List of problems on discrete subgroups of Lie groups and their computational aspects

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Abstract. In this paper we present a problem list pertaining to discrete subgroups of Lie groups and their computational aspects, consisting mostly of the problems collected during the ICERM workshop "Computational Aspects of Discrete Subgroups of Lie Groups" held in June of 2021.

In this paper we present a problem list, consisting mostly of the problems collected during the ICERM workshop held in June of 2021. However, some of the problems are older, some of these go back to the 1970s. Many of the problems are purely theoretical, while some have an obvious computational flavor.

1. Background

In this section we collect definitions and basic facts about abstract groups and discrete subgroups of Lie groups that are used in what follows.

Group theory. We begin with a discussion of some group-theoretic notions. Most of these notions deal with the *subgroup structure* of abstract groups.

An abstract group G is said to satisfy a property P *virtually* if there exists a finite-index subgroup of G which satisfies P.

An abstract group G is said to be *a surface group* if it is isomorphic to the fundamental group of a closed (i.e. compact with empty boundary) surface of negative Euler characteristic.

An abstract group G is said to be *coherent* if every finitely generated subgroup of G is also finitely-presentable.

A subgroup H of a group G is called *maximal* if there is no proper subgroup between H and G. Some maximal subgroups have finite index in G (for instance, subgroups of prime index are always maximal). Of interest to us are maximal subgroups of infinite index; we will refer to these as *strictly maximal*.

A group G is said to satisfy the *Howson property* if the intersection of any two finitely generated subgroups is again finitely generated.

The property is named after A. G. Howson, who proved in [**31**] that free groups satisfy this property. In contrast, if F_r is the free group of rank $r \geq 2$, then $F_r \times \mathbb{Z}$ does not satisfy the Howson property (see Example 1.1 below). In particular, $SL(n, \mathbb{Z}), n \geq 4$, does not satisfy the Howson property either (since it always contains $F_r \times \mathbb{Z}$). On the other hand, all discrete subgroups of $PSL(2, \mathbb{R})$ satisfy the Howson property (see e.g. [**27**] for surface groups). More generally, every finitely generated discrete subgroup of $PSL(2, \mathbb{C})$ which is not a lattice satisfies the Howson property; see e.g. [**30**].

Even more generally, if Γ_1, Γ_2 are geometrically finite subgroups of a discrete subgroup Γ in a rank 1 Lie group (see below), then the intersection $\Gamma_1 \cap \Gamma_2$ is again geometrically finite, hence, finitely generated. A proof of this result again appears in Hempel's paper [**30**]: While he only works with discrete subgroups of $PSL(2, \mathbb{C})$, his proof is also valid for subgroups of other rank 1 Lie groups. In contrast, the Howson property fails *for all* lattices in $PSL(2, \mathbb{C})$: It was noted by Hempel in [**30**] that the property fails for the fundamental groups of 3-dimensional manifolds fibering over the circle. Due to the work of Agol and Wise, it is known that all finite volume hyperbolic 3-manifolds admit finite-sheeted covering spaces which fiber over the circle.

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Distortion. Let G, H be finitely generated groups equipped with word metrics d_G, d_H respectively. Assume that H is a subgroup of G. Then the *distortion function* $\delta(n)$ for the inclusion map $H \to G$ is defined as follows:

$$\delta(n) = \max\{d_H(1,h)) : h \in H, d_G(1,h) \le n\}.$$

For instance, $\delta(n)$ is linear if and only if the inclusion $H \to G$ is *bi-Lipschitz*, i.e. there exists a constant A such that

$$d_H(1,h) \le A d_G(1,h)$$

for all $h \in H$. Subgroups with linear distortion are said to be *undistorted*. The same concept applies in the case of an isometric group action $H \times X \to X$ of a finitely generated group H on a metric space X. The distortion function (relative to a point $x \in X$) of this action is

$$\delta_x(n) = \sup\{d_H(1,h)) : h \in H, d_X(x,hx) \le n\}.$$

Geometric finiteness. We now turn to the discussion of discrete subgroups of Lie groups.

