Quantization of bending deformations of polygons in $\mathbb{E}^3$, hypergeometric integrals and the Gassner representation

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August 27, 1999

Abstract

The Hamiltonian potentials of the bending deformations of n-gons in $\mathbb{E}^3$ studied in [KM] and [Kly] give rise to a Hamiltonian action of the Malcev Lie algebra $\mathcal{P}_n$ of the pure braid group P_n on the moduli space M_r of n-gon linkages with the side-lengths $r = (r_1, ..., r_n)$ in $\mathbb{E}^3$. If $e \in M_r$ is a singular point we may linearize the vector fields in $\mathcal{P}_n$ at e. This linearization yields a flat connection ∇ on the space $\mathbb{C}_*^n$ of n distinct points on $\mathbb{C}$. We show that the monodromy of ∇ is the dual of a quotient of a specialized reduced Gassner representation.

1 Introduction

In [KM] and [Kly] certain Hamiltonian flows on the moduli space M_r of n-gon linkages in $\mathbb{E}^3$ were studied. In [KM] these flows were interpreted geometrically and called *bending deformations of polygons.* In [Kly], Klyachko pointed out that the Hamiltonian potentials of the bending deformations gave rise to a Hamiltonian action of $\mathcal{P}_n$, the Malcev Lie algebra of the pure braid group P_n (see §3), on M_r. It is a remarkable fact, see [K1, Lemma 1.1.4], that a representation $\rho : \mathcal{P}_n \to End(V)$, $dim(V) < \infty$, gives rise to a flat connection ∇ on the vector bundle $\mathbb{C}_*^n \times V$ over $\mathbb{C}_*^n$, the space of distinct points in $\mathbb{C}$. Accordingly the monodromy representation of ∇ yields a representation $\hat{\rho} : P_n \to Aut(V)$.

We see then that if we can find a finite dimensional representation of the Lie algebra $\mathcal{B} \subset C^\infty(M_r)$ generated by the bending Hamiltonians under the Poisson bracket , i.e. if we can "quantize" $\mathcal{B}$, then we will obtain a representation of P_n. Klyachko suggested using a geometric quantization of M_r to quantize $\mathcal{B}$. This appears to be difficult to carry out because the bending flows do not preserve a polarization. Note however that the problem of quantizing a Poisson subalgebra of $C^\infty(M_r)$ can be solved immediately if the functions in the subalgebra have a common critical point $x \in M_r$. For in this case we may simultaneously linearize all the Hamiltonian fields at x. We are fortunate that simultaneous critical points for the algebra $\mathcal{B}$ exist if M_r is singular. Indeed, a degenerate n-gon (i.e. an n-gon which is contained in a line L) is a critical point of all bending Hamiltonians.

The point of this paper is to compute the representation $\hat{\rho}_{\epsilon,r} : P_n \to Aut(T_{\epsilon,r})$ associated to a degenerate n-gon P. Here $T_{\epsilon,r} = T_P(M_r)$ and $\epsilon = (\epsilon_1, ..., \epsilon_n)$, $\epsilon_i \in \{\pm 1\}$, and $r = (r_1, ..., r_n), r_i \in \mathbb{R}_+$, are defined as follows. Fix an orientation on L. The number r_i is the length of the i-th edge of P. Define ϵ_i to be $+1$ if the i-th edge is positively oriented and $\epsilon_i = -1$ otherwise. We call $\epsilon = (\epsilon_1, ..., \epsilon_n)$ the vector of edge-orientation of P.

Our formula for $\rho_{\epsilon,r} : \mathcal{P}_n \to End(T_{\epsilon,r})$ is in terms of certain $n \times n$ matrices $J_{ij}(\lambda)$ which are called *Jordan-Pochhammer* matrices. Let $\lambda = (\lambda_1, ..., \lambda_n)$ be an n-tuple of complex numbers. Define matrices $J_{ij}(\lambda)$ for $1 \leq i < j \leq n$ by

$$\begin{array}{cc}
 & \begin{array}{ccccc} & i-\text{th column} & & j-\text{th column} & \end{array} \\
\begin{array}{c} \\ i-\text{th row} \\ \\ \\ j-\text{th row} \\ \\ \\ \end{array} & \left(\begin{array}{ccccc}
0\ldots0 & 0 & 0\ldots0 & 0 & 0\ldots0 \\
0\ldots0 & \lambda_j & 0\ldots0 & -\lambda_j & 0\ldots0 \\
0\ldots0 & 0 & 0\ldots0 & 0 & 0\ldots0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0\ldots0 & -\lambda_i & 0\ldots0 & \lambda_i & 0\ldots0 \\
0\ldots0 & 0 & 0\ldots0 & 0 & 0\ldots0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)
\end{array} = J_{ij}(\lambda).$$

