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Actions of discrete groups on nonpositively curved spaces

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1 Introduction

We are interested in properties of groups which admit sufficiently nice actions by isometries on spaces of nonpositive curvature. Let us call a group *Hadamard* if it admits discrete actions by non-parabolic isometries on Hadamard spaces. These are synthetic analogues of Hadamard manifolds, they are complete geodesic metric spaces which are nonpositively curved in the sense of distance comparison. Typical examples of Hadamard groups are subgroups of fundamental groups of closed Riemannian manifolds of nonpositive sectional curvature. Various algebraic and geometric properties of Hadamard groups are well-known, such as: Solvable subgroups are virtually abelian [GW, LY] and centralizers virtually split [E, BH]. If G is a finitely generated Hadamard group then the stable norm (translation number) $\|g\| := \lim_{n\to\infty} \frac{|g^n|}{n}$ of a non-periodic element $g \in G$ is always non-zero. This excludes for instance Baumslag-Solitar subgroups.

The class of semisimple (i.e. non-parabolic) actions on Hadamard spaces has better functorial properties than the subclass of cocompact actions. Note however that there are groups which admit semisimple, but no cocompact discrete actions, e.g. certain infinitely generated groups or finitely generated groups with infinite-dimensional cohomology. Examples of such groups can be found in direct products of free groups, thus they have semisimple actions on products of hyperbolic planes.

The main goal of this note is to find new obstructions for the existence of semisimple actions. In Sect. 2, we derive general properties of Hadamard groups, in particular:

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Theorem 2.3 *Being Hadamard is a commensurability invariant for groups.*

Of basic importance for this paper is the observation that a Haken 3-manifold group admits discrete semisimple actions if and only if it admits cocompact actions, cf. [L1].

Theorem 2.4 *Let M be a compact Haken 3-manifold whose boundary has zero Euler characteristic. If $\pi_1(M)$ is a Hadamard group, then M admits a smooth metric of nonpositive sectional curvature with totally geodesic flat boundary.*

Corollary 2.7 *Let N be a closed manifold and M a graph-3-manifold with no metric of nonpositive curvature. If $\pi_1(N)$ contains $\pi_1(M)$ as a subgroup then N does not admit metrics of nonpositive curvature.*

For Seifert manifolds M, Theorem 2.4 has been proven by Bridson.

In Sect. 3, we discuss the existence of nonpositively curved metrics on Haken 3-manifolds.

Theorem 3.2 *Graph-manifolds containing a non-orientable Seifert component admit metrics of nonpositive curvature.*

This theorem was independently proven in [BK1]. According to [L1, L2] only graph-manifolds, i.e. Haken manifolds glued from Seifert components, can yield non-existence examples.

Example *(cf. Theorem 3.7) There are reducible diffeomorphisms of closed hyperbolic surfaces whose mapping tori do not admit metrics of nonpositive curvature.*

Special examples of this kind are first given in [L1] and are also discussed in [L2]. After this work was done, the authors learnt that Buyalo and Kobelski constructed a numerical invariant which detects the existence of nonpositively curved metrics on graph-manifolds in the general case, see [BK2].

In Sect. 4 we give new examples of non-Hadamard groups. We discuss the borderline case of the mapping class group Mod_S of a finite area hyperbolic surface S which is very close to be Hadamard: It admits a properly discontinuous action on a convex negatively curved space, namely Teichmüller space $T(S)$ equipped with the Weil-Petersson metric dW; however, $T(S)$ is not complete and Dehn twists act by parabolic isometries. The metric completion of $(T(S), dW)$ is a Hadamard space but the extended action of Mod_S is not discrete because it has infinite point stabilizers. Nevertheless:

Theorem 4.2 *Let S be a surface of finite type. If S is orientable we assume that either the genus of S is at least 3 or the genus of S is two and S has at least 1 puncture. If S is nonorientable we assume that either the genus of S is at least 4 or the genus of S is three and S has at least 2 punctures. Then the mapping class group Mod_S is not Hadamard.*

We conclude by giving an example of a non-Hadamard group for which the previously known obstructions to being Hadamard vanish.