Many problems which are open for higher rank lattices (and their subgroups) are well-understood in the case of discrete subgroups of rank 1 Lie groups G, or, at least, for *geometrically finite* subgroups of G. Hence, we begin by reviewing some elements of the theory of discrete subgroups of rank 1 Lie groups.

Geometric finiteness. Let G be a rank 1 Lie group (with finite center and finitely many connected components). Then the symmetric space corresponding to G is the quotient $X = G/K$, where $K < G$ is a maximal compact subgroup. One equips X with the projection of a Riemannian metric on G which is K-right-invariant and G-left-invariant. The Riemannian manifold X is then complete, simply-connected and has sectional curvature in some interval $[-b, -a]$, where $a > 0$. A subgroup $\Gamma < G$ is discrete if and only if it acts properly discontinuously on X.

There are two most tractable classes of discrete subgroups $\Gamma < G$: *Convex-cocompact* and, more generally, *geometrically finite.*

Definition 1.1. *A discrete subgroup $\Gamma < G$ is said to be* convex-cocompact *if there exists a nonempty closed convex Γ-invariant subset $C \subset X$ such that C/Γ is compact.*

Every convex-cocompact subgroup is finitely-presentable and, moreover, is Gromov-hyperbolic. Examples of convex-cocompact subgroups are given, for instance, by uniform lattices in Γ. We refer the reader to the surveys [**32, 34**] for other interesting examples.

Here is a useful criterion of convex cocompactness: A subgroup $\Gamma < G$ is convex-cocompact if and only if the following two properties hold (see [**12**]):

(a) Γ is finitely generated. We let d_Γ denote the word metric on Γ with respect to some finite generating set.

(b) For one (equivalently, every) $x \in X$, the *orbit map*

$$o_x : \Gamma \to \Gamma x \subset X, \quad o_x(\gamma) = \gamma x$$

is a *quasi-isometric embedding* $(\Gamma, d_\Gamma) \to X$, where X is equipped with its Riemannian distance function d_X. In the case at hand, the map o_x is a quasi-isometric embedding if and only if there exists a constant L such that for each $\gamma \in \Gamma$

$$L^{-1} d_\Gamma(\gamma, 1_\Gamma) - L \le d_X(x, \gamma x) = d_X(x, o_x(\gamma)).$$

One also says that such subgroups $\Gamma < G$ are *undistorted* (the action of Γ on X is undistorted).

Definition 1.2. *A subgroup $\Gamma < G$ is said to be* geometrically finite *if the following two conditions hold:*
(a) There exists a nonempty closed convex subset $C \subset X$ such that C/Γ has finite and positive volume.
(b) Orders of finite-order elements in Γ are bounded from above.

Note that, in view of Selberg's lemma, the second condition is automatically satisfied if Γ is finitely generated. Examples of geometrically finite subgroups of G are given by lattices in G, in which case $C = X$. While geometrically finite subgroups of G are, in general, not undistorted, the distortion of the word metric of Γ with respect to the metric d_X is *at worst exponential*: There exists a constant A such that

$$\log_A(d_\Gamma(\gamma, 1_\Gamma)) - A \le d_X(x, \gamma x) = d_X(x, o_x(\gamma))$$

for all $\gamma \in \Gamma$.

Discrete subgroups of higher rank Lie groups.

A subgroup in a lattice Γ in an algebraic group G is called *thin* if it is Zariski dense but has infinite index in Γ.

An element of $SL(n, \mathbb{R})$ that is diagonalizable over $\mathbb{R}$ is said to be *regular* if it has distinct eigenvalues; it is called *singular* otherwise. A rank 2 free abelian subgroup of $SL(n, \mathbb{R})$ is said to be *supersingular* if it is generated by two singular elements, whose product is also singular. More generally, an element of a Lie group with finitely many connected components is called regular if its image under the adjoint representation is regular. In the theory of P-Anosov subgroups $\Gamma < G$ (which we will briefly discuss in a moment) one also meets a *relative* notion of regularity, relative to the parabolic group $P < G$. In particular, all infinite order elements of P-Anosov subgroups $\Gamma < G$ are P-regular.

