare Section 11.2). If $j = r+1$ then we choose a surface of the genus $\xi(p)$ or $\xi'(p)$ (according to the Case 1 or Case 2 of the Section 11.2). Let $F(j)$ be a Schottky subgroup of $\mathrm{Isom}(\mathbb{H}^2) < \mathbf{Mob}_3$ such that $S(j)$ isometric to the Nielsen's kernel of $\mathbb{H}^2/F(j)$. Let $\varphi'_j \in F(j)$ be an element corresponding to the generator of $\pi_1(\partial S(j))$. Furthermore we let $A(\varphi_j) \subset R(F(j))$be a complementary segment in $\partial\mathbb{H}^2$ to $Fix(\varphi'_j)$, let $\mathcal{K}(\varphi_j)$ be the cone with the axis $A(\varphi'_j)$ and the vertex angle $\alpha = \pi/2$ (in the case $j \neq r+1$) and $\alpha = \pi - \vartheta$ (in the case $j = r+1$).

When $j \neq r+1$ the intersection $\mathcal{K}(\varphi'_j) \cap \mathbb{H}^2$ is contained in the complement to the Nielsen's convex hill $N(j)$ of $F(j)$; if $j = r+1$ then $\mathcal{K}(\varphi'_j) \cap N(j)$ lies in the $w$-collar of $\partial N(j)$. In any case, the cone $\mathcal{K}(\varphi'_j)$ is precisely invariant under $\langle\varphi_j\rangle \subset F(j)$.

Surfaces of the genus 10 (which has been glued to $\tilde{S}$) correspond to $r$ copies of the group $G(10,1) = F(j)$, that had been constructed in Section 4. We will use $r$ copies of the ball $B \subset S^3$ as precisely invariant cones $\mathcal{K}(\varphi'_j)$, where $\varphi'_j := h \in G(10,1)$ (see Section 4), $j = 1, ..., r$.

In the exceptional cases $e = 0$, $|e| = 1$ we replace $r - e$ copies of the group $G(10,1)$ by the Fuchsian Schottky groups $F(j)$. These groups uniformize the genus 10 surface in the same way as $F(i), i \geq r+1$ (see the generic case). The vertex angles $\alpha$ in these cases are $\pi/2$ .

## 11.6 Construction of fundamental sets $P(j)$ for the groups $F(j)$.

By the choice of vertex angles for the cones $\mathcal{K}(\varphi'_j)$ there exist Moebius transformations $\zeta_j$ which map $\mathrm{ext}(\mathcal{K}(\varphi'_j))$ onto $\mathrm{int}(\mathcal{K}(\varphi_j))$. These transformations may be chosen with the additional property: $\zeta_j$ maps the attractive fixed points of $\varphi'_j$ to the attractive fixed points of $\varphi_j, 1 \leq j \leq \hat{p}(r+1)$. Hence we have:

$$(\zeta_j^*(\varphi'_j) := \zeta_j \varphi'_j \zeta_j^{-1} = \varphi_j$$

The domain $P\langle\varphi_j\rangle := \zeta^{-1}(P\langle\varphi_j\rangle)$ is a fundamental domain for the group $\langle\varphi'_j\rangle$. Then we set

$$P(j) \cap \mathcal{K}(\varphi'_j) = P\langle\varphi\rangle \cap \mathcal{K}(\varphi'_j)$$

Furthermore, choose an arbitrary fundamental set

$$P(j) \cap (S^3 \backslash \mathcal{K}(\varphi'_j))$$

for the action of the group $F(j)$ on $S^3 \backslash F(j)(\mathcal{K}(\varphi'_j))$ Thus we get the fundamental set

$$P(j) := (P\langle\varphi'_j\rangle \cap \mathcal{K}(\varphi'_j)) \cup (P\langle\varphi_j\rangle \cap (S^3 \backslash \mathcal{K}(\varphi'_j)))$$

for the action of the group $F(j)$ on $S^3$. This fundamental set satisfies for the conditions of the first Maskit Combination Theorem with the amalgamated subgroup $\langle\varphi'_j\rangle$.

