Invent. math. 128, 393–416 (1997)

Inventiones mathematicae

Quasi-isometries preserve the geometric decomposition of Haken manifolds

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Oblatum 27-III-1996 & 5-IX-1996

Abstract. We prove quasi-isometry invariance of the canonical decomposition for fundamental groups of Haken 3-manifolds with zero Euler characteristic. We show that groups quasi-isometric to Haken manifold groups with nontrivial canonical decomposition are finite extensions of Haken orbifold groups. As a by-product we describe all 2-dimensional quasi-flats in the universal covers of non-geometric Haken manifolds.

1 Introduction

Let Γ be a finitely generated group. A geodesic metric space X, on which Γ acts properly discontinuously and cocompactly by isometries, can be regarded as a geometric model for Γ. Important examples are Cayley graphs associated to finite generating sets and universal covers of compact Riemannian manifolds with fundamental group Γ. All such model spaces X are quasi-isometric to one another and their quasi-isometry invariants are called *geometric invariants* of Γ, cf. [Gr1]. It is a basic question in this context to classify all finitely generated groups up to quasi-isometry. Note that commensurable groups have the same geometric invariants, whereas the converse is in general not true.

This paper deals with the geometry of 3-manifold groups. Our main result concerns the canonical decomposition of Haken manifolds M with boundary of

This research was partially supported by the SFB "Nichtlineare partielle Differentialgleichungen" in Bonn and the NSF grant DMS-9306140 (Kapovich)

zero Euler characteristic. Jaco, Shalen, Johannson and Thurston proved that M can be cut along flat surfaces into finitely many geometric components which are either Seifert or hyperbolic. This canonical decomposition of M is unique up to isotopy and it corresponds to an algebraic decomposition of $\pi_1(M)$ as a graph of groups which is invariant under group automorphisms. We prove that the canonical decomposition is more generally invariant under all quasi-isometries and therefore it is a geometric invariant of the fundamental group. To make this precise, put a Riemannian metric on M and take the universal cover $X = \tilde{M}$ as a geometric model for $\pi_1(M)$. The canonical decomposition of M lifts to a decomposition of X where a geometric component of X is the universal cover of a geometric component of M. Let $X' = \tilde{M}'$ be the universal cover of another Haken manifold M' of the same kind, decomposed in the same way.

Main Theorem 1.1 *Let $\phi : X \to X'$ be a quasi-isometry. Then ϕ preserves the geometric decompositions of X and X' in the following sense: For any geometric component Y of X there exists a geometric component Y' of X' within uniformly bounded Hausdorff-distance from $\phi(Y)$. The components Y and Y' have the same type (Seifert or hyperbolic). Moreover ϕ preserves the ordering of geometric components and therefore induces an isomorphism of the trees dual to the geometric decompositions of X and X'.*

We did a first step in this direction in our earlier paper [KL2] where we proved that the quasi-isometry class of $\pi_1(M)$ detects whether M has a Seifert component. Theorem 1.1 implies that also the existence of a hyperbolic component is "visible" in the geometry of $\pi_1(M)$. This had first been proven by N. Brady and Gersten using different techniques; they showed that the divergence of $\pi_1(M)$ is exponential if and only if M has a hyperbolic component, see [Ge]. Note that there are non-geometric Haken manifolds whose fundamental groups are quasi-isometric but not commensurable, see [KL1, KL3].

Our main application is a geometric characterization of Haken manifold groups:

Theorem 1.2 *Suppose that Γ is a finitely generated group whose Cayley graphs are quasi-isometric to the universal cover X of a non-geometric Haken manifold M with zero Euler characteristic. Then there is a short exact sequence*

$$1 \longrightarrow \textit{finite group} \longrightarrow \Gamma \longrightarrow \pi_1(O) \longrightarrow 1$$

where O is a compact 3-dimensional orbifold which is finitely covered by a Haken manifold of the same kind as M. In particular, if Γ is torsion-free then Γ is isomorphic to the fundamental group of a Haken manifold N.

Results analogous to Theorem 1.2 were previously known in several cases when M is geometric: The result for *Nil*-manifolds is due to Gromov, see [Gr2], and the euclidean case is due to Bridson and Gersten. Rieffel [R] proved Theorem 1.2 when M is a Seifert manifold with hyperbolic base orbifold. The case when M is a *Sol*-manifold remains open. Note that there are obvious examples of self quasi-isometries of *Sol* which are not within bounded distance from any isometry. Regarding the hyperbolic case, Schwartz [Sch] proved for finite volume noncompact hyperbolic manifolds H of dimension ≥ 3 that the quasi-isometry group of $\pi_1(H)$ is naturally isomorphic to the commensurator of $\pi_1(H)$; this shows in particular that any finitely generated group quasi-isometric to $\pi_1(H)$ is a finite extension of a group commensurable with $\pi_1(H)$.

As a by-product of the proof of Theorem 1.1 we also give a classification of 2-dimensional quasi-flats in X, cf. Theorem 4.10. We prove that each 2-quasi-flat in X is contained in a tubular neighborhood of a finite union of *isolated flats* in X. Besides quasi-flats which are Hausdorff close to a flat there are also *twisted* quasi-flats which spread through a finite chain of consecutive Seifert components. We describe canonical models for these quasi-flats in Sect. 4.3.

Our approach is based on the strong link between Haken 3-manifolds and the geometry of nonpositive curvature. Based on Thurston's Hyperbolization Theorem, it is shown in [L] that Haken manifolds M of zero Euler characteristic generically admit metrics of nonpositive curvature with totally geodesic flat boundary. Moreover, we prove in [KL3] that their fundamental groups are geometrizable in the following weak sense: If M is neither a *Sol*- nor *Nil*-manifold, then there exists a Haken manifold of nonpositive curvature M' such that the universal covers $X = \tilde{M}$ and $X' = \tilde{M}'$ are bilipschitz homeomorphic by a homeomorphism which preserves the geometric decomposition. In particular, the geometric components of the universal cover X are quasi-isometrically embedded and X is bicombable. Therefore in the present paper we shall only consider nonpositively curved Haken manifolds M.

We already mentioned that single 2-quasi-flats are generally not Hausdorff-close to flats. Our idea is to show that sufficiently complicated patterns of 2-flats are quasi-isometrically rigid. As in [KL2], we use the concept of *asymptotic cone* of a metric space. A quasi-isometry $X \to X'$ becomes "more continuous" when one rescales the metrics with a small factor, and in the ultralimit it induces a bilipschitz homeomorphism $X_\omega \to X'_\omega$ of asymptotic cones. By analysing the geometry and topology of the asymptotic cones we prove that their induced geometric decompositions are preserved by bilipschitz homeomorphisms. This is done by classifying bilipschitz embedded 2-flats and topologically characterizing the flats separating geometric components. The Divergence Lemma 4.1 is the key tool for translating the topological rigidity of the asymptotic cones into a statement about quasi-isometric rigidity for the geometric decompositions of X and X'. This lemma implies that a 2-dimensional quasi-flat Q in X which is not uniformly close to a convex set C diverges from C at a linear rate and hence the distinction between Q and C becomes visible in the asymptotic cone X_ω.

Regarding Theorem 1.2, a finitely generated group, which is quasi-isometric to X, admits an action by uniform quasi-isometries on X. The Main Theorem implies that the canonical decomposition of X is quasi-preserved and there is an induced action on the dual tree for the canonical decomposition. By a general argument, the vertex and edge stabilizers act quasi-transitively on the corresponding components and flats. Using work of Schwartz and Tukia we conclude that the vertex stabilizers are finite extensions of fundamental groups of 3-dimensional hyperbolic and Seifert orbifolds and the edge stabilizers are their peripheral subgroups.

2 Preliminaries

2.1 Notations and conventions

We will use different notions of distance between subsets A, B of metric spaces: the Hausdorff distance and the distance of closest points denoted by $dist(A, B)$. The distance of a point x from a set A will be sometimes denoted by $d_A(x)$. If A is a subset of a topological space X then $\bar{A}$ will denote the closure of A in X. We use the notation $[xy]$ for the geodesic segment connecting points x, y in a metric space X. If Z is a subset in a metric space X and $R > 0$ then $N_R(Z)$ denotes the R-neighborhood of Z in X, we will refer to $N_R(Z)$ as a *tubular neighborhood* of Z. We assume that all segments, rays and geodesics are parameterized by unit speed.