Currently, there is no clarity on what higher-rank analogues of convex-cocompactness and geometric finiteness should be, i.e. generalizations of these rank 1 notions to discrete subgroups Γ of semisimple Lie groups G (with finitely many components and finite center), such that the real rank of G is ≥ 2. One of the generalizations of the class of convex-cocompact subgroups is given by *P-Anosov subgroups*, where P is a parabolic subgroup of G. We refer the reader to the paper [**35**, section 11] in this volume for some discussion of these and references. There is even less clarity regarding geometric finiteness; initial steps in this direction are taken in [**36**], where various relativizations of Anosov subgroups are proposed and relations between them are established. However, none of these classes contains any lattices in higher rank Lie groups. Another approach to generalizing convex-cocompactness in higher rank appears in [**18**].

The next example shows that the Howson property fails in higher rank, even for intersections of Anosov subgroups of lattices.

EXAMPLE 1.1. *There exists a discrete subgroup $\Gamma < SL(3,\mathbb{R})$ isomorphic to $F_2 \times \mathbb{Z}$ which contains two Anosov subgroups whose intersection is not finitely generated.*

To find such a subgroup, consider the standard embedding $SO(2,1) < SL(3,\mathbb{R})$ and note that it commutes with a subgroup C (isomorphic to $\mathbb{R}$) consisting of singular matrices. Let $\Gamma_1 < SO(2,1)$ be a Schottky subgroup isomorphic to the rank 2 free group F_2 and generated by elements a, b. Let c be a non-trivial element of C. The subgroup Γ generated by a, b and c is discrete, and isomorphic to $F_2 \times \mathbb{Z}$. Define $\Gamma_2 < \Gamma$ to be the subgroup generated by a and the product bc. Then the intersection $\Gamma_1 \cap \Gamma_2$ is the normal closure of $\langle a \rangle$ in F_2 (see [**51**]*), hence, it is not finitely generated (see also an explanation in* [**30**]*).*

At the same time, Γ_1 is an Anosov subgroup of the rank 1 Lie group $SO(2,1)$, hence, it is an Anosov subgroup in $SL(3,\mathbb{R})$ (see e.g. [**28**]*). With a bit more work, it follows that Γ_2 is also Anosov. For instance, if c is sufficiently close to $1 \in SL(3,\mathbb{R})$, then the Anosov property of Γ_2 follows from the stability of Anosov subgroups; see again* [**28**] *or* [**37**].

2. $SL(2,\mathbb{Z})$-related problems

Take the congruence subgroup $\Gamma(2) < SL(2,\mathbb{Z})$ and let Λ denote the commutator subgroup of $\Gamma(2)$. Then Λ is free of infinite rank.

PROBLEM 2.1 (A. Kontorovich). *Which integers are traces of elements of Λ? Is it true that the local obstruction is the only obstruction?*

Here, $z \in \mathbb{Z}$ is *locally* a trace of an element of Λ provided that for each natural number n, $z \pmod n$ is the trace of an element of Λ (also taken mod n). Note that B. Ogrodnik precisely identified all the local obstructions in this problem [**53**], and studied extensive numerics and other related considerations for this problem. We do not currently know that a positive proportion of numbers arise as traces! For progress on related "local-global"-type problems, see [**7, 8, 41**]. There has also been recent progress on traces in very thin (having critical exponent anything above 1/2) subgroups of $SL(2,\mathbb{Z})$, assuming said subgroups contain parabolic elements (which the above Λ does not): see [**42**].

3. $SL(3,\mathbb{Z})$-related problems

This is a series of general questions about structure of subgroups of $SL(3,\mathbb{Z})$.

3.1. Intrinsic properties of thin subgroups of $SL(3,\mathbb{Z})$.

PROBLEM 3.1. *What are finitely generated thin subgroups of $SL(3,\mathbb{Z})$ as abstract groups?*

Note that all currently known thin subgroups of $SL(3,\mathbb{Z})$ are either virtually free or virtually surface groups. Examples of free subgroups are given by Tits' ping-pong argument; see [**65**]. Examples of thin surface subgroups of $SL(3,\mathbb{Z})$ are constructed in [**44**].