Example *(cf. Theorem 4.5) There are closed 4-manifolds M which are fibered over hyperbolic surfaces with hyperbolic fiber so that $\pi_1(M)$ is not Hadamard.*

2 Hadamard groups

A *Hadamard space*, or CAT(0)-space, is a complete geodesic metric space which has nonpositive curvature in the sense of distance comparison, cf. [Ba1, Ba2, KlL]; it is not required to be locally-compact. Hadamard spaces are a synthetic generalization of Hadamard manifolds, i.e. complete simply-connected Riemannian manifolds of nonpositive sectional curvature.

An action ρ of a group Γ by isometries on a Hadamard space X is called *semisimple* if the displacement function of each isometry $\rho(\gamma)$ attains its infimum. This means that each isometry $\rho(\gamma)$ is either loxodromic (preserves a geodesic and acts on it as a translation) or elliptic (has a fixed point). The action ρ is called *discrete* if each metric ball B in X intersects only finitely many of its translates $\rho(\gamma)B$.

Definition 2.1 *We call a group Γ a* **Hadamard group** *if it admits a discrete semisimple action on a Hadamard space.*

Fundamental groups of closed nonpositively curved manifolds and their subgroups are examples of Hadamard groups.

2.1 Commensurability invariance

The Hadamard property is inherited by subgroups. In this section we show that it also passes to finite index supgroups. Let H be a finite index normal subgroup in G and suppose that ρ is a semi-simple action of H on a Hadamard space X. Consider the space of functions

$$V = \{f : G \to X \mid f(hg) = \rho(h)f(g) \quad \text{for all} \quad h \in H\}$$

Analoguosly to the construction of the induced representation we let the group G act on V by the rule

$$(g \cdot f)(x) := f(xg).$$

We equip V with the metric

$$d(f_1,f_2)^2 = \sum_{\bar{g} \in H \backslash G} d(f_1(g),f_2(g))^2$$

A choice of representatives for the cosets in $H\backslash G$ identifies V isometrically with $X^{|G:H|}$. In particular, V is also a Hadamard space. The action of H on V preserves the product decomposition and each factor. The action on the factor corresponding to the coset Hg is given by

$$(h \cdot f)(g) = \rho(ghg^{-1}) \cdot f(g).$$

Therefore the action of H on V is discrete and semisimple. This implies immediately that the action of G is discrete. To check that it is also semisimple we need the

Lemma 2.2 *Let Y be a Hadamard space and α an isometry. If α^k is semisimple for some $k \neq 0$ then α is semisimple.*

Proof. The set A of minimal displacement for the isometry α^k is a Hadamard space and it is preserved by α. If α^k is elliptic then α acts as an isometry of finite order on A. If α^k is loxodromic then A splits isometrically as a product $A = B \times \mathbb{R}$ and α acts as an element of finite order on B. In both cases the claim follows if we can show that isometries of finite order on a Hadamard space Z are elliptic, i.e. have a fixed point. This is true because for any finite subset F of Z the function $\sum_{z\in F} d^2(z, \cdot)$ is uniformly strictly convex and therefore has a unique minimum. □

This concludes the proof that the action of G on V is discrete and semisimple. Recall that two discrete groups are called *commensurable* if they contain isomorphic subgroups of finite index. Our discussion shows that being Hadamard is a commensurability invariant:

Theorem 2.3 *Suppose that G and H are commensurable groups. If H is a Hadamard group, then G is a Hadamard group as well.*

Proof. G contains a finite index normal subgroup which is isomorphic to a subgroup of H. □

For finitely generated groups there is a natural geometric equivalence relation which is weaker than the algebraic equivalence relation of being commensurable: Two groups are called *quasi-isometric* if their Cayley graphs are bilipschitz on the large scale, see [KL1] for a more precise definition. Observe that the Hadamard property is not a quasi-isometry invariant: E.g. fundamental groups of all closed Seifert 3-manifolds M with hyperbolic base orbifold are quasi-isometric to each other (as proven independently by Epstein, Gersten and Mess, see [Ge2]), but $\pi_1(M)$ is Hadamard if and only if M is finitely covered by a product (Bridson). Other examples are provided by fundamental groups of graph manifolds, cf. [KL3].