A map $f : (X_1, d_1) \to (X_2, d_2)$ of metric spaces is a (K, ϵ)-*quasi-isometric embedding* with $K, \epsilon > 0$ if

$$K^{-1}d_1(x, y) - \epsilon \leq d_2(f(x), f(y)) \leq Kd_1(x, y) + \epsilon$$

for each $x, y \in X_1$. (One should think of ϵ as a large positive number.) Note that quasi-isometric embeddings are not necessarily injective or continuous. A map $f_1 : (X_1, d_1) \to (X_2, d_2)$ is a *quasi-isometry* if there are two constants C_1, C_2 and another map $f_2 : (X_2, d_2) \to (X_1, d_1)$ such that both f_1, f_2 are quasi-isometric embeddings and

$$d_1(f_2 f_1(x), x) \leq C_1, d_2(f_1 f_2(y), y) \leq C_2$$

for every $x \in X_1, y \in X_2$. Such spaces X_1, X_2 are called *quasi-isometric*.

Hadamard (or $CAT(0)$) spaces are complete geodesic metric spaces with non-positive curvature in the distance comparison sense, cf. [Ba, KlL]. They are not assumed to be locally compact. In Hadamard spaces one can define the *angle* between geodesic segments $[ab], [ac]$, see [KL2]. We shall denote by $\partial_{geo} X$ the *geometric* or *ideal boundary* of the Hadamard space X. For a closed convex subset C in a Hadamard space X, $proj_C$ will denote the closest-point projection to C. These projections are distance-nonincreasing. A *flat, bilipschitz flat*, respectively *quasi-flat* in X is (the image of) an isometric, bilipschitz, respectively quasi-isometric embedding of the euclidean 2-plane into X.

Convention 2.1 *All flats, bilipschitz flats and quasi-flats considered in the present paper are* **2-dimensional***.*

2.2 3-manifolds and their canonical decomposition

We refer to [S] for information about the geometrization of 3-manifolds and a description of the eight 3-dimensional homogeneous geometries. Here we only recall a few facts which are important for this paper.

A compact smooth 3-manifold P is called *geometric* if its interior admits a geometric structure, i.e. a complete locally homogeneous Riemannian metric. If P is aspherical and has nonempty boundary, then the occurring homogeneous spaces will be $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$. In the latter case, the manifold P is *Seifert* and itself admits a metric modelled on $\mathbb{H}^2 \times \mathbb{R}$ with totally geodesic boundary, unless it has almost abelian fundamental group. Manifolds P locally modelled on $\mathbb{H}^3$-geometry are called *hyperbolic*. Note that according to our definition the solid torus $D^2 \times \mathbb{S}^1$ and $\mathbb{S}^1 \times \mathbb{S}^1 \times [0,1]$ are geometric manifolds which are hyperbolic and Seifert simultaneously.

In this paper we will only consider aspherical 3-manifolds of zero Euler characteristic, equivalently the boundary of such manifolds is a (possibly empty) collection of tori and Klein bottles. Instead of giving the definition of Haken manifolds, we remind that all non-geometric *Haken manifolds* of zero Euler characteristic admit a *canonical decomposition* into finitely many geometric manifolds which are glued along boundary tori or Klein bottles, see [JS, J, Th, Ka, O]. This decomposition is unique up to isotopy if these geometric submanifolds are chosen to be maximal up to isotopy. Note that geometric components of a Haken manifold of this kind never have almost abelian fundamental group, thus the classes of hyperbolic and Seifert components become disjoint.

If the Haken manifold M of zero Euler characteristic carries a Riemannian metric g of nonpositive sectional curvature with totally-geodesic flat boundary, then the canonical topological decomposition of M into hyperbolic and Seifert components can be realized geometrically by cutting along totally-geodesically embedded flat tori and Klein bottles Σ, cf. [L, LS]. The metric g can be chosen (once differentiable) so that all Seifert components are in their interior locally isometric to $\mathbb{H}^2 \times \mathbb{R}$. Call a flat in the universal cover X of M an *isolated flat* if it either covers one of the flat surfaces Σ in the canonical decomposition of M or is a boundary flat of X. This terminology is motivated by the fact that a flat is parallel to an isolated flat if and only if no other flat intersects it transversally. The isolated flats decompose X into convex subsets with totally-geodesic flat boundary which we call *geometric components* of X; they are universal covers of the geometric components of M. Each Seifert component Y of X splits isometrically as the product of the real line and a 2-dimensional factor which is isometric to a convex domain in $\mathbb{H}^2$ bounded by disjoint geodesics. The fibration of Y by parallel geodesics covers a Seifert fibration of the corresponding Seifert component of M by closed geodesics.

2.3 Ultralimits and asymptotic cones

For a discussion of ultralimits of metric spaces we refer to [DW, Gr2, KL2, KlL], here we recall some of their basic properties.

Fix a non-principal ultrafilter ω on $\mathbb{N}$. For any sequence of pointed metric spaces (X_n, x_n^0) the ultralimit (X_ω, x_ω^0) is a metric space. Points in the ultralimit are represented by sequences (x_n) of points in X_n. If the sequence (X_n) is precompact in the Gromov-Hausdorff topology, then the ultralimit is the Gromov-Hausdorff limit of a ω-large subsequence. Ultralimits of Hadamard spaces are Hadamard spaces. The ultralimit of a sequence of (K_n, ϵ_n)-quasi-isometries $f_n : (X_n, x_n^0) \to (Y_n, y_n^0)$ is a (K, ϵ)-quasi-isometry $f_\omega : (X_\omega, x_\omega^0) \to (Y_\omega, y_\omega^0)$ with $K = \omega\text{-}\lim K_n$ and $\epsilon = \omega\text{-}\lim \epsilon_n$.

To form the *asymptotic cone* X_ω of a metric space X, one chooses sequences of scale factors $\lambda_n > 0$ with $\omega\text{-}\lim \lambda_n = 0$ and basepoints $x_n^0 \in X$ and takes the ultralimit $(X_\omega, x_\omega^0) := \omega\text{-}\lim_n(\lambda_n \cdot X, x_n^0)$. Here, $\lambda_n \cdot X$ denotes the metric space obtained by rescaling the metric of X with the factor λ_n. When we speak of "the asymptotic cone X_ω of X", we mean one of these ultralimits, suppressing the choices of λ_n, x_n^0 in our notation. In general, the isometry type of X_ω depends on these choices. However, in our applications various asymptotic cones will share the same geometric properties. The asymptotic cone X_ω of a Hadamard space X is a Hadamard space and in general not locally compact. If X admits a cocompact group of isometries then X_ω is homogeneous.

The asymptotic cone is a useful tool for the study of quasi-isometries, because a (K, ϵ)-quasi-isometric embedding of metric spaces becomes continuous in the rescaling process and induces a K-bilipschitz embedding of their asymptotic cones.

2.4 Busemann functions

Suppose that Y is a Hadamard space. Busemann functions measure the relative distance from points at infinity. Pick a point ξ in the geometric boundary of Y, i.e. an equivalence class of parallel geodesic rays. Take a unit speed geodesic ray $\rho : [0, \infty) \to X$ asymptotic to ξ. The Busemann function B_ξ corresponding to ρ is the monotonic limit :

$$B_\xi(\cdot) := \lim_{t\to\infty} [d(\cdot, \rho(t)) - t]$$

B_ξ is a convex function which is well-defined up to an additive constant. For every ray r asymptotic to ξ we have:

$$B_\xi \circ r(t) = -t + const \tag{1}$$

If r_1, r_2 are two geodesic rays asymptotic to ξ, then $B_\xi \circ r_1(t) = B_\xi \circ r_2(t)$ holds iff the ideal triangle with vertices $r_1(t), r_2(t), \xi$ has angles $\leq \pi/2$ for all t. These two properties characterize Busemann functions.

Level sets of Busemann functions are called *horospheres*. The sublevel sets are called *horoballs* and they are convex.

We prove two facts about Busemann functions for later reference.