PROBLEM 3.2 (M. Kapovich). *Give an example of a finitely generated thin subgroup of $SL(3,\mathbb{Z})$ which is neither virtually free nor is virtually a surface group. For instance, does the free product of two surface groups embed? Does the free product $\mathbb{Z}^2 \star \mathbb{Z}$ embed?*

Note that $SL(4,\mathbb{Z})$ contains subgroups isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ and free products of surface groups.

The existence of subgroups of $SL(3,\mathbb{Z})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ was claimed by G. Soifer in [**63**], but the proof is known to be wrong. In the subsequent paper [**64**], G. Soifer constructs subgroups in $SL(3,\mathbb{Q})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$. However, the construction depends on the existence of *singular* diagonalizable elements in $SL(3,\mathbb{Q}) \setminus \{1\}$ and such elements do not exist in $SL(3,\mathbb{Z})$. In fact, Soifer's construction requires the existence of *supersingular* diagonalizable subgroups in $SL(3,\mathbb{Q})$ (see section 1).

In view of Soifer's construction, it makes sense to ask a slightly more general question:

Problem 3.3 (M. Kapovich). *Does there exist a discrete subgroup $\Gamma < SL(3,\mathbb{R})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ and containing only regular diagonalizable elements?*

Similarly:

Problem 3.4 (K. Tsouvalas). *Does there exist a discrete subgroup $\Gamma < SL(3,\mathbb{R})$ isomorphic to $\Gamma_0 \star \mathbb{Z}$, where Γ_0 is a surface group?*

Note that it is impossible to find an Anosov subgroup $\Gamma < SL(3,\mathbb{R})$ isomorphic to $\Gamma_0 \star \mathbb{Z}$ with this property, since every Anosov subgroup of $SL(3,\mathbb{R})$ is either virtually free or a virtually surface group, [**16**].

Problem 3.5. *Is $SL(3,\mathbb{Z})$ coherent?*

This open problem goes back to Serre (1974), [**62**]. It is known that virtually free groups and virtually surface groups are coherent. More generally, fundamental groups of 3-dimensional manifolds are coherent. In particular, discrete subgroups of $PSL(2,\mathbb{C})$ are coherent. More examples of coherent groups come from combinatorial group theory; see the survey [**70**] by Wise. On the other hand, the groups $SL(n,\mathbb{Z})$, $n \geq 4$, are known to be noncoherent since they contain a copy of $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$, hence, of $F_2 \times F_2$, and the latter is known to be noncoherent; see e.g. [**50**]. (Here F_2 is the rank 2 free group.)

3.2. Extrinsic properties of thin subgroups of $SL(3,\mathbb{Z})$. In their pioneering paper [**47**], Margulis and Soifer proved that every finitely generated matrix group is either virtually polycyclic or contains a strictly maximal subgroup. However, very little is known about the algebraic structure of such subgroups.

Problem 3.6. *Is there a virtually free strictly maximal subgroup in $SL(n,\mathbb{Z})$, $n \geq 3$?*

Note that the proof of existence of strictly maximal subgroups in the work of Margulis and Soifer starts with construction of a profinitely dense free subgroup. But the next step of the construction is to extend such a subgroup to a maximal subgroup and it is totally unclear what happens to the algebraic structure of the subgroup in the process.

The next problem is due to G. Prasad and J. Tits:

Problem 3.7. *Is every strictly maximal subgroup of $SL(3,\mathbb{Z})$ virtually free?*

According to Margulis and Soifer [**48**], Prasad and Tits asked this question for $SL(n,\mathbb{Z})$, $n \geq 3$. It was proven by Margulis and Soifer [**48**] that the answer is negative for $n \geq 4$. The remaining open case is for $n = 3$.