2.2 3-manifold subgroups and obstructions

In general, it is a stronger property for a group to admit cocompact discrete actions on Hadamard spaces than to admit semisimple ones. In this section we prove that for Haken 3-manifold groups both properties are equivalent. This will impose restrictions on possible Haken 3-manifold subgroups of Hadamard groups. For a definition of Haken manifolds and their canonical (geometrical) decomposition see [He, Sc].

Theorem 2.4 ([L1]) *Let M be a compact Haken 3-manifold whose boundary has zero Euler characteristic (∂M may be empty). If $\pi_1(M)$ is a Hadamard group, then M admits a smooth metric of nonpositive sectional curvature with totally geodesic flat boundary.*

Proof. If a hyperbolic component occurs in the canonical decomposition of M, then M admits a smooth metric of nonpositive curvature, see [L1]. Thus we can assume that M is a graph-manifold. Moreover, we can exclude the cases that M is a *Sol*-, *Nil*- or Euclidean manifold and suppose that all Seifert components of M have a hyperbolic base orbifold.

Let ρ be a discrete semisimple action of $\pi_1(M)$ on the Hadamard space X. Let S be a splitting surface in the canonical decomposition of M, i.e. a torus or Klein bottle. Because the action of $\pi_1(S)$ on X is discrete and loxodromic, there exist $\pi_1(S)$-invariant 2-flats, i.e. totally-geodesically embedded Euclidean planes. Any two invariant flats are parallel and the restricted actions of $\pi_1(S)$ are isometrically conjugate. Hence a well-defined flat metric g_S is induced on S.

Consider a Seifert component Z of M. The fundamental group fits into the exact sequence

$$0 \longrightarrow \mathbb{Z} = \langle f \rangle \longrightarrow \pi_1(Z) \longrightarrow \pi_1(O) \longrightarrow 1$$

where O is the base orbifold and f is represented by the generic fiber of the Seifert fibration. The isometry $\rho(f)$ is loxodromic and the union of all axes of $\rho(f)$ forms a convex subset C of X which splits isometrically as $C = Y \times \mathbb{R}$. The subset C as well as its product decomposition are preserved by the action of $\pi_1(Z)$. We have induced semisimple actions ϕ and ψ on the factors:

- ϕ is an action of $\pi_1(Z)$ on Y. f acts by the identity and ϕ descends to a discrete semisimple action $\bar{\phi}$ of $\pi_1(O)$ on Y.
- ψ is an action of $\pi_1(Z)$ on $\mathbb{R}$.

We put a nonpositively curved metric on Z which extends the flat metrics on the boundary surfaces by replacing ϕ with an action on a 2-dimensional nonpositively curved manifold as follows. Let the elements $a_i \in \pi_1(O)$ represent the boundary components of the orbifold O. They have infinite order and we denote by l_i the minimal displacement of a_i acting on Y. Choose a smooth metric of nonpositive curvature on O which is flat near the boundary so that the lengths of the boundary components are l_i. This yields an action of $\pi_1(O)$ on the universal cover $\tilde{O}$ which is a nonpositively curved surface. Lift this action to $\pi_1(Z)$ and form the product with the action ψ. The result is a discrete cocompact action without fixed points of $\pi_1(Z)$ on the nonpositively curved Riemannian 3-manifold $\tilde{O} \times \mathbb{R}$. It yields a smooth nonpositively curved metric on Z which is flat near the boundary and for each boundary component S of Z, the induced metric coincides with the flat metric g_S. Hence we can glue the metrics on the Seifert components to obtain a smooth nonpositively curved metric on M. □

Combining Theorems 2.3 and 2.4, we obtain:

Corollary 2.5 *Suppose that M is a closed 3-manifold of nonpositive curvature and N is finitely covered by M. Then N admits a metric of nonpositive curvature as well.*

Theorem 2.4 yields a new obstruction for a group to be Hadamard:

Corollary 2.6 *Suppose that Γ is a Hadamard group which contains a subgroup isomorphic to the fundamental group $\pi_1(M)$ of a closed Haken 3-manifold. Then M admits a smooth metric of nonpositive sectional curvature.*

In particular, we obtain an obstruction for nonpositively curved metrics on closed manifolds:

Corollary 2.7 *Let N be a closed smooth manifold and M a graph 3-manifold with no metrics of nonpositive curvature. If there is an embedding $\pi_1(M) \to \pi_1(N)$ of fundamental groups, then N admits no Riemannian metric of nonpositive sectional curvature.*

In the Sect. 3.2 we will provide examples of closed graph-manifolds with no metrics of nonpositive curvature.