Lemma 2.2 *Suppose that C is a convex subset of Y such that the Busemann function B_ξ is constant on C. Then the union U of geodesic rays emanating from points of C and asymptotic to ξ is convex and splits isometrically as the direct product*

$$C \times \mathbb{R}_+.$$

Proof: For a point $q \in C$ let l_q denote the geodesic ray emanating from q in the direction ξ. Let x, y be a pair of distinct points in C, the function B_ξ is constant on $[xy]$. It follows from the definition of the Busemann function that the angles between l_x, l_y and the geodesic segment $[xy]$ are at least $\pi/2$. However Y is a Hadamard space, thus the ideal triangle with vertices x, y, ξ has angles $\pi/2, \pi/2, 0$ and spans an isometrically embedded flat half-strip, namely the union

$$\cup_{q \in [xy]} l_q.$$

The Lemma follows. □

Lemma 2.3 *Let (X_n, x_n^0) be a sequence of based Hadamard spaces with the ultralimit (X_ω, x_ω^0). Let $Y_n \subset X_n$ be convex subsets with $\omega\text{-}\lim_n d(x_n^0, Y_n) = \infty$. Then $f := \omega\text{-}\lim_n (d_{Y_n} - d_{Y_n}(x_n^0))$ is a Busemann function on X_ω.*

Proof: For $x_n \in X_n - Y_n$, denote by $\rho_{x_n} : [0, d_{Y_n}(x_n)] \to X_n$ the perpendicular from x_n to Y_n. For any points $x_n, x_n' \in X_n$, the function $t \mapsto d(\rho_{x_n}(t), \rho_{x_n'}(t))$ decreases monotonically. If $x_\omega = (x_n)$ is a point in X_ω then the ultralimit $\rho_{x_\omega} := \omega\text{-}\lim \rho_{x_n} : [0, \infty) \to X_\omega$ is a geodesic ray which does not depend on the choice of the sequence (x_n) representing x_ω, and all rays ρ_{x_ω} are asymptotic to the same ideal point $\xi_\omega \in \partial_{geo} X_\omega$. Applying the triangle inequality, we obtain

$$d(\rho_{x_n}(t), x_n') \geq \underbrace{d(x_n', proj_{Y_n}(x_n))}_{\geq d_{Y_n}(x_n')} - \underbrace{d_{Y_n}(\rho_{x_n}(t))}_{= d_{Y_n}(x_n) - t} \geq t + (d_{Y_n}(x_n') - d_{Y_n}(x_n)).$$

Passing to the ultralimit yields

$$d(\rho_{x_\omega}(t), x_\omega') \geq t + (B(x_\omega') - B(x_\omega))$$

where $B := \omega\text{-}\lim d_{Y_n}$. We rewrite the previous inequality:

$$d(x_\omega', \rho_{x_\omega}(t)) - t \geq B(x_\omega') - B(x_\omega)$$

and send t to infinity:

$$B_{\xi_\omega}(x_\omega') - B_{\xi_\omega}(x_\omega) \geq B(x_\omega') - B(x_\omega).$$

Since we may exchange the roles of x_ω and x_ω', the equality holds and we conclude that the function $B_{\xi_\omega} - B$ is constant on X_ω. □

Remark 2.4 *Similarly, one shows that the ultralimit of Busemann functions on X_n is a Busemann function on X_ω (or constant $\pm\infty$).*

2.5 Quasi-isometric embeddings into piecewise Euclidean spaces

Suppose that X is a geodesic metric space and $E \subseteq X$ is an open subset which is isometric to a convex subset in the Euclidean m-space $\mathbb{R}^m$.

Lemma 2.5 *There is a constant $R = R(K, \epsilon, m)$ such that the following is true: Let $qf : \mathbb{R}^m \to X$ be a (K, ϵ)-quasi-isometric embedding with the image Q. Then either $N_R(Q) \supseteq E$ or $Q \cap E \subseteq N_R(\partial E)$.*

Proof: If the conclusion of the lemma is not satisfied then the convex subset $E' := E - N_{\frac{2R}{3}}(\partial E)$ satisfies $Q \cap E' \neq \emptyset$ and $E' \not\subseteq N_{\frac{R}{3}}(Q)$. Thus there exists a point $x \in E$ with

$$d(x, Q) = \frac{R}{3} \qquad \text{and} \qquad d(x, \partial E) \geq \frac{2R}{3}.$$

If the assertion of the lemma were not true, then we can find sequences of subsets $E_n \subseteq X_n$, (K, ϵ)-quasi-isometries $qf_n : \mathbb{R}^m \to X_n$, points $x_n \in E_n$ and numbers R_n tending to infinity, so that: $d(x_n, qf_n(\mathbb{R}^m)) = d(x_n, qf_n(0)) = \frac{R_n}{3}$ and $d(x_n, \partial E_n) \geq \frac{2R_n}{3}$. Now we rescale $\mathbb{R}^m$ with the center 0 and X_n with the center x_n using the scale factor R_n^{-1}. The ultralimit of the $(K, R_n^{-1}\epsilon)$-quasi-isometric embeddings $qf_n : (R_n^{-1} \cdot \mathbb{R}^m, 0) \to (R_n^{-1} \cdot X_n, x_n)$ is a K-bilipschitz embedding $\flat : \mathbb{R}^m \to X_\omega$. The subset $E_\omega := \omega\text{-}\lim E_n \subset X_\omega$ is isometric to a convex subset of $\mathbb{R}^m$. Moreover its interior (with respect to an isometric embedding into $\mathbb{R}^m$) is open in X_ω, because the X_n were assumed to be geodesic. By construction, the image of $\flat$ intersects the interior of E_ω, but does not contain it. This is impossible, because the restriction of $\flat$ to $\flat^{-1}(int(E_\omega))$ is a bilipschitz map between open subsets of $\mathbb{R}^m$ and therefore a local homeomorphism. □

Corollary 2.6 *(H. Fürstenberg) Every (K, ϵ)-quasi-isometric embedding $\mathbb{R}^m \to \mathbb{R}^m$ is a (K, ϵ')-quasi-isometry with a constant $\epsilon' = \epsilon'(K, \epsilon, m)$.*

3 Asymptotic cones of universal covers of Haken manifolds

Let $\mathcal{H}_{npc}$ be the class of all compact non-geometric Haken manifolds which are equipped with a nonpositively curved Riemannian metric such that the boundary is totally-geodesic and flat. These manifolds have geometric rank one and they contain totally-geodesically immersed flat 2-tori. Throughout this section, M (resp. M') shall denote a Riemannian manifold in $\mathcal{H}_{npc}$ and X (resp. X') its universal cover. The decomposition of X into geometric components (cf. section 2.2) induces a corresponding decomposition of X_ω and we will prove that this decomposition is preserved by bilipschitz homeomorphisms $X_\omega \to X'_\omega$. A complete description of bilipschitz embedded flats will be given in section 3.5; besides flats there are non-trivial twisted examples of bilipschitz flats. Please keep in mind convention 2.1: all flats and bilipschitz flats are assumed to be 2-dimensional!

3.1 Geometric components

Let $Y_n \subset X$, $n \in \mathbb{N}$, be geometric components with $\omega\text{-}\lim \lambda_n^{-1} \cdot d(Y_n, x_n^0) < \infty$. We call their ultralimit $Y_\omega = \omega\text{-}\lim \lambda_n^{-1} \cdot Y_n \subset X_\omega$ a *geometric component* in the asymptotic cone X_ω. Since there are only finitely many isometry types of geometric components in X, Y_ω is isometric to the asymptotic cone of a geometric component Y of X. We call Y_ω Seifert, respectively hyperbolic, according to the type of Y. We collect some properties of the geometric components proven in [KL2].

A sequence of flats $F_n \subset Y_n$ with $\omega\text{-}\lim \lambda_n^{-1} \cdot d(F_n, x_n^0) < \infty$ determines a flat $F_\omega \subset Y_\omega$. In fact, every flat in Y_ω arises in this way. If Y_ω is hyperbolic, then the flats F_n can be chosen in ∂Y_n. Moreover, by Lemma 2.14 and Corollary 4.6 in [KL2], each bilipschitz embedded flat in Y_ω is totally-geodesic:

Lemma 3.1 *Any bilipschitz flat in X_ω which is contained in a geometric component is a flat.*

Seifert components are isometric to the product of $\mathbb{R}$ with a metric tree which branches everywhere and is geodesically complete. A key property of hyperbolic components is that any two distinct flats share at most one point, cf. Proposition 4.3 in [KL2]. It follows that Seifert components cannot be embedded into hyperbolic components:

Lemma 3.2 *Let T be a metric tree with at least 3 ideal end points. Then $T \times \mathbb{R}$ cannot be bilipschitz embedded into a hyperbolic component Y_ω.*

Later we will need the following lemma concerning the separation of Seifert components by lines:

Lemma 3.3 *Let T be a geodesically complete tree and $C \subseteq \mathbb{R}$ a closed subset. Assume that $f : C \to T \times \mathbb{R}$ is a bilipschitz embedding whose image l separates. Then $C = \mathbb{R}$ and $proj_T(l)$ is contained in a segment with no branch point in its interior. In particular, if T branches everywhere then l is a fiber $\{t\} \times \mathbb{R}$.*

Proof: Suppose that $proj_T(l)$ contains two points a, b which form a tripod together with a third point c. If $F \subset T \times \mathbb{R}$ is a flat which doesn't contain a and b, then $F \cap l$ is a proper subset of l and consequently doesn't separate F (Alexander duality). For any points $x, y \in T \times \mathbb{R}$ there are flats F_x, F_y containing c so that $x \in F_x$ and $y \in F_y$. Therefore x and y can be connected in $(F_x \cup F_y) - l$, a contradiction. Hence, $proj_T(l)$ is contained in a segment with no branch point in its interior. If $C \neq \mathbb{R}$, then $f(C)$ cannot separate any flat in $T \times \mathbb{R}$ and we again obtain a contradiction. □

3.2 Separation by flats

Suppose that F is an isolated flat adjacent to a geometric component $Y \subset X$. We refer to the pair (F, Y) as a *cooriented flat*. For each cooriented isolated flat (F, Y), we introduce a partition $\mathscr{L}_F$ of F into disjoint subsets: if Y is a Seifert component, then $\mathscr{L}_F$ consists of the parallel lines corresponding to Seifert fibers; if Y is hyperbolic, then $\mathscr{L}_F$ just consists of the points in F. A cooriented flat (F, Y) defines a signed distance function sd_F on X: we set $sd_F(x) := \pm dist(x, F)$ with the positive sign if x belongs to the same connected component of $X - F$ as Y and the negative sign otherwise.