The following open problem also goes back to the work of Margulis and Soifer [**47**], where they proved the existence of strictly maximal subgroups in finitely generated non-polycyclic matrix groups:

Problem 3.8. *Are there finitely generated strictly maximal subgroups of $SL(3,\mathbb{Z})$? The same question for $SL(n,\mathbb{Z})$, $n \geq 4$. (The expected answer is negative.)*

Problem 3.9 (J.-P. Serre). *Is there a profinitely dense non-virtually free subgroup in $SL(3,\mathbb{Z})$?*

Note that the key ingredient in proof of existence of strictly maximal subgroups in $SL(n,\mathbb{Z})$, $n \geq 4$, given in [**48**], is the existence of profinitely dense subgroups containing $\mathbb{Z}^2$.

Problem 3.10. *Is it true that Anosov subgroups of $SL(3,\mathbb{Z})$ are never maximal?*

Remark 3.11. *The only known results about nonexistence of strictly maximal finitely generated subgroups are in rank 1:*

1. It is an easy consequence of the ping-pong argument that if M is a maximal geometrically finite subgroup of a lattice in a rank 1 Lie group Γ, then M has finite index in Γ.

2. A much harder theorem is that every maximal finitely generated subgroup in a lattice $\Gamma < O(3,1)$ necessarily has finite index in Γ; see [**26**]. *(This is an application of deep structural results about finitely generated discrete subgroups of $O(3,1)$.)*

Remark 3.12. *It is known that every Anosov surface subgroup Γ of $SL(3,\mathbb{R})$ is* virtually *a maximal Anosov subgroup, i.e. if $\Lambda < SL(3,\mathbb{R})$ is any Anosov subgroup containing Γ, then $|\Gamma : \Lambda| < \infty$. At the same time, free Anosov subgroups of a semisimple Lie group G are never virtually maximal as Anosov subgroups, cf.* [**20**].

PROBLEM 3.13. *Does $SL(3,\mathbb{Z})$ have the Howson property?*

REMARK 3.14. *1. The answer to the previous question is negative for $SL(n,\mathbb{Z})$, $n \geq 4$, and positive for $SL(2,\mathbb{Z})$; see Section 1.*

2. The Howson property is unclear even for intersections of Anosov subgroups of $SL(3,\mathbb{Z})$; cf. the discussion in Section 1.

PROBLEM 3.15. *Is there a lattice $\Gamma < SL(3,\mathbb{R})$ containing a singular diagonal element?*

Note that such a lattice will also necessarily contain a product subgroup $F_2 \times \mathbb{Z}$. Then this lattice will not have the Howson property with respect to Anosov subgroups; see Example 1.1.

PROBLEM 3.16. *Give an example of a finitely generated thin subgroup of $SL(3,\mathbb{Z})$ which is not (relatively) Anosov.*

Note that all currently known constructions of finitely-generated thin subgroups of $SL(3,\mathbb{Z})$ are relatively Anosov. It is known that all finitely generated discrete subgroups of $SL(2,\mathbb{R})$ are geometrically finite, hence, relatively Anosov.

The next problem is motivated by the Howson property: It is possible that it is easier to prove this property by restricting to the class of Anosov subgroups:

PROBLEM 3.17 (M. Kapovich). *Suppose that Γ_1, Γ_2 are Anosov subgroups of $SL(3,\mathbb{Z})$. Is $\Gamma_1 \cap \Gamma_2$ finitely generated?*

4. Problems on higher rank lattices

4.1. Profinite density.

PROBLEM 4.1 (G. Soifer). *Does there exist a thin profinitely dense subgroup of $SL(n,\mathbb{Z})$, $n \geq 3$, generated by two elements?*

Note that Aka, Gelander and Soifer [**1**] proved that there exists a uniform constant k such that for every n, $SL(n,\mathbb{Z})$ contains a k-generated thin profinitely dense subgroup.