Define $J_{ii} = 0$ and $J_{ij}(\lambda) = J_{ji}(\lambda)$ for $i > j$. We have (as can be verified easily)

Lemma 1.1 *The matrices $\{J_{ij}(\lambda)\}$ satisfy the infinitesimal braid relations:*

- $[J_{ij}(\lambda), J_{kl}(\lambda)] = 0$ *if* $\{i,j\} \cap \{k,l\} = \emptyset$.
- $[J_{ij}(\lambda), J_{ij}(\lambda) + J_{jk}(\lambda) + J_{ki}(\lambda)] = 0$, i,j,k *are distinct.*

Consequently the assignment $\rho_\lambda(X_{ij}) = J_{ij}(\lambda)$ (see Section 3 for the meaning of X_{ij}) yields a representation $\rho_\lambda : \mathcal{P}_n \to M_n(\mathbb{C})$ and a flat connection ∇ on $\mathbb{C}^n_* \times \mathbb{C}^n$. Here we realize $\mathbb{C}^n$ as the space of *row* vectors with n components. It is immediate that the subspace $\mathbb{C}^n_0 \subset \mathbb{C}^n$ defined by

$$\mathbb{C}^n_0 = \{z \in \mathbb{C}^n : \sum_i z_i = 0\}$$

is invariant under ρ_λ, in fact $\rho_\lambda(\mathcal{P}_n)(\mathbb{C}^n) \subset \mathbb{C}^n_0$. Now we assume $\sum_{i=1}^n \lambda_i = 0$. Then $\lambda \in \mathbb{C}^n_0$ and we see that $\rho_\lambda(\mathcal{P}_n)(\lambda) = 0$. Thus we have a $\mathcal{P}_n$-invariant filtration

$$\mathbb{C}\lambda \subset \mathbb{C}^n_0 \subset \mathbb{C}^n.$$

Define $W_\lambda = \mathbb{C}^n_0/\mathbb{C}\lambda$. Now let P be a degenerate n-gon with side-lengths $r = (r_1, ..., r_n)$ and edge-orientations $\epsilon = (\epsilon_1, ..., \epsilon_n)$. Our first main theorem is

Theorem A. *There is a $\mathcal{P}_n$-invariant almost complex structure J^ϵ on $T_{\epsilon,r}$ such that there is an isomorphism of $\mathcal{P}_n$-modules $T^{1,0}_{\epsilon,r} \cong W_\lambda$ for $\lambda := (\sqrt{-1}\epsilon_1 r_1, ..., \sqrt{-1}\epsilon_n r_n)$.*

Here $T^{1,0}_{\epsilon,r} = \{w \in T_{\epsilon,r} \otimes \mathbb{C} : J^\epsilon w = \sqrt{-1}w\}$. We have

Corollary. *The flat connection on $\mathbb{C}^n_* \times T^{1,0}_{\epsilon,r}$ has the connection form*

$$\omega = \sum_{1 \le i < j \le n} \frac{dz_i - dz_j}{z_i - z_j} \otimes J_{ij}(\lambda)$$

with λ as above.

We then adapt the methods of [K1] to give formulae for multivalued parallel sections of ∇ in terms of hypergeometric integrals and to compute the monodromy of ∇.

Before stating our first formula for the monodromy of ∇ we need more notation. Let γ_j, $1 \le j \le n$, be the free generators of the free group $\mathbb{F}_n$. Define the character $\chi : \mathbb{F}_n \to \mathbb{C}^*$ by $\chi(\gamma_j) = e^{2\pi i \lambda_j}$, $1 \le j \le n$ (recall that $\lambda_j = \sqrt{-1}\epsilon_j r_j$). Let $\mathbb{C}_{\chi^{-1}}$ be the 1-dimensional module (over $\mathbb{C}$) in which the free group $\mathbb{F}_n$ acts by χ^{-1}. The pure braid group P_n acts by automorphisms on $\mathbb{F}_n$ so that the character χ is fixed. Thus we have the associated action of P_n on $\mathrm{H}_1(\mathbb{F}_n, \mathbb{C}_{\chi^{-1}})$. We let $\Gamma_n = \pi_1(\mathbb{CP}^1 - \{z_1, ..., z_n\})$ be the fundamental group of the n times punctured sphere. Hence Γ_n is the quotient of $\mathbb{F}_n$ by the normal subgroup generated by $\gamma_1 \ldots \gamma_n$. Since $\chi(\gamma_1 \ldots \gamma_n) = 1$, the character χ induces a character of Γ_n. The group P_n fixes $\gamma_1 \ldots \gamma_n$ and consequently acts on Γ_n and on $\mathrm{H}_1(\Gamma_n, \mathbb{C}_{\chi^{-1}})$. We can now state

Theorem B. *The monodromy representation of* ∇ *is equivalent to the representation of* P_n *on* $\mathrm{H}_1(\Gamma_n, \mathbb{C}_{\chi^{-1}})$.