There are corresponding notions in the asymptotic cone. Namely, let (F_n, Y_n) be a sequence of cooriented flats with signed distance functions sd_{F_n}. It gives rise to a cooriented flat $(F_\omega, Y_\omega) := \omega\text{-}\lim(F_n, Y_n)$ and a signed distance function $sd_{F_\omega} := \omega\text{-}\lim sd_{F_n}$ on X_ω. We call the set $\{sd_{F_\omega} > 0\}$ the *positive* side of F_ω. Note that the positive side is not connected at all! $Y_\omega := (Y_n)$ is the geometric component *adjacent* to F_ω on the positive side. If the negative side of F_ω is non-empty (equivalently, if F_n is not parallel to a boundary flat of X for ω-all n), then there is a geometric component Z_ω of X_ω adjacent to F_ω on the negative side as well. In this situation we say that the negative side of F_ω is the F_ω*-side* of Y_ω. Similarly, the positive side of F_ω is the F_ω*-side* of Z_ω.

The flat F_ω inherits a partition $\mathscr{L}_{F_\omega}$ into points or parallel lines. If the geometric component on the positive side is Seifert, then $\mathscr{L}_{F_\omega}$ consists of lines parallel to the $\mathbb{R}$-fiber, otherwise it consists of points. If F_ω has adjacent Seifert components on both sides, then the two families of lines which correspond to both coorientations of F_ω are transversal.

We say that a flat or a geometric component F_ω of X_ω *essentially separates* two sets $S_1, S_2 \subset X_\omega$ if the sets $S_i - F_\omega$ lie on distinct sides of F_ω (we allow $S_i \subset F_\omega$). A set $S \subset X_\omega$ is said to be *essentially split* by F_ω if there are points of $S - F_\omega$ on both sides of F_ω. There is a dual tree to the decomposition of X into geometric components and this tree-like order persists in the asymptotic cone: For any three geometric components $Y_{1\omega}, Y_{2\omega}, Y_{3\omega}$ of X_ω either one of them essentially separates the other two or there is a unique geometric component Y_ω which essentially separates any two of the components $Y_{i\omega}$. Observe also that the set of isolated flats which essentially separate two given isolated flats F_ω and F'_ω is totally ordered. We call isolated flats $F_{1\omega}, \ldots, F_{n\omega}$ in X_ω *consecutive* if $F_{i\omega}$ and $F_{i+1,\omega}$ belong to one component and $F_{i\omega}$ essentially separates the flats $F_{i\pm1,\omega}$ for all i (where it makes sense). Note that three consecutive isolated flats share at most one point. The same is true for four consecutive geometric components.

Lemma 3.4 *Assume that the subset $A \subset X_\omega$ is not essentially split by any isolated flat. Then A is contained in a single geometric component of X_ω.*

Proof: Let us first consider the case that A consists of two points x_ω and y_ω. There are isolated flats $F_{x\omega} \supset x_\omega$ and $F_{y\omega} \supset y_\omega$. We are done if $F_{x\omega}$ and $F_{y\omega}$ coincide. Otherwise the set $\mathscr{C}$ of geometric components which essentially separate $F_{x\omega}$ and $F_{y\omega}$ is non-empty and totally ordered. By assumption, any component in $\mathscr{C}$ contains exactly one of the points x_ω and y_ω, because otherwise we are again done. At most finitely many components in $\mathscr{C}$ can contain interior points of the segment $[x_\omega y_\omega]$, since 4 consecutive components share at most one point. Hence there exist adjacent components $Y_\omega, Y'_\omega \in \mathscr{C}$ each of which contains exactly one of the points x_ω, y_ω. Then the isolated flat $Y_\omega \cap Y'_\omega$ essentially separates x_ω and y_ω, a contradiction. Thus our claim holds if A consists of two points.

Consider now the general case. Let a, b be any two points of A. As shown above, they lie in some geometric component Y_0. If A is not already contained in Y_0, there is an isolated flat $F_1 \subset Y_0$ which separates $Y_0 - F_1$ from a point $c \in A$. Since F_1 does not split A, all of $A - F_1$ lies on the same side of F_1. In particular a, b belong to the geometric component $Y_1 \neq Y_0$ adjacent to F_1. If A is not contained in Y_1, we can continue this argument inductively and construct four consecutive geometric components Y_0, Y_1, Y_2, Y_3 which contain the points a, b. However, the intersection of four consecutive geometric components contains at most one point. □

3.3 Projections to flats

Basic to our understanding of the topology of X_ω is the study of projections to isolated flats. Inside geometric components, we have:

Lemma 3.5 *Let F_ω be an isolated flat in the geometric component Y_ω and let $\sigma \subset Y_\omega$ be a geodesic segment disjoint from F_ω. Then $proj_{F_\omega}(\sigma)$ is contained in a set $l \in \mathscr{L}_{F_\omega}$.*

Proof: The assertion for hyperbolic components is included in Lemma 4.4 of [KL2]. The Seifert case follows from the following corresponding statement for trees:

Sublemma 3.6 *Let c be a geodesic in a metric tree and $[uv]$ a geodesic segment disjoint from c. Then the nearest-point-projection to c maps $[uv]$ to a point.*

Proof: Let $p \in c$ be the point closest to $[uv]$ and $q \in [uv]$ be the point closest to c. Recall that if $[rs]$ and $[st]$ are geodesic segments in a tree with $[rs] \cap [st] = \{s\}$ then $[rs] \cup [st] = [rt]$. This implies that any segment from a point on σ to a point on c contains $[pq]$ and the claim follows. □

We extend the previous lemma to projections of the entire asymptotic cone X_ω.

Proposition 3.7 *Let F_ω be a cooriented isolated flat in X_ω and suppose that A is a connected component of $X_\omega - F_\omega$ on the positive side of F_ω. Then $proj_{F_\omega}(A) \subseteq l$ for some $l \in \mathscr{L}_{F_\omega}$ and hence $\bar{A} \cap F_\omega \subseteq l$.*

Proof: Let $x_\omega, z_\omega \in X_\omega - F_\omega$ be points on the positive side of F_ω so that $[x_\omega z_\omega] \cap F_\omega = \emptyset$. Let Y_ω be the geometric component of X_ω adjacent to F_ω on the positive side.

Case 1: If $[x_\omega z_\omega] \cap Y_\omega = \emptyset$ then there exists a unique isolated flat $F'_\omega \subset Y_\omega$ such that F'_ω essentially separates $[x_\omega z_\omega]$ from Y_ω. Hence $proj_{F_\omega}[x_\omega z_\omega] \subseteq proj_{F_\omega} F'_\omega$ which is contained in a set $l \in \mathscr{L}_{F_\omega}$ by the previous lemma.

Case 2: If $[x_\omega z_\omega] \cap Y_\omega = [x'_\omega z'_\omega]$ then the previous lemma implies that $proj_{F_\omega}[x'_\omega z'_\omega] \subseteq l \in \mathscr{L}_{F_\omega}$ and by the same reasoning as in the Case 1 one has $proj_{F_\omega}[x_\omega x'_\omega] \subseteq l$ and $proj_{F_\omega}[z_\omega z'_\omega] \subseteq l$.

Hence each geodesic segment on the positive side of F_ω and disjoint from F_ω projects into a set $l \in \mathscr{L}_{F_\omega}$. We conclude that the sets $\{sd_{F_\omega} > 0\} \cap proj^{-1}_{F_\omega}(l)$ are open for all $l \in \mathscr{L}_{F_\omega}$ and our claim follows. □

3.4 Rigidity of bilipschitz homeomorphisms

We first look at the position of a bilipschitz embedded flat $B = f(\mathbb{R}^2)$ in X_ω relative to a cooriented isolated flat F_ω. Suppose that $B - F_\omega$ consists of several connected components and let B_0 be a component on the positive side of F_ω. By Proposition 3.7, ∂B_0 is contained in some $l \in \mathscr{L}_{F_\omega}$ and therefore is homeomorphic to a closed subset of the real line. Since ∂B_0 separates B, Alexander duality yields that $l = \partial B_0$ is a line and the geometric component adjacent to F_ω on the positive side is Seifert. Note that the pair $(B_0, \partial B_0)$ is homeomorphic to the pair $(\mathbb{R}_+ \times \mathbb{R}, 0 \times \mathbb{R})$.