4.2. Commutator map problems.

PROBLEM 4.2 (A. Shalev). *Is it true that for $n \geq 3$ the commutator map of $SL(n,\mathbb{Z})$ is surjective?*

Note that $SL(n,\mathbb{Z})$ is a perfect group (equal to its own commutator subgroup), which implies that every element is a product of commutators. Not every perfect group has surjective commutator map. One measure of failure of surjectivity of the commutator map in a group Γ is given by the *commutator length* and *stable commutator length*:

Given $\gamma \in [\Gamma,\Gamma]$, let $\ell(\gamma)$ denote the least number k such that γ is the product of k commutators in Γ. The number $\ell(\gamma)$ is called the *commutator length* of γ. This quantity has an asymptotic counterpart, the *stable commutator length*:

$$\ell_\infty(\gamma) = \lim_{n\to\infty} \frac{\ell(\gamma^n)}{n}.$$

There are perfect groups which have elements of positive stable commutator length: For instance, each hyperbolic van Dyck group with the presentation

$$\langle a, b, c \mid a^p = b^q = c^r = abc = 1\rangle, \quad p^{-1} + q^{-1} + r^{-1} < 1$$

is perfect whenever the numbers p, q, r are pairwise coprime. However, such a group (as any nonelementary hyperbolic group) contains elements of positive stable commutator length since it admits unbounded quasimorphisms, [**4, 22**]. At the same time, if Γ is a lattice in a simple Lie group of rank ≥ 2, then $\ell_\infty(\Gamma) = \{0\}$; see [**15**].

REMARK 4.3. *For a group Γ, a map $f : \Gamma \to \mathbb{R}$ is said to be a* quasimorphism *if there is a constant C such that for all $\alpha, \beta \in \Gamma$,*

$$|f(\alpha\beta) - f(\alpha) - f(\beta)| \leq C.$$

In other words, quasimorphisms are approximate *additive characters of a group. Trivial examples of quasimorphisms are given by bounded maps $f : \Gamma \to \mathbb{R}$. Quasimorphisms form a real vector space. The quotient, denoted $QM(\Gamma)$, of this space by the subspace of bounded quasimorphisms detects "richness" of the space of quasimorphisms of Γ. There are many groups which do not admit nontrivial additive characters, but do admit unbounded quasimorphisms. For instance, for every* nonelementary hyperbolic group Γ, *the space $QM(\Gamma)$ is infinite-dimensional,* [**22**].

Note also that if $\Gamma = SL(2, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a quadratic field with infinitely many units, then Γ contains elements which are commutators *locally but not globally* [**24**]. Here, an element γ is *locally a commutator* if its image in every congruence-quotient of Γ is a commutator. An element of Γ is a commutator *globally* if it equals the commutator $[\alpha, \beta]$ for some $\alpha, \beta \in \Gamma$. Clearly, every global commutator is also a local commutator.

4.3. Characterization of higher rank lattices.

PROBLEM 4.4 (M. Kapovich). *What algebraic properties distinguish higher rank (irreducible uniform) lattices among abstract groups?*

One such characterization was given by Lubotzky and Venkataramana [**46**], in terms of profinite completions. There are some indirect signs that other algebraic characterizations of lattices are also possible:

(1) Higher rank lattices are quasi-isometrically rigid (Kleiner and Leeb [**40**], Eskin [**23**]).
(2) Higher rank lattices are rigid in the sense of the 1st order logic (Avni, Lubotzky, Mieri [**2**]).
(3) Appearance of Serre relators in profinite completions (Prasad, Rapinchuk [**58**]).

In the case of groups Γ of integer points of split semisimple algebraic groups over $\mathbb{Z}$, a defining feature is the *Serre relators*. However, Serre relators are for unipotent elements, which do not exist in uniform lattices. Uniform higher rank lattices satisfy *approximate* Serre relators.

PROBLEM 4.5. *Do these determine whether a discrete linear group is a higher rank lattice?*

An alternative approach to a characterization of lattices is via the *Prasad–Raghunathan rank*:

DEFINITION 4.1 (Prasad–Raghunathan rank). *Let Γ be a group. Let A_i denote the subset of Γ that consists of those elements whose centralizer contains a free abelian group of rank at most i as a subgroup of finite index. Thus, $A_0 \subset A_1 \subset \dots$. The* Prasad–Raghunathan rank*, $\mathrm{PRrank}(\Gamma)$, of Γ is the minimal number i such that $\Gamma = \gamma_1 A_i \cup \dots \cup \gamma_m A_i$ for some $\gamma_1, \dots, \gamma_m \in \Gamma$.*

For instance, if Γ is a lattice in a semisimple Lie group of rank n, then $\mathrm{PRrank}(\Gamma) = n$. If M is a compact Riemannian manifold of nonpositive curvature with $\Gamma = \pi_1(M)$, then $\mathrm{PRrank}(\Gamma)$ equals the geometric rank of M, i.e. the largest n such that every geodesic in M is contained in an immersed n-dimensional flat. We refer to [**56**] and [**3**] for details.