Let $M^*(F(j))$ denote the manifold

$$(R(F(j))\backslash F(j) \cdot \mathcal{K}(\varphi_j'))/F(j).$$

## 11.7 Construction of the group $\mathfrak{G}$.

Pick a prime number $p > p_1$. We will combine the following Kleinian groups:

$$H(p), F^*(j) := \zeta_j^*(F(j)), j = 1, ..., (r+1)\hat{p} \; .$$

The group $H(p)$ has the fundamental domain $P(p)$ (see Section 11.4). The groups $F^*(j)$ have the fundamental domains $P^*(j) := \zeta_j(P(j))$ such that:

$$(P^*(j)\backslash \operatorname{ext}(\mathcal{K}(\varphi_j)) \cap (P^0(p)\backslash \operatorname{int}(\mathcal{K}(\varphi_j))$$

is a fundamental domain for action of the group $\langle \varphi_j \rangle$. Hence the conditions of Maskit's first Combination Theorem are satisfied and the group

$$G = \langle H(p), F(j)^*, 1 \le j \le (r+1)\hat{p} \rangle$$

is a Kleinian group and the set

$$\Phi := (P(p)\backslash \bigcup_{j=1}^{(r+1)\hat{p}} \mathcal{K}(\varphi_j)) \cup \bigcup_{j=1}^{(r+1)\hat{p}} (P^*(j)\backslash \operatorname{ext}(\mathcal{K}(\varphi_j))$$

is a fundamental domain for this group. Clearly $S^3\backslash\Phi$ is contained in the solid torus $\mathfrak{T}(m)$.

Next we let $\overrightarrow{Y_j} := \overrightarrow{c_j}(\epsilon_p)$ and $\mathcal{K}_j := \mathcal{K}(c_j(\epsilon_p))$. By the construction of the set $\Phi$ and the results of Section 11.4, it follows that Property (4) of Theorem 11 holds for the set $\Phi$. Below we verify Property (5) of that theorem.

By the first Maskit's Combination Theorem, see [4], we have: The manifold $M^*(\mathfrak{G})$ is obtained by gluing the manifolds $M^*(H(p))$ and $M^*(F(j)), 1 \le j \le \hat{p}(r+1)$. All these manifolds are trivial Seifert fibered spaces. The gluing homeomorphisms are lifted to the maps $\zeta_j$ of the cones $K(\varphi_j), K(\varphi_j')$. Hence, the fibers of these bundles are glued to each other (so that the orientation of the fibers is preserved). Therefore, the manifold $M^*(\mathfrak{G})$ is a trivial circle bundle as well. By the construction of $\mathfrak{G}$ and the results of Sections 11.1, 11.2, the base of this bundle is homeomorphic to the surface $\mathfrak{R}$ (see Section 11). Moreover, $m+1$ boundary curves of $\mathfrak{R}$ correspond to the cones $K(Y_j), 1 \le j \le s$ and to the torus $\partial\mathfrak{T}(m)$.

Now we compute the relative Euler number for the group $G$. Recall that in the Section 6 we defined the *infinite shortest arcs* $\gamma_j \subset K(h_j(\epsilon_p))$ and $\beta_l \subset K(Y_l)$ so that:

$$e(H(\epsilon_p), \tilde{\beta}_1 \cup \ldots \cup \tilde{\beta}_s \cup \tilde{\gamma}_1 \cup \ldots \cup \tilde{\gamma}_{r+1}) = 0.$$

When we pass to the index $p$ subgroup in $H(\epsilon_p)$ the arcs

$$c_1(\epsilon_p)^q(\tilde{\gamma}_j') = \tilde{\gamma}_j, 0 \le q \le p-1$$

become the shortest arcs corresponding to $\varphi_j$. We set

$$\tilde{\beta} := \tilde{\beta}_1 \cup \ldots \cup \tilde{\beta}_s$$

Therefore the Euler number

$$e(H(p), \tilde{\beta} \cup \tilde{\gamma}_1 \cup \ldots \cup \tilde{\gamma}_{\hat{p}(r+1)})$$

is equal to zero. All $\tilde{\gamma}_j$ are circular arcs in $S^3$, hence $e(F(j), \zeta_j^{-1}(\tilde{\gamma}_j)) = 0$ for any $j > |e|$ (since $F(j)$ is a Fuchsian group for these $j$). If $j \leq |e|$ then $e(F(j), \zeta_j^{-1}(\tilde{\gamma}_j)) = 1$ because $F(j) = G(10, 1)$ (see the Section 6). Thus, $e(G, \tilde{\beta}) = |e|$ according to Section 3.