Lemma 3.8 *Any flat F_ω in X_ω is contained in a single geometric component. Moreover, F_ω arises as the ultralimit of a sequence of flats in X.*

Proof: If F_ω is not contained in a geometric component then it is essentially split by some isolated flat F'_ω (Lemma 3.4). The geometric components on the both sides of F'_ω must be Seifert. Moreover, $F_\omega \cap F'_\omega$ contains two transversal lines and therefore $F_\omega = F'_\omega$. Since any flat in a geometric component arises as the ultralimit of a sequence of flats, the claim follows. □

Next we give a topological characterization of isolated flats which are not adjacent to Seifert components of X_ω.

Lemma 3.9 *Let B be a bilipschitz flat in X_ω. The following two properties are equivalent:*

1. *The intersection of B with any other bilipschitz flat B' contains at most one point.*
2. *B is an isolated flat which is not adjacent to any Seifert component.*

Proof: If B is a bilipschitz flat which satisfies the first property then B cannot be essentially split by any isolated flat. By Lemmata 3.4 and 3.1, B is a flat contained in a geometric component. The component must be hyperbolic, so B is an isolated flat and moreover the geometric components on the both sides of B must be hyperbolic. (Note that it may happen that B has only one side!)

Vice versa, assume now that F is an isolated flat satisfying the second property and let B' be a bilipschitz flat intersecting F. Then for any connected component B_0 of $B' - F$, $\bar{B}_0 \cap F$ is a point in F. Since B cannot be disconnected by one point, $B' - F$ consists of one component and $B \cap F$ is a point. □

Lemma 3.10 *Let T be a geodesically complete tree which branches at every point. Then for any bilipschitz embedding $f : T \times \mathbb{R} \to X_\omega$, the image is contained in a Seifert component and the map f preserves the Seifert fibration.*

Proof: Suppose that an isolated flat F_ω essentially splits $f(T \times \mathbb{R})$. Let $\Omega^\pm$ be connected components of $f(T \times \mathbb{R}) - F_\omega$ which lie on different sides of F_ω. Then their boundaries are transversal straight lines $l_\pm$ in F_ω. On the other hand, the inverse images $f^{-1}(l_\pm)$ separate $T \times \mathbb{R}$ and hence they are parallel lines by Lemma 3.3. This is impossible, because f is bilipschitz.

Hence $f(T \times \mathbb{R})$ is not essentially split by any isolated flat and therefore lies in a Seifert component by Lemmata 3.4 and 3.2. The second assertion follows from Lemma 3.1. □

We apply the above observations to show that homeomorphisms of asymptotic cones are rigid in the sense that they preserve the decomposition into geometric components.

Proposition 3.11 *Let $X, X' \in \mathscr{H}_{npc}$ and let $\phi : X_\omega \to X'_\omega$ be a bilipschitz homeomorphism. Then:*

(i) ϕ maps flats to flats.

(ii) Each isolated flat which is not adjacent to a Seifert component is mapped via ϕ to an isolated flat of the same kind.

(iii) The image of each Seifert component of X_ω is a Seifert component of X'_ω.

Proof: Assertion (ii) follows from Lemma 3.9 and assertion (iii) from Lemma 3.10. According to Lemma 3.8, any flat in X_ω lies in a geometric component. Thus for isolated flats between hyperbolic components Assertion (i) follows again from Lemma 3.9 and for flats contained in Seifert components from Lemma 3.10. □

3.5 Structure of bilipschitz-embedded flats

Let $f : \mathbb{R}^2 \hookrightarrow X_\omega$ be a C-bilipschitz embedding. We will now take a closer look at the position of the bilipschitz flat $B := f(\mathbb{R}^2)$ relative to an isolated flat F which separates B, i.e. $B - F$ is disconnected. We observed in the beginning of Section 3.4 that, for each component B_0 of $B - F$, $proj_F(B_0)$ is a straight line contained in $B \cap F$. It follows that

$$proj_F|_F : F \to B \cap F$$

is a retraction and $B \cap F$ is contractible.

Assume that B_1 and B_2 are two components of $B - F$ on the *same* side of F. Then $l_i := \overline{B}_i \cap F$ are parallel lines bounding a flat strip $S \subset F$. Any points $p_1 \in l_1$ and $p_2 \in l_2$ can be connected inside $B \cap F$ by a rectifiable curve of length at most $C^2 \cdot d(p_1, p_2)$. Indeed, connect the points $x_1 = f^{-1}(p_1), x_2 = f^{-1}(p_2)$ by the geodesic segment $[x_1 x_2] \subset \mathbb{R}^2$, its length is at most $Cd(p_1, p_2)$. Then the projection of $f([x_1x_2])$ to F has length at most $C^2 \cdot d(p_1, p_2)$ and lies inside of $B \cap F$.

Since $B \cap F$ is contractible we conclude that $S \subset B \cap F$ and

$$B = B_1 \cup S \cup B_2.$$

Consider now the case that B_1 and B_2 are components of $B - F$ on *distinct* sides of F. Then $l_i := \bar{B}_i \cap F$ are transversal straight lines in F and, by the above,

$$B = B_1 \cup B_2 \cup (B \cap F).$$

The set $D := f^{-1}(F - (l_1 \cup l_2)) = f^{-1} proj^{-1}(F - (l_1 \cup l_2))$ is open in $\mathbb{R}^2$, and therefore $f|_D : D \hookrightarrow F - (l_1 \cup l_2)$ is a local homeomorphism. On the other hand, $f(\mathbb{R}^2)$ is closed because f is proper, and $f(D)$ must be a union of connected components of $F - (l_1 \cup l_2)$. The lines l_1, l_2 divide F into four "quadrants" and we conclude that $B \cap F$ is a union of two opposite closed quadrants.

Now we are ready to discuss the structure of a bilipschitz flat B which is not contained in a single geometric component. We describe an inductive process of geometric decomposition of B. According to Lemma 3.4, B is essentially split by a flat F_0. Let B^+ be the component of $B - F_0$ on the positive side of F_0. If B^+ is contained in the Seifert component S_1 adjacent to F_0 on the positive side then it is a vertical half-plane, as follows for instance from Lemma 3.1. In this case, we stop the decomposition on the right side of F_0. Otherwise, another isolated flat $F_1 \subset S_1$ essentially splits B. Between the pairs of quadrants $f^{-1}(F_0 \cap B)$ and $f^{-1}(F_1 \cap B)$ there is a strip A_1 whose image $f(A_1)$ is a flat strip in S_1. (This strip could degenerate to a single line.) We continue this process of decomposition on the both sides of F_0 and obtain a sequence of consecutive Seifert components $\ldots, S_{-1}, S_0, S_1, \ldots$. The union of these Seifert components is a convex subset of the asymptotic cone. The transition of Q between adjacent Seifert components contributes a definite amount of stretch to the bilipschitz embedding f, and this leads to:

Lemma 3.12 *The number of possible Seifert components S_j occurring in the decomposition is finite and bounded uniformly in terms of the bilipschitz constant of f and the geometry of M.*

Proof: Observe that fibres $l_i : \mathbb{R} \to S_i$ and $l_j : \mathbb{R} \to S_j$ in different Seifert components, which are parameterized by unit speed, have uniform divergence. Namely there is a positive constant α, which depends on the angles between fibers of adjacent Seifert components in M, so that

$$\lim_{t \to \infty} \frac{d(l_i(t), l_j(t))}{t} \geq \alpha$$

We denote the bilipschitz constant of f by C and restrict our attention to a finite number N of Seifert components S_j. The points $(1/t) \cdot f^{-1} \circ l_j(t)$ are contained in a disk of radius $C+o(t)$ in the Euclidean plane and they are $(\alpha/C+o(t))$-separated. Hence N is bounded in terms of α, C and the assertion follows. □

We summarize the above discussion.