PROBLEM 4.6. *Are there discrete linear groups Γ which are not virtually nontrivial direct products and are not lattices, satisfying $\mathrm{PRrank}(\Gamma) \geq 2$?*

PROBLEM 4.7 (G. Prasad). *Does there exist a discrete Zariski dense subgroup $\Gamma < G$ (with G a simple real algebraic group) such that Γ is not a lattice but $\mathrm{PRrank}(\Gamma) = rank_{\mathbb{R}}(G)$?*

Another group-theoretic property closely related to lattices is the *bounded generation property*:

DEFINITION 4.2 (BGP, Bounded Generation Property). *A group Γ is said to have BGP if there exist elements $\gamma_1, \dots, \gamma_k$ such that every $\gamma \in \Gamma$ can be written as a product*

$$\gamma = \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_k^{n_k}$$

for some $n_1, \dots, n_k \in \mathbb{Z}$. (Note that a power of each γ_i appears only once.)

Many classes of higher rank nonuniform lattices satisfy the BGP; see the references in [**17**]. Nonlinear groups that satisfy the BGP were constructed by A. Muranov [**52**]. On the other hand, it was recently proven in [**17**] that uniform lattices in semisimple Lie groups *never* satisfy the BGP. More generally, they prove that a subgroup of $SL(n, \mathbb{C})$ boundedly generated by semisimple elements has to be virtually solvable.

PROBLEM 4.8 (M. Kapovich). *Suppose that Γ is an abstract (infinite) $\mathbb{R}$-linear group satisfying the BGP. Is it isomorphic to a lattice in a Lie group?*

4.4. Why are higher rank lattices super-rigid? One way to say that an abstract group Γ is *super-rigid* is to require that for every field F and $n \in \mathbb{N}$, there are only finitely many conjugacy classes of representations $\Gamma \to GL(n, F)$. Of course, some groups do not admit any nontrivial linear representations, so it makes sense to restrict the discussion to finitely generated linear groups Γ.

Loosely speaking, such a group is (super) rigid if it satisfies some peculiar relators. There are many proofs of rigidity and super-rigidity of (higher rank irreducible) lattices, but none of these proofs (in the setting of uniform lattices) use relators satisfied by lattices, likely because such relators are simply unknown (see previous section). In contrast, there are known proofs of super-rigidity of some classes of higher rank non-uniform lattices (see [**60**] and references therein) which use explicit relators.

PROBLEM 4.9 (M. Kapovich). *What are group-theoretic reasons that make higher rank uniform lattices (super)-rigid? Are the approximate Serre relators responsible for this? Or high Prasad-Raghunathan rank?*

One known result in this direction is that the BGP implies super-rigidity [**55**]. Another group-theoretic property implying super-rigidity is given by Lubotzky in [**45**].

5. Algorithmic problems

PROBLEM 5.1 (M. Kapovich). *For which classes of algebraic semisimple Lie groups G is the discreteness problem decidable for Zariski dense finitely generated subgroups?*

Here, decidability of discreteness is understood in the sense of BSS formalism of computations over the real numbers, as it is discussed for instance in [**25**] and [**33**]. The input for a possible BSS algorithm consists of a finite tuple of elements of G which generate a Zariski dense subgroup. The algorithm is supposed to determine if these elements generate a discrete subgroup. The Zariski density assumption is imposed to eliminate "trivial" counter-examples, which show that discreteness is undecidable already in the case of cyclic subgroups of S^1. It is known that discreteness is decidable for finitely generated subgroups of $G = PSL(2,\mathbb{R})$ (see for instance, Gilman's paper [**25**] and references therein) and is undecidable for subgroups of $PSL(2,\mathbb{C})$ (see [**33**]). The simplest case where the answer is unclear is, as usual, $G = SL(3,\mathbb{R})$.