3 Nonpositively curved metrics on Haken 3-manifolds

3.1 Existence

We consider compact Haken manifolds with boundary of zero Euler characteristic. For Riemannian metrics of nonpositive curvature, we shall always assume that the boundary is totally-geodesic. We recall the following existence result:

Theorem 3.1 ([L1],[L2]) *Let M be a Haken 3-manifold with boundary of zero Euler characteristic which satisfies either of the following conditions:*

1. *M contains a hyperbolic component.*
2. *∂M is non-empty.*

Then M admits a smooth metric of nonpositive curvature.
Addendum. *In the second case, let $\{\gamma_i\}$ be a collection of homotopically nontrivial loops in the boundary, one on each component, so that none of them represents the fibre of the adjacent Seifert component. Then, given positive numbers l_i, there exists a smooth nonpositively curved metric on M so that the loops γ_i are geodesics of length l_i.*

The class of Haken manifolds which are not covered by Theorem 3.1 consists of closed Seifert and graph-manifolds. These manifolds do not always admit metrics of nonpositive curvature, compare the examples in Sect. 3.2. However, we have the following existence theorem for this class of 3-manifolds:

Theorem 3.2 *If a closed graph-manifold contains a nonorientable Seifert component then it admits a metric of nonpositive curvature.*

We recall that a graph manifold is compact 3-manifold obtained from gluing Seifert manifolds along incompressible boundary surfaces. We exclude from the class of graph manifolds all *Sol*- and Seifert manifolds.

Lemma 3.3 *Suppose E is a nonorientable Euclidean Seifert manifold with a single boundary component. Then any flat metric on ∂E can be extended to a flat metric on E.*

Proof. Either ∂E is a Klein bottle or ∂E is a torus and E is homotopy equivalent to a torus. In both cases one checks the claim easily. □

Lemma 3.4 *Suppose that Z is a connected Seifert manifold whose base orbifold O is orientable hyperbolic (without boundary reflectors). If Z is nonorientable, we prescribe a flat metric on ∂Z, so that fibres are represented by geodesics of the same length. If Z is orientable, we prescribe such a metric on all but one boundary component. Then this metric can be extended to a nonpositively curved metric on Z.*

Proof. Consider the case when the base O is a pair of pants and Z is nonorientable. Exactly two boundary components of Z are Klein bottles. The fundamental group has the presentation

$$\pi_1(Z) = \langle a, b, c, t \mid ata^{-1} = t^{-1}, btb^{-1} = t^{-1}, ctc^{-1} = t, abc = 1 \rangle.$$

The prescribed flat metric h on ∂Z assigns lengths to the boundary curves of the pair of pants which can be realized by a unique hyperbolic metric. Nonpositively curved metrics on Z correspond to representations $\psi : \pi_1(Z) \to Isom(\mathbb{R})$. h determines the translations $\psi(c)$ and $\psi(t)$ and does not restrict the reflections $\psi(a)$ and $\psi(b)$. We can choose the reflections $\psi(a)$ and $\psi(b)$ so that their product equals $\psi(c^{-1})$. The argument in the cases that O is an annulus with one cone point and/or Z is orientable are similar. By gluing such building blocks we finish the proof of the lemma in the general case. □

Lemma 3.5 *Let N be a nonorientable Seifert manifold. Then any flat metric on ∂N so that Seifert fibers have equal length can be extended to a nonpositively curved metric on N.*