Description of bilipschitz 2-flats B in X_ω: Either B is contained in a geometric component and is a genuine flat. If this is not the case, we call B *twisted*. B is then contained in a finite collection of consecutive Seifert components $S_0, \ldots, S_k$ with $k \geq 1$. The consecutive isolated flats $F_i := S_{i-1} \cap S_i$ are the isolated flats which essentially split B. We describe the intersections as we move through the chain of Seifert pieces: $(B \cap S_0) - F_1$ and $(B \cap S_k) - F_k$ are vertical half-planes H_0 and H_k. Let l_i^+ and l_i^- be the lines in F_i which consists of points closest to F_{i+1} and F_{i-1}. Furthermore, let l_1^- and l_k^+ be the boundaries of the half-planes H_0 and H_k. Then the intersection $B \cap F_i$ is the union of two opposite quadrants bounded by $l_i^{\pm}$. Finally, the intersection $B \cap S_i$, $0 < i < k$, consists of the vertical strip V_i bounded by l_i^+, l_{i+1}^- and four quadrants. The convex hull $ch(B)$ of the bilipschitz flat B is given by:

$$ch(B) = H_0 \cup F_1 \cup V_1 \cup F_2 \cup \ldots \cup F_k \cup H_k$$

Lemma 3.13 *No bilipschitz flat in X_ω is contained in a horoball.*

Proof: It follows from the description of bilipschitz flats in X_ω that if a convex set contains a bilipschitz flat then it also contains a flat. Therefore, if a horoball contains a bilipschitz flat, the corresponding Busemann function is bounded from above and hence is constant on a 2-flat. Lemma 2.2 implies that X_ω must contain a 3-dimensional Euclidean half-space H. We know that any 2-flat in X_ω is contained in a geometric component. Since parallel 2-flats must be contained in the same geometric component, H itself lies in a geometric component, which is absurd. □

4 Quasi-isometries of universal covers of Haken manifolds

In this section, X, X' will denote the universal covers of nonpositively curved Riemannian 3-manifolds $M, M' \in \mathscr{H}_{npc}$.

4.1 Linear divergence of quasi-disks

We want to understand the position of quasi-flats relative to convex subsets in X. The following local statement will be our basic tool. A *quasi-disk* is defined as (the image of) a (K, ϵ)-quasi-isometric embedding

$$qd : B_R(0) \subset \mathbb{R}^2 \to X$$

of a Euclidean 2-disk for positive constants R, K and ϵ.

Divergence Lemma 4.1 *There are positive functions $\rho = \rho(\epsilon, K), \alpha = \alpha(\epsilon, K)$ and $r_0 = r_0(\epsilon, K)$ with the following property: If $C \subset X$ is a convex subset, $R > 0$ and $qd : B_R(0) \to X$ is a (K, ϵ)-quasi-disk such that $d_C(qd(0)) \geq \rho$ then for every $r \in [r_0, R]$ the quasi-disk $qd(B_r(0))$ is not contained in the $\alpha r + d_C(qd(0))$-neighborhood of C. (Thus, $qd(B_R(0))$ is linearly divergent from C.)*

Proof: It is enough to prove the following assertion: There exist positive numbers D, R such that for any quasi-disk $qd : B_R(0) \to X$, whose center $qd(0)$ lies at distance at least D away from a convex set $C \subset X$, there is a point $q \in qd(B_R(0))$ with $d_C(q) \geq 1 + d_C(qd(0))$.

Assume that the assertion is not true. Then we have a sequence of convex sets C_n, sequences of positive numbers (R_n) and (D_n) tending to infinity and a sequence of quasi-disks $qd_n : B_{R_n} \to X$ satisfying:

$$d_{C_n}(qd_n(0)) \geq D_n \qquad \text{and} \qquad d_{C_n}|_{qd_n(B_{R_n}(0))} \leq 1 + d_{C_n}(qd_n(0))$$

We pick $\lambda_n^{-2} := min(R_n, D_n)$ and form the ultralimit X_ω of the sequence of based metric spaces $(\lambda_n \cdot X, qd_n(0))$. The sequence of quasi-disks yields a bilipschitz flat B in X_ω. According to Lemma 2.3, the ultralimit of the functions $\lambda_n \cdot (d_{C_n} - d_{C_n}(qd_n(0)))$ is the Busemann function B_ξ associated to an ideal boundary point ξ of X_ω. By construction, B_ξ is nonpositive on B. This contradicts Lemma 3.13. □

As a consequence we see: If the boundary of a quasi-disk lies close (relative to its radius) to a convex set C, then most of the interior of the quasi-disk lies uniformly close to C. More precisely:

Corollary 4.2 *There is a positive constant $\delta = \delta(K, \epsilon)$ such that every (K, ϵ)-quasi-disk $qd : B_R(0) \to X$ satisfies:*

$$qd(B_R(0)) \subset \{d_C \leq \delta \cdot R\} \Longrightarrow qd(B_{\frac{R}{2}}(0)) \subset \{d_C \leq \rho\}$$

Proof: Choose $\delta := min(\frac{\rho}{2r_0}, \frac{\alpha}{2})$. If $R < 2r_0$, then $qd(B_R(0))$ is contained in the ρ-neighborhood of C. Assume that $R \geq 2r_0$ and there is a point $p \in B_{\frac{R}{2}}(0)$ with $d_C(qd(p)) > \rho$. Since $\frac{R}{2} \geq r_0$, the previous lemma implies that the quasi-disk $qd(B_{\frac{R}{2}}(p)) \subset qd(B_R(0))$ is not contained in the $\frac{\alpha R}{2} \geq \delta R$-neighborhood of C, a contradiction. □

4.2 Rigidity of quasi-isometries

Let $qf : \mathbb{R}^2 \to X$ be a (K, ϵ)-quasi-isometric embedding.

Definition 4.3 *We call a quasi-flat $Q := f(\mathbb{R}^2) \subset X$* asymptotically flat, *if for some sequence of scale factors $\lambda_n \to 0$ and some base point $q_0 \in Q$, $Q_\omega = \omega\text{-}\lim(\lambda_n \cdot Q, q_0)$ is a flat in the asymptotic cone X_ω.*

Proposition 4.4 *Let $\rho(K,\epsilon)$ be as in Lemma 4.1. If the (K,ϵ)-quasi-flat Q is* asymptotically flat*, then it is contained in the $\rho(K,\epsilon)$-neighborhood of a flat F.*

Proof: By Lemma 3.8, each flat F_ω in X_ω is represented by a sequence (F_n) of flats in X. If $Q_\omega = F_\omega$, we have $qf(B_{\lambda_n^{-1}}(0)) \subset N_{\delta\cdot\lambda_n^{-1}}(F_n)$ for ω-all n. Corollary 4.2 implies that $qf(B_{\lambda_n^{-1}/2}(0)) \subset N_\rho(F_n)$ for ω-all n. Consequently the flats F_n subconverge to a flat F which contains Q in its ρ-neighborhood . □

Corollary 4.5 *The following properties are equivalent for (K,ϵ)-quasi-flats Q in X:*

1. *Q is asymptotically flat.*
2. *Q is contained in the $\rho(K,\epsilon)$-neighborhood of a flat.*
3. *Q is contained in a tubular neighborhood of a geometric component.*

Proof: We already proved the implication $1 \Longrightarrow 2$. $2 \Longrightarrow 3$ holds, because flats are contained in geometric components. Assume that Q satisfies property 3. Then the asymptotic cone Q_ω is a bilipschitz flat which is contained in a geometric component of X_ω. Lemma 3.1 implies that Q_ω is a flat. □

Note that if the quasi-flat Q is contained in the ρ-neighborhood of the flat F, then Q and F have finite Hausdorff-distance bounded in terms of the quasi-isometry constants, cf. Corollary 2.6.

We now can control the effect of quasi-isometries $\phi : X \to X'$ on flats $F \subset X$. Although quasi-flats in X' are in general not Hausdorff-close to a flat, we have:

Theorem 4.6 *Suppose that $\phi : X \to X'$ is a quasi-isometry. Then the image under ϕ of any flat F in X lies within uniformly bounded Hausdorff distance from a flat F' in X'.*

Proof: We proved in Lemma 3.11 that the induced bilipschitz homeomorphism $\phi_\omega : X_\omega \to X'_\omega$ maps flats to flats. Hence $\phi(F)$ is an asymptotically-flat quasi-flat in X'. Thus Corollary 4.5 implies that $\phi(F)$ is Hausdorff-close to a flat. □

Let $\phi_\#(F)$ denote a flat $F' \subset X'$ which is Hausdorff-close to $\phi(F)$. Note that F' is essentially unique, any other flat with the same property is parallel to F'.

Lemma 4.7 *Let F_1, F_2, F_3 be pairwise nonparallel isolated flats in X which do not separate each other. Then the flats $F'_1 = \phi_\#(F_1), F'_2 = \phi_\#(F_2), F'_3 = \phi_\#(F_2)$ also do not separate each other.*

Proof: For any $r > 0$ we can connect F_2 and F_3 outside the r-neighborhood $N_r(F_1)$ by a curve γ. If r is chosen sufficiently large, then the image $\phi(\gamma)$ lies on one side of $\phi_\#(F_1)$. Therefore $\phi_\#(F_2)$ and $\phi_\#(F_3)$ lie on the same side of $\phi_\#(F_1)$. □

Corollary 4.8 *Let F be a boundary flat in X. Then $\phi_\#(F)$ is a boundary flat in X' as well.*

Lemma 4.7 implies our Main Theorem 1.1.