PROBLEM 5.2 (A. Detinko). *Is freeness decidable for finitely generated subgroups of arithmetic groups?*

There is a practical algorithm testing whether a finitely generated linear group over an arbitrary (infinite) field contains a free non-abelian subgroup ([**19**, Section 6.2]).

Note that freeness is undecidable for subsemigroups in linear groups; see [**39**]. Freeness is decidable for subgroups of $SL(2,\mathbb{Z})$ and, more generally, for discrete subgroups of $SL(2,\mathbb{R})$. It is also decidable for some special classes of subgroups of arithmetic groups:

(a) Anosov subgroups.

(b) Subgroups which admit finitely-sided Dirichlet domains in associated symmetric spaces.

Freeness is likely to be, at least effectively, undecidable. The reason is the existence of *badly distorted* finitely generated free subgroups of $SL(n,\mathbb{Z})$ for large n: these are free subgroups whose distortion function is comparable to the k-th Ackermann function (for any k); see [**21, 13**] for the description of embeddings of such free groups in free-by-cyclic groups, and [**29, 69**] for embeddings into $SL(n,\mathbb{Z})$.

PROBLEM 5.3 (A. Detinko). *Is arithmeticity decidable? More precisely, is there an algorithm that decides if a finitely generated Zariski dense subgroup Λ (given by its set of generators) of an* irreducible *arithmetic group Γ (say, $SL(n,\mathbb{Z})$, $n \geq 3$) has finite index in Γ (cf.* [**19**, Section 5.3]*)?*

Note that this problem is semidecidable: There is an algorithm which will terminate if $\Lambda < \Gamma$ has finite index. The problem is known to be decidable for subgroups of $SL(2,\mathbb{Z})$ and undecidable for subgroups of $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$.

PROBLEM 5.4 (M. Kapovich). *Is the membership problem for finitely generated subgroups of $SL(3,\mathbb{Z})$ decidable?*

Note that the membership problem for a finitely generated subgroup H of a finitely generated group G is decidable if and only if the *distortion function* of H in G is recursive. All *known* finitely generated subgroups of $SL(3,\mathbb{Z})$ have at most exponential distortion, hence, have decidable membership problem.

In contrast, the membership problem is undecidable for finitely generated subgroups of $SL(4,\mathbb{Z})$. The reason is that this group contains $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$, which, in turn, contains a direct product of two free groups of large ranks. The latter contains finitely generated normal subgroups with undecidable membership problem (*Mihailova subgroups*, [**50**]). However, in this case, the ambient lattice is reducible.

PROBLEM 5.5. *Are there irreducible arithmetic groups Γ such that for Zariski dense subgroups $\Lambda < \Gamma$ the membership problem is undecidable?*

Very likely, such arithmetic subgroups Γ can be found in $SO(p,q)$ for suitable p,q. The existence of Λ is an application of the Rips construction of small cancellation groups with non-recursively distorted normal subgroups [**61**], combined with the Cubulation Theorem of Dani Wise [**68**] and the embedability of cubulated groups in RACGs (Right-Angled Coxeter groups) [**69**], which, in turn, admit Zariski dense representations in $\Gamma := O(p,q) \cap GL(p+q,\mathbb{Z})$ [**5**].

Recall that the membership problem is decidable for quasi-isometrically embedded subgroups, such as Anosov subgroups and finite-index subgroups in lattices.

PROBLEM 5.6. *Suppose that Γ is an irreducible lattice in a higher rank semisimple Lie group. Is it decidable that $\gamma \in \Gamma$ is a commutator?*

Note that this question is a special case of decidability of equations in Γ. In the last 20 or so years there was a great deal of progress in understanding equations in (relatively) hyperbolic groups (which includes lattices in rank 1 Lie groups). In contrast, decidability of equations in higher rank lattices is very poorly understood.

Here is a similar number-theoretic problem:

PROBLEM 5.7. *Is every integer $n \in \mathbb{Z}$ a sum of three cubes, where n is not 4 nor 5 modulo 9? Is it even decidable if the given integer is a sum of three cubes?*

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