Proof. We need only consider the case that the base orbifold O of N is hyperbolic. Pick a maximal family of disjoint simple one-sided loops in O and consider the inverse images Σ of these loops and the boundary reflectors under the natural projection $N \to O$. The surfaces Σ are one-sided tori and Klein bottles. We see that N arises from gluing Euclidean Seifert manifolds with one boundary component and a Seifert manifold N' with orientable hyperbolic base. At least one of these pieces must be nonorientable. The claim follows from the previous two lemmas. □

Proof of Theorem 3.2: The theorem is implied by the previous lemma and the Addendum to Theorem 3.1. □

3.2 Non-existence

It is well-known that an aspherical Seifert manifold admits a metric of nonpositive curvature if and only if it is finitely covered by a product of surface and circle [E]. We now give a few examples of closed graph-manifolds which do not admit metrics of nonpositive curvature. Note that in graph manifold groups all solvable subgroups are virtually abelian and all centralizers split.

We start with the simplest example of closed graph-manifolds. Take a pair of compact orientable surfaces S_j with single incompressible boundary component γ_j. We consider the products $Z_j = S_j \times \mathbb{S}^1$ as fiber bundles with base S_j and fiber $\mathbb{S}^1$. There is a canonical pair of "base-fiber" subgroups $\langle b_j \rangle$ and $\langle f_j \rangle$ in $\pi_1(\partial Z_j)$. f_j is represented by the fiber of the Seifert fibration.

Theorem 3.6 *Let $h : \partial Z_1 \to \partial Z_2$ be a homeomorphism. Then the manifold M obtained by gluing Z_j along h admits a metric of nonpositive curvature if and only if the induced isomorphism h_* preserves the pair of "base-fiber" subgroups:*

$$\{\langle h_* b_1 \rangle, \langle h_* f_1 \rangle\} = \{\langle b_2 \rangle, \langle f_2 \rangle\}$$

(h may switch base and fiber).

Proof. Each nonpositively curved metric on M induces a flat metric on the separating torus $T = \partial Z_j$. In our situation only special metrics can be induced, namely such metrics which are rectangular with respect to the canonical bases $\{b_j, f_j\}$. To see this, consider as in the proof of Theorem 2.4 the set C_j of minimal displacement of the deck transformation f_j on the universal cover $\tilde{M}$. As above, there is a $\pi_1(Z_j)$-invariant isometric splitting $C_j = Y_j \times \mathbb{R}$. Since b_j is homologically trivial in Z_j, the action of b_j on the $\mathbb{R}$-factor must be trivial. The claim follows, because a rectangular basis for a lattice in Euclidean plane is unique (up to change of signs). □

To construct another example, suppose that S is a closed oriented hyperbolic surface. The orientation on S allows to distinguish "right" and "left" Dehn twists along simple closed geodesics. Choose a decomposition of S by disjoint simple closed geodesics $L = \{\gamma_1, \dots, \gamma_q\}$ with the weights $n_j \in \mathbb{Z} - \{0\}$. In what follows we shall denote by D_γ the right Dehn twist along γ.

Theorem 3.7 *Suppose that f is a homeomorphism of S which is a composition of iterates of Dehn twists $D_{\gamma_j}^{n_j}$ along γ_j, $n_j \in \mathbb{Z} - \{0\}, 1 \le j \le q$. If all the numbers n_j have the same sign (i.e. all the Dehn twists are either left or right) then the mapping torus $M = S \times_f [0, 1]$* does not admit *a metric of nonpositive curvature.*

Proof. The mapping tori T_j of $f_{|\gamma_j}$ decompose M into Seifert pieces. The fibration of M gives us two bases $\{f_j, b_j\}$ and $\{f_j', b_j'\}$ of $H_1(T_j, \mathbb{Z})$. Here the elements f_j, f_j' correspond to the Seifert fibers of the Seifert components adjacent to T_j. The elements b_j, b_j' are defined by the fibration of M over S^1 and the orientation of S. These bases are related by:

$$b_j' = -b_j \qquad \text{and} \qquad f_j' = f_j + n_j \cdot b_j.$$

Suppose that all the numbers n_j have the same sign, say positive. Assume that M carries a metric of nonpositive curvature. Then the flat metrics on the boundary tori of a Seifert component Z satisfy the condition