**Remark 11.4.** *The orientation of fibers of the manifold $M^*(G)$ is induced by the orientation of the loop $\tilde{\theta} \subset \partial\mathfrak{T}(m)$ (see Section 7). This orientation is consistent with the orientation of fiber of the manifold $\mathfrak{N}(10)$ (see Sections 3, 4).*

We need groups $G$ with negative Euler numbers as well. Thus, consider the reflection $J$ in the plane $\Pi'$ (see Section 2.5 of [4]). Then $J \circ h = h \circ J$, $J(B) = B$ and the group $G(10, -1) := JG(10, 1)J$ has the relative Euler number

$$e(G(10, -1), \tilde{\sigma}) = -1$$

if orientation of the fibers is given by the maps

$$\zeta_j : K(\varphi_j) \longrightarrow \bar{\Pi} = K(\varphi_j'), \quad j = 1, \ldots |e|$$

Then the group

$$G = G(-|e|) := \langle H(p), F(j)^*, |e| \leq j \leq (r+1)\hat{p}, \zeta_i^*(G(10, -1)), 1 \leq i \leq |e| \rangle$$

has all the properties of the group $\mathfrak{G}(|e|)$ but its relative Euler number $e(G, \tilde{\beta})$ is equal to $-|e|$.

So, for all $e \in \mathbb{Z}$ we have constructed the group $G$ such that $e(G, \tilde{\beta}) = e$. The relative Euler number $e(S^3 \backslash \mathfrak{T}(m), \tilde{\delta})$ is equal to zero (where we consider $S^3 \backslash \mathfrak{T}(m)$ as a trivial circle bundle with the typical fiber $\tilde{\theta}$). Then we have:

$$e(M^*(\mathfrak{G}, \beta_1 \cup \ldots \cup \beta_s \cup \delta)) = e \ .$$

The group $\mathfrak{G}$ satisfies the properties (1)-(6). This concludes the proof of Theorem 11.2. □

# 12 Conformal gluing

Suppose that $M$ is a smooth $n$-dimensional manifold with boundary. A *conformal thickening* of $M$ is an open conformally flat manifold $(M', C')$ such that $\partial M \subset M'$. Two conformal thickenings $(M', C')$ , $(M'', C'')$ of $M$ are *equivalent* if the restrictions of $C'$ and $C''$ to int$(M)$ coincide. Finally a conformally flat manifold with boundary $(M, C)$ is an equivalence class of conformal thickenings of $M$. For any conformally flat manifold with boundary $(M, C)$ we have the canonical coorientation $v$ on $\partial M$ (a nonvanishing field of normal vectors directed "inward" $M$). In what follows we will consider only the case of conformally flat manifold with compact boundary.

For any conformally flat manifold with boundary $(M, C)$ we can (and will) choose a conformal thickening $(M', C')$ such that $M$ is a deformation retract of $M'$. Let

$\tilde{M} \subset \tilde{M}'$ be the universal covering of $M$ embedded in the universal covering of $M'$. Then we define the developing map $d : \tilde{M} \to S^n$ to be the restriction of the developing map of $(M', C')$ to $\tilde{M}$. The holonomy representation of $(M, C)$ is the same as holonomy representation of $(M', C')$. It's easy to see that the developing map and holonomy representation are independent on the choice of conformal thickening.

Suppose that $(M_j, C_j)$ $(j = 1, 2)$ are two conformally flat manifolds with boundary, $(M_j', C_j')$ are their conformal thickenings, $D_j \subset \partial M_j$ are boundary components. A diffeomorphism $h : D_1 \to D_2$ is called *Moebius* iff $h$ extends to a conformal diffeomorphism $f : U_1 \to U_2$ of open neighborhoods $U_j$ of $D_j$ in $M_j'$ such that $f$ *reverses* the canonical coorientation.

**Remark 12.1.** *If $g_j$ are Riemannian metrics on $M_j$ corresponding to the flat conformal structures $C_j$, then a conformal map between $D_j$ with respect to the induced metrics is not (in general) Moebius according to our definition.*

Suppose that $(M_j, C_j)$ are two conformally flat manifolds with boundary, $D_j \subset \partial M_j$, $h : D_1 \to D_2$ is a Moebius diffeomorphism. Let $N$ be a manifold obtained by gluing $M_1, M_2$ via $h$. Then the flat conformal structures $C_1, C_2$ extend to a flat conformal structure $C$ on the manifold $N$. This idea of *conformal gluing* is due to R. Kulkarni [7]. We will use conformal gluing of manifolds constructed in this paper in [6].

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