4.3 Structure of quasi-flats

In this section we will completely describe the quasi-flats in X. Asymptotically flat quasi-flats were treated in Corollary 4.5. Let us start by constructing **examples** of quasi-flats which are *twisted*, i.e. not asymptotically flat: Take a chain $S_0, \ldots, S_k$ of successive Seifert components in X. They are separated by a chain of consecutive isolated flats $F_1, \ldots, F_k$ where $F_i = S_{i-1} \cap S_i$. For $0 < i < k$, there is a vertical flat strip $V_i \subset S_i$ which connects and is orthogonal to the successive flats F_i, F_{i+1}: V_i is a union of Seifert fibers in S_i and can be described as the union of all shortest geodesic segments whose endpoints lie in F_i, respectively F_{i+1}. Finally we take two vertical flat half-planes $H_0 \subset S_0$, $H_k \subset S_k$ which are orthogonal to and whose boundary line is contained in F_1, respectively F_k. Note that

$$A := H_0 \cup F_1 \cup V_1 \cup F_2 \cup \ldots \cup F_k \cup H_k$$

has finite Hausdorff distance $< d$ from its convex hull, and A, equipped with the path metric, is $(1, L)$-quasi-isometrically embedded in X, where the positive constants d, L depend on the geometry of M. Each flat F_i contains a pair of distinguished transversal lines arising as intersection with adjacent strips or half-planes. They divide F_i into 4 quadrants. Remove from each flat F_i one pair of opposite open quadrants. What remains from A is a quasi-flat whose quasi-isometry constants are uniformly bounded in terms of k and the geometry of M.

Let $Q = qf(\mathbb{R}^2)$ be a twisted (K, ϵ)-quasi-flat. Based on our analysis of the structure of bilipschitz flats in X_ω, cf. Section 3.5, we will show that Q is uniformly close to one of the model quasi-flats just constructed.

Definition 4.9 *We say that a flat F* essentially splits *the (K, ϵ)-quasi-flat Q if Q contains points at distance $\geq \rho = \rho(K, \epsilon)$ on the both sides of F. Otherwise we say that Q lies* essentially on one side *of F. A set A* essentially contains *Q if Q is contained in the ρ-neighborhood of A.*

Since Q is twisted, Corollary 4.5 implies that there are isolated flats which essentially split Q. We denote by $Q_\omega \subset X_\omega$ the bilipschitz flat represented by the constant sequence (Q). (Here we consider ultralimits with a constant sequence of base points.) Due to the Divergence Lemma 4.1, a flat F essentially splits Q in X, if and only if the flat $F_\omega := (F)$ essentially splits $Q_\omega = (Q)$ in X_ω. According to our discussion in Section 3.5, there are finitely many consecutive isolated flats which essentially split Q_ω. Consequently, the collection of all isolated flats essentially splitting Q is finite and forms a chain $F_1, \ldots, F_k$ of consecutive isolated flats in X. There is a chain of consecutive Seifert components $S_0, \ldots, S_k$ such that $F_i = S_{i-1} \cap S_i$. Their union Z is a convex set which essentially contains Q and therefore $Q_\omega \subset Z_\omega = (Z)$. Q_ω is the union of pairs of opposite quadrants in the flats $F_{i\omega}$ and half-planes $H_{0\omega} \subset S_{0\omega}$, $H_{k\omega} \subset S_{k\omega}$. (Any two successive isolated flats $F_{i\omega}$ have a line in common and the vertical strips inbetween therefore degenerate.) We represent the half-planes $H_{0\omega}, H_{k\omega}$ by sequences of half-planes $H_{0n} \subset S_0$, respectively $H_{kn} \subset S_k$, which are orthogonal to and whose boundary line is contained in F_1, respectively F_k. We denote by $C_n \subset X$ the convex hull of $H_{0n} \cup \bigcup_{i=1}^{k} F_i \cup H_{kn}$. Since Q_ω is contained in $C_\omega = (C_n)$, we conclude using Corollary 4.2 that for all $q \in Q$ and $R > 0$ there is a ω-large set of values n such that:

$$Q \cap B_R(q) \subset N_\rho(C_n)$$

Observe that Q contains points in S_0, S_k which are arbitrarily far away from the boundary flats F_1, F_k. It follows that the sequences $(H_{0n}), (H_{kn})$ subconverge to half-planes H_0, H_k. We denote by $V_i \subset S_i$, $0 < i < k$, the vertical strips orthogonal to F_i, F_{i+1}. The set

$$A := H_0 \cup F_1 \cup V_1 \cup F_2 \cup \ldots \cup F_k \cup H_k$$

is uniformly Hausdorff-close to its convex hull, and we conclude from the previous discussion that Q is contained in a uniformly bounded tubular neighborhood of A. After replacing Q by a quasiflat at uniformly bounded Hausdorff distance, we may assume that Q is contained in A. Moreover Q is a (K', ϵ')-quasiflat in A equipped with the path metric, with K', ϵ' depending on K, ϵ and M, because the path metric on A and the metric induced from X are $(1, L)$-quasi-isometric with a constant $L = L(M)$. The intersection lines with adjacent strips or half-planes divide each flat F_i into four quadrants. By Lemma 2.5, each of these quadrants is either contained in the r-neighborhood of Q or the intersection of the quadrant with Q is r-close to its boundary, with a constant $r = r(K', \epsilon', M)$. It follows from the description of bilipschitz flats in X_ω that for each F_i exactly two quadrants are contained in the r-neighborhood of Q. Similarly, the half-planes H_0 and H_k are contained in the r-neighborhood of Q. This concludes the proof of:

Theorem 4.10 (Classification of quasi-flats) *There is a constant $d = d(K, \epsilon, M)$ so that each (K, ϵ)-quasi-flat lies at Hausdorff distance at most d from a flat or a twisted model quasi-flats as described in the beginning of this section.*

Corollary 4.11 *(1) Any (K, ϵ)-quasi-flat Q in X lies within uniform distance from a finite union of flats. (2) The number of necessary flats is uniformly bounded in terms of K. (3) The limit set of Q in the ideal boundary $\partial_{geo} X$ is a simple loop which is continuous with respect to the Tits metric. (4) There is a constant $K_0 = K_0(M) > 1$ such that if $K \leq K_0$ then Q is asymptotically flat.*

Proof: The first and third claim follow directly from our previous discussion.

The asymptotic cone Q_ω is isometric to a complete Euclidean cone over a circle of length $l \geq 2(\pi + k\alpha)$ where k is the number of isolated flats essentially separating Q and $\alpha > 0$ is the minimal possible angle of intersection between the fibers of adjacent Seifert components. There is a K-bilipschitz homeomorphism $\flat : \mathbb{R}^2 \to Q_\omega$ and we assume without loss of generality that $\flat$ maps the origin to the tip of the cone Q_ω. Let γ be the image of the unit circle. γ has length at most $2\pi K$ and circumvents the disk of radius K^{-1} centered at the tip of Q_ω. Therefore $2\pi K \geq \frac{2(\pi+k\alpha)}{K}$ and

$$k \leq \frac{(K^2-1)\pi}{\alpha}.$$

This implies the second and fourth claim. □

5 Groups quasi-isometric to fundamental groups of Haken manifolds

5.1 Quasi-actions of groups on metric spaces

Suppose that Γ is a group and ρ is a map from Γ to the set of all (K, ϵ)-quasi-isometries of a metric space X.

Definition 5.1 *We call ρ a* quasi-action *or* under-representation *of Γ on X if for some constant L and all $\gamma_1, \gamma_2 \in \Gamma$ the quasi-isometries $\rho(\gamma_1\gamma_2)$ and $\rho(\gamma_1) \circ \rho(\gamma_2)$ are L-close. The quasi-action is called* quasi-transitive *if for some constant M all orbits $\rho(\Gamma) \cdot x$ are M-close to X. The* kernel *(or* under-represented subgroup*) of the action ρ is the subgroup of Γ which consists of elements whose action on X is Hausdorff-close to the identity. A quasi-action is called* properly discontinuous *if for each bounded subset $C \subset X$ there are only finitely many elements $\gamma_j \in \Gamma$ so that $\rho(\gamma_j)(C) \cap C \neq \emptyset$.*

To simplify notations, we will denote $\rho(\gamma) \cdot x$ by γx.

A typical example of properly discontinuous quasi-transitive quasi-actions appears as follows: Assume that the finitely generated group Γ is quasi-isometric to a metric space X, i.e. there is a quasi-isometry q from a Cayley graph of Γ to X. Then q transfers the isometric action of Γ on the Cayley graph to a quasi-action on X. If Γ, equipped with a word metric, can be injectively and quasi-isometrically embedded into X, then there is an honest action of Γ on X by quasi-isometries with uniform constants. This is the case if X is a geodesic metric space (and Γ infinite).