$$\sum_{T_j \subseteq \partial Z} \langle f_j, b_j \rangle = 0.$$

Summing this equation over all Seifert components yields

$$\sum_j (\langle f_j, b_j \rangle + \langle f_j', b_j' \rangle) = 0.$$

On the other hand, the positivity of the n_j implies

$$\sum_j (\langle f_j, b_j \rangle + \langle f_j', b_j' \rangle) = \sum_j -n_j \|b_j\|^2 < 0,$$

a contradiction. □

We have a combinatorial criterion for the existence of a nonpositively curved metric on a closed graph-manifold M when the dual graph to the canonical decomposition of M is a tree. Loosely speaking, the more Seifert components M has, the more likely it admits a metric of nonpositive curvature. The criterion extends the special case of a linear dual graph which is discussed in [L1, L2].

4 Examples of non-Hadamard groups

4.1 Mapping class groups

If Σ is a closed connected hyperbolic surface and $P \subset \Sigma$ is a finite set of points then the *genus* of the surface $S = \Sigma - P$ is either the genus of Σ (if it is orientable) or the number of projective planes in the prime decomposition of Σ (if Σ is not orientable). The surface $S = \Sigma - P$ is called a surface of *finite type*. We denote by $UT(S)$ the unit tangent bundle and by Mod_S the mapping class group.

The following argument is essentially due to Birman [Bi]. For a unit tangent vector $v \in UT(\Sigma)$, $Diff(\Sigma, v)$ will be the group of diffeomorphisms of Σ which fix the vector v. The evaluation map

$$\epsilon : Diff(\Sigma) \to UT(\Sigma), \quad \epsilon(f) = f(v)$$

is a locally trivial fibration with fiber $Diff(\Sigma, v)$, see [Bi]. We have the long exact homotopy sequence of a fibration

$$\ldots \to \pi_1(Diff(\Sigma)) \to \pi_1(UT(\Sigma)) \to \pi_0(Diff(\Sigma, v)) \to \pi_0(Diff(\Sigma)) \to$$

$$\to \pi_0(UT(\Sigma)) = 1$$

where we use the natural group structures on the mapping class group $\pi_0(\mathit{Diff}(\Sigma))$ and the relative mapping class group $\pi_0(\mathit{Diff}(\Sigma, v))$. Since Σ is hyperbolic, components of $\mathit{Diff}(\Sigma)$ are contractible, see [EE], and we get the short exact sequence

$$1 \to \pi_1(UT(\Sigma)) \to \pi_0(\mathit{Diff}(\Sigma, v)) \to \pi_0(\mathit{Diff}(\Sigma)) \to 1$$

Suppose now that γ is a simple homotopically nontrivial loop on S which splits S in two components S^+ and S^- so that:

- all the punctures of S are contained in S^-,
- the genus of S^+ is at least 2 (in the orientable case) or at least 3 (in the nonorientable case),
- $\pi_1(S^-)$ is not abelian.

Under these assumptions the surface F obtained by attaching a disc D to the boundary of S^+ is hyperbolic. Pick a unit tangent vector $v \in UT(D)$. Then the group $\pi_0(\mathit{Diff}(F, v))$ is naturally isomorphic to $\pi_0(\mathit{Diff}(S^+, \partial S^+))$ where $\mathit{Diff}(S^+, \partial S^+)$ consists of diffeomorphisms which fix ∂S^+ pointwise. Therefore $\pi_0(\mathit{Diff}(F, v))$ embeds in the mapping class group $Mod_S = \pi_0 \mathit{Diff}(S)$ (we extend each diffeomorphism in $\mathit{Diff}(S^+, \partial S^+)$ into S^- by the identity). Thus we proved the following fact which is due to Mess [Me] in the case of closed orientable surfaces (our proof is essentially the same):

Proposition 4.1 *Let S be a surface of finite type. If S is orientable we assume that either the genus of S is at least 3 or the genus of S is two and S has at least 2 punctures. If S is nonorientable we assume that either the genus of S is at least 4 or the genus of S is three and S has at least 2 punctures.*

Then the mapping class group Mod_S contains as a subgroup $\pi_1(UT(F))$ where F is a closed hyperbolic surface.