In §10 we define the *Gassner representation* of the pure braid group, the *reduced Gassner representation* and their *specializations* via characters of the free group. Let $\mathcal{L}$ is the $\mathbb{C}$-algebra of Laureant polynomials on $t_1, .., t_n$.

Theorem C. *The monodromy representation of* ∇ *is dual to the quotient of the reduced Gassner representation* $Z^1(\Gamma_n, \mathcal{L})$ *specialized at* $t_j = e^{-2\pi\epsilon_j r_j}$, *where we quotient by the 1-dimensional subspace* $B^1(\Gamma_n, \mathbb{C}_\chi)$ *fixed by* P_n.

Our results appear to be related to those of [DM] and [Lo] but there are significant differences. In [Lo], D. D. Long linearizes the action of P_n on the moduli space of n-gon linkages in S^3 obtained from the action of P_n on

$$\mathrm{Hom}(\pi_1(S^2 - \{z_1, ..., z_n\}), SU(2))/SU(2)$$

by precomposition. The corresponding action of P_n on M_r is trivial in our case, see [KM, Remark 5.1]. In [DM], Deligne and Mostow arrive at the Gassner representation by considering a variation of Hodge structure over $\mathbb{C}_*^n/PGL_2(\mathbb{C}) \subset M_r$. They obtain the quotient (by the 1-coboundaries) of the reduced Gassner representation specialized at $(e^{2\pi i r_1}, ..., e^{2\pi i r_n})$; we obtain the dual of the quotient of the reduced Gassner representation specialized at $(e^{-2\pi\epsilon_1 r_1}, ..., e^{-2\pi\epsilon_n r_n})$. Here we must assume $\sum_{i=1}^n r_i = 2$ to be consistent with [DM]. Our representation lies in $GL(n-2, \mathbb{R})$; their representation is in $U(n-3, 1)$.

Acknowledgements. It is our pleasure to thank Ragnar Buchweitz and Richard Hain for helpful conversations.

2 The moduli space of n-gon linkages in $\mathbb{E}^3$.

Let $Pol_n(\mathbb{E}^3)$ be the space of (closed) n-gons with distinguished vertices in the Euclidean space $\mathbb{E}^3$. An n-gon P is defined to be an ordered n-tuple of points $(v_1, ..., v_n) \in (\mathbb{E}^3)^n$. The point v_i is called the i-th vertex of P. The vertices are joined in cyclic order by edges $e_1, ..., e_n$ where e_i is the oriented segment from v_i to v_{i+1}. We think of e_i as a vector in $\mathbb{R}^3$. Two polygons $P = (v_1, ..., v_n)$ and $Q = (w_1, ..., w_n)$ are identified if and only if there exists an orientation-preserving isometry g of $\mathbb{E}^3$ such that $g(v_i) = w_i$, $1 \leq i \leq n$. Let $r = (r_1, ..., r_n)$ be an n-tuple of positive real numbers. Then M_r is defined to be the moduli space of n-gons with the side-lengths $r_1, ..., r_n$ modulo isometries as above. An element of M_r will be called a *closed n-gon linkage.*

We will also need the moduli space space N_r of "open" n-gon linkages. To obtain N_r we repeat the above construction of M_r except we do not assume the end vertex v_{n+1} of the edge e_n is equal to v_1.

The starting point of [KM] was the observation that

$$M_r = \{e = (e_1, ..., e_n) \in \prod_{i=1}^n S^2(r_i) : e_1 + ... + e_n = 0\}/SO(3).$$

This equality exhibits M_r as the symplectic quotient of $\prod_{i=1}^n S^2(r_i)$ and has many consequences. First M_r is a complex analytic space with isolated (quadratic) singularities. The smooth part of M_r is a Kähler manifold. The singular points of M_r are the equivalence classes of degenerate n-gons. Thus M_r is singular if and only if r is the set of side-lengths of a degenerate n-gon.