We need the next lemma for decomposing quasi-actions on trees of spaces. Let $\mathscr{A}$ be a collection of subsets $A \subset X$ such that:

- Every bounded subset $B \subset X$ intersects only finitely many sets in $\mathscr{A}$.
- Any two distinct sets in $\mathscr{A}$ have infinite Hausdorff distance.
- There is a constant H such that for all $\gamma \in \Gamma$ and $A \in \mathscr{A}$ the set γA is H-Hausdorff close to another set in $\mathscr{A}$.

In this situation, we can speak of the *stabilizer* in Γ of a set $A \in \mathscr{A}$: it consists of all elements $\gamma \in \Gamma$ such that γA and A have finite Hausdorff distance. Clearly the stabilizer is a subgroup of Γ.

Lemma 5.2 *If the quasi-action ρ is quasi-transitive then the stabilizer of any set $A \in \mathscr{A}$ acts quasi-transitively on A, i.e. orbits of points in A are uniformly close to A.*

Proof: Let $B \subset X$ be a ball so that $X = \Gamma \cdot B$. By assumption, only finitely many sets $\gamma_1 A, \ldots, \gamma_l A \in \mathscr{A}$ intersect B. Let $C = \max\{d(\gamma_j \circ \gamma_i^{-1}(B), B) : 1 \le i, j \le l\}$. For $x, y \in A$ there are $\gamma_x, \gamma_y \in \Gamma$ so that $\gamma_x(x), \gamma_y(y) \in B$. $\gamma_x A$ and $\gamma_y A$ are Hausdorff close to some subsets $\gamma_i A$ and $\gamma_j A$ respectively. Then $\gamma_{xy} := \gamma_y^{-1}\gamma_j\gamma_i^{-1}\gamma_x$ is in the stabilizer of A and carries x uniformly close to y: $d(\gamma_{xy}(x), y) \le C + diam(B)$ □

5.2 Quasi-actions on geometric components

We first consider the case of hyperbolic components. Let Y be the universal cover of a hyperbolic component of M and suppose that we have a quasi-transitive action of a group G on Y by (K, ϵ)-quasi-isometries. Richard Schwartz [Sch] proves:

– The group G fits into a short exact sequence
$$1 \longrightarrow Fin(G) \longrightarrow G \longrightarrow \bar{G} \longrightarrow 1 \qquad (2)$$
with $Fin(G)$ finite and $\bar{G}$ a nonuniform lattice in $Isom(\mathbb{H}^3)$. Hence $Fin(G)$ is the unique maximal finite normal subgroup of G and $\bar{G}$ is the fundamental group of a compact hyperbolic 3-orbifold with flat boundary.
– If F is a boundary flat of Y then the quasi-action of the stabilizer of F in G is within bounded distance from an isometric action of a Euclidean lattice on $F \cong \mathbb{R}^2$. The stabilizers of boundary flats in G correspond to peripheral subgroups of the orbifold fundamental group. $Fin(G)$ is also the unique maximal finite normal subgroup of the stabilizer of F.

Remark 5.3 *It is unknown whether a group G satisfying (2) admits a torsion-free subgroup of finite index.*

Now we turn to the case of Seifert components. Let $S = \Sigma \times \mathbb{R}$ be the universal cover of a Seifert component of M with hyperbolic base orbifold and consider a properly discontinuous quasi-transitive quasi-action action of a group G on S by (K, ϵ)-quasi-isometries. Σ is a convex domain of the hyperbolic plane whose boundary is a non-empty union of disjoint geodesics. For our purposes, we are interested in the case when the collection of boundary flats of S is invariant under this action, i.e. boundary flats are carried to within uniformly bounded distance of boundary flats. Using reflections in faces of S we extend this quasi-action to a properly discontinuous quasi-transitive quasi-action of a bigger group H on $\mathbb{H}^2 \times \mathbb{R}$ by (K', ϵ')-quasi-isometries; the new constants K', ϵ' depend on K, ϵ and the geometry of Σ. The convex domain $S \subset \mathbb{H}^2 \times \mathbb{R}$ is quasi-preserved by G.

Proposition 5.4 *Any (K, ϵ)-quasi-isometry ϕ of a Seifert component $S = \Sigma \times \mathbb{R}$ quasi-preserves the Seifert fibration, i.e. there is a number $r = r(K, \epsilon)$ such that for any $s \in \Sigma$ the image $\phi(\{s\} \times \mathbb{R})$ is r-Hausdorff close to another fiber $\{\bar{\phi}(s)\} \times \mathbb{R}$.*

Proof: Any fiber $\{s\} \times \mathbb{R}$ is the intersection of two orthogonal flats F, F' in S and according to Corollary 4.5 the images of F, F' are Hausdorff-close to flats $\phi_\#(F), \phi_\#(F')$ in Y. For any $R > 0$, the intersection of tubular neighborhoods $N_R(\phi_\#(F)) \cap N_R(\phi_\#(F'))$ is a union $C' \times \mathbb{R}$ of Seifert fibers. The diameter of C' is bounded above in terms of K, ϵ and R, because F and F' are orthogonal. The assertion follows. □

This Proposition was first proven by E. Rieffel in [R] who used quite different arguments.

As a consequence the quasi-action of H on $\mathbb{H}^2 \times \mathbb{R}$ descends to a quasi-transitive quasi-action on the hyperbolic plane by quasi-isometries with bounded constants. Let K be the kernel of this quasi-action and $\bar{H} = H/K$. The induced action of $\bar{H}$ on the ideal circle $\partial_{geo}\mathbb{H}^2$ by homeomorphisms is effective and a convergence group action in the sense of Gehring and Martin. Moreover, it satisfies the "simple axis condition" of Tukia and is topologically conjugate to an action of a Moebius group by Theorem 6B in [Tu]. This Moebius group acts cocompactly on $\mathbb{H}^2$ and also properly discontinuously, because it preserves a locally finite pattern of geodesics. This implies that the group $\bar{G} = G/K$ is the fundamental group of a compact 2-dimensional hyperbolic orbifold O with boundary. We therefore have an exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow \pi_1(O) \longrightarrow 1$$

Peripheral subgroups of $\pi_1(O)$ correspond to stabilizers of boundary geodesics of Σ.

Now we want to determine the structure of the kernel K of the quasi-action on hyperbolic plane. K stabilizes each fiber (up to uniformly bounded distance) and acts quasi-transitively and properly discontinuously on each fiber.

Lemma 5.5 *K has a unique maximal finite normal subgroup $Fin(G)$ and the quotient group $K/Fin(G)$ is isomorphic to $\mathbb{Z}$ or the infinite dihedral group D_∞.*

Proof: There is an element $k \in K$ which is far from the identity and preserves the orientation of the fibres on the large scale. k is quasi-isometrically conjugate to a translation and generates an infinite cyclic subgroup of K. The subgroup $\langle k \rangle \cong \mathbb{Z}$ has finite index in K because K acts properly discontinuously on fibers. This implies assertion of the lemma. □

Since $\pi_1(O)$ does not have nontrivial finite normal subgroups, $Fin(G)$ is algebraically characterized as the unique maximal finite normal subgroup of G. The quotient group $\bar{G} := G/Fin(G)$ fits into an exact sequence

$$1 \longrightarrow \mathbb{Z} \text{ or } D_\infty \longrightarrow \bar{G} \longrightarrow \pi_1(O) \longrightarrow 1$$

and is isomorphic to the fundamental group of a Seifert orbifold. The peripheral subgroups of the Seifert orbifold correspond to the stabilizers of boundary flats of S. $Fin(G)$ is also the unique maximal finite normal subgroup of the stabilizers of boundary flats in G.

5.3 The general case

Suppose that Γ is a finitely generated group which is quasi-isometric to the universal cover X of a Haken manifold M with nontrivial canonical decomposition. We can assume without loss of generality that M is nonpositively curved. Let T be the simplicial tree dual to the geometric decomposition of X. We have a quasi-transitive properly discontinuous quasi-action of Γ on X. By Theorem 1.1 this action induces an action of Γ by automorphisms on the tree T. The quotient T/Γ is a finite graph and Γ therefore decomposes as a finite graph of groups. The vertex and edge stabilizers were described in section 5.2. The unique maximal finite normal subgroups of all vertex and edge stabilizers coincide and therefore coincide with the kernel $Fin(\Gamma)$ of the action of Γ on T. The vertex stabilizers for the action of $\bar{\Gamma} := \Gamma/Fin(\Gamma)$ are fundamental groups of 3-dimensional hyperbolic and Seifert orbifolds with flat boundary. We recall that the edge stabilizers are peripheral subgroups of these orbifolds. We glue these orbifolds along boundary components according to the graph T/Γ. The fundamental group of the resulting orbifold O is isomorphic to $\bar{\Gamma}$. The orbifold O is finitely covered by a Haken manifold, cf. [MM]. This proves Theorem 1.2.

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