Theorem 4.2 *Under the conditions above or if S is an orientable surface of genus 2 with one puncture, the group Mod_S is not a Hadamard group. In particular, there are no effective cocompact discrete actions of Mod_S on Hadamard spaces.*

Proof. Assume first that S satisfies the assumptions of Proposition 4.1. Then the closed manifold $UT(F)$ is modelled on the $SL(2, \mathbb{R})$-geometry. Thus $UT(F)$ does not admit a metric of nonpositive curvature. Corollary 2.6 implies that Mod_S is not a Hadamard group.

This argument cannot be applied if S is an orientable surface of genus 2 with one puncture. Here is another way to construct 3-manifold subgroups inside the mapping class group of a once punctured orientable hyperbolic surface S: There are natural isomorphisms $Mod_S \cong Aut(\pi_1(\Sigma))$ and $Mod_\Sigma \cong Out(\pi_1(\Sigma))$, and therefore the short exact sequence

$$1 \to \pi_1(\Sigma) \to Mod_S \to Mod_\Sigma \to 1.$$

Let D_γ be a Dehn twist along a simple closed loop in Σ. Denote by $[D_\gamma]$ the corresponding element in Mod_Σ. The inverse image in Mod_S of the cyclic group generated by $[D_\gamma]$ is isomorphic to the fundamental group of the mapping torus M of D_γ. M is a closed graph manifold which does not admit a metric of nonpositive curvature by Theorem 3.6 or 3.7. Hence, Mod_S is not a Hadamard group. □

Remark 4.3 Any solvable subgroup of Mod_S is almost abelian [BLM].

We do not know whether the mapping class groups of surfaces which are not covered by Theorem 4.1 are Hadamard (for example the braid group Mod_{S^2-P}). It is clear that we must exclude almost free groups $Mod_{T^2-\{p\}}$ and Mod_{S^2-P} for $\#(P) \leq 4$. Gersten [Ge1] proved that $Out(F_r)$ is not a Hadamard group for free groups of rank $r \geq 4$.

4.2 4-manifolds fibered over surfaces

In this section we consider closed 4-manifolds M which are total spaces of surface bundles over surfaces. We require the base B and the fiber F to be hyperbolic. M yields a representation $r : \pi_1(B) \to Mod_F$. Not much is known about the existence of metrics of nonpositive (negative) curvature on such manifolds. First of all, there are trivial examples which are finitely covered by products of hyperbolic surfaces. They have geometric rank 2 and admit locally symmetric metrics. Next, there are Kodaira-type examples of geometric rank 1 which are ramified covers of products branched along totally-geodesic surfaces, see [Ko]. These manifolds carry singular metrics of nonpositive curvature but it is unknown whether they admit smooth metrics of nonpositive curvature. This problem has already been posed as Exercise 1 on page 2 in [BGS]. In all known examples, the representation r either has a non-trivial kernel or the image contains reducible elements of the mapping class group.

Problem 4.4 *Find a bundle $M = B \times_r F$ so that the representation r is injective and the image consists of pseudo-Anosov elements only. Give examples where the fundamental groups are word-hyperbolic. Give examples where M admits a metric of (constant) negative curvature.*

We apply the obstruction criterion 2.7 to construct surface bundles without metrics of nonpositive curvature.

Theorem 4.5 *There are closed 4-manifolds M which are fibered over hyperbolic surfaces with hyperbolic fiber so that $\pi_1(M)$ is not a Hadamard group. In particular M doesn't carry a metric of nonpositive curvature.*

Proof. We consider a homomorphism r obtained by composing an epimorphism $\phi : \pi_1(B) \to \mathbb{Z} = \langle z \rangle$ and an action of z on F by a Dehn twist D. Then $\pi_1(M)$ contains the fundamental group of the mapping torus N of D. According to Theorem 3.6 or 3.7, the manifold N admits no metric of nonpositive curvature. Hence by Corollary 2.6 the group $\pi_1(M)$ is not a Hadamard group. □

Remark 4.6 Any solvable subgroup of $\pi_1(M)$ is abelian. All centralizers in $\pi_1(M)$ split as products.

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