In [KM] we introduced bending deformations of closed polygonal linkages in $\mathbb{E}^3$, see also [Kly]. Suppose $P = e = (e_1, ..., e_n)$. Let $I \subset \{1, ..., n\}$ be a subset and define $f_I \in C^\infty(M_r)$ by

$$f_I(e) = \| \sum_{i \in I} e_i \|^2.$$

Then f_I is the Hamiltonian potential of a Hamiltonian vector field B_I. The vector $e_I = \sum_{i \in I} e_i$ is constant along an integral curve of B_I. By [KM, Lemma 3.5], $B_I(e) = (\delta_1, ..., \delta_n)$, where $\delta_i = e_I \times e_i$, $i \in I$, and $\delta_i = 0$ for $i \notin I$. The integral curves of B_I are obtained as follows. Define an element $ad(e_I) \in so(3)$ by

$$ad(e_I)(v) = e_I \times v$$

and a one-parameter group $R_I(t) \subset SO(3)$ by

$$R_I(t) = \exp(t\ ad(e_I)).$$

Then the integral curve $e(t)$ of B_I passing through e is given by

$$e_i(t) = R_I(t) e_i, i \in I$$

$$e_j(t) = e_j, j \notin I.$$

This motion of a polygon P has a simple geometric interpretation if the elements of I are consecutive. In this case e_I is a diagonal and it divides the polygon into two parts. Keep one part fixed and *bend* the polygon by rotating the other part around the diagonal with the angular speed $\|e_I\|$. For this reason we call the above motion a bending deformation of the polygon. We will be specifically interested in the case $I = \{i, j\}$, $i < j$. We abbreviate $f_{\{i,j\}}$ to f_{ij} and $B_{\{i,j\}}$ to B_{ij} . We have:

$$f_{ij}(e) = \|e_i + e_j\|^2.$$

Lemma 2.1 *Let $e \in M_r$ be a degenerate polygon. Then $B_{ij}(e) = 0$ for all i, j.*

Proof: The bending field B_{ij} is given by

$$B_{ij}(e) = (0, ..., (e_i + e_j) \times e_i, 0, ..., (e_i + e_j) \times e_j, 0, ...) = (0, ..., e_j \times e_i, 0, ..., e_i \times e_j, 0, ...).$$

If e is degenerate then e_i and e_j are linearly dependent, so $e_i \times e_j = 0$. $\square$

Remark 2.2 *In fact $B_I(e) = 0$ for all I if e is degenerate.*

Define $\tilde{N}_r := \prod_{i=1}^n S^2(r_i)$ where $S^2(r_i)$ is the round 2-sphere of the radius r_i. We also define $\tilde{M}_r \subset \tilde{N}_r$ by

$$\tilde{M}_r = \{e \in \tilde{N}_r : \sum_{i=1}^n e_i = 0\}.$$

Hence N_r is the quotient of $\tilde{N}_r$ by $SO(3)$ and M_r is the quotient of $\tilde{M}_r$ by $SO(3)$.

3 The Malcev Lie algebra of the pure braid group.

Let P_n be the pure braid group on n strands in $\mathbb{C}$ (see [C, §1]). Let $\mathbb{C}^n_*$ denote the subset of $\mathbb{C}^n$ consisting of distinct n-tuples. Then P_n is isomorphic to the fundamental group of $\mathbb{C}^n_*$.

Let $\mathcal{P}_n$ be the Malcev Lie algebra of P_n, see [ABC]. Kohno found the following presentation for $\mathcal{P}_n$ in [K2] (see also [I, Proposition 3.2.1]).

Lemma 3.1 *The Lie algebra $\mathcal{P}_n$ is the quotient of the free Lie algebra over $\mathbb{Q}$ generated by $X_{ij}, 1 \leq i,j \leq n$, subject to the relations:*

1. $X_{ii} = 0$, $1 \leq i \leq n$.
2. $X_{ij} = X_{ji}$, $1 \leq i,j \leq n$
3. $[X_{ij}, X_{kl}] = 0$ *if* $\{i,j\} \cap \{k,l\} = \emptyset$.
4. $[X_{ij}, X_{ij} + X_{jk} + X_{ki}] = 0$, *$i,j,k$ are distinct.*

We will now see that any finite dimensional representation of $\mathcal{P}_n$ induces a finite dimensional representation of P_n on the same vector space. This remarkable fact is an immediate consequence of the following lemma of Kohno [K1, Lemma 1.1.4].

Lemma 3.2 *Suppose V is a finite dimensional vector space and $A_{ij}, 1 \leq i,j \leq n$, are elements of $End(V)$ such that $A_{ii} = 0$ and $A_{ij} = A_{ji}$. Let ∇ be the connection on the trivial V bundle over $\mathbb{C}^n_*$ with connection form*

$$\omega = \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} \otimes A_{ij}.$$

Then ∇ is flat if and only if the relations (3) and (4) for $\mathcal{P}_n$ are satisfied by the A_{ij}'s.

Thus there is a 1-1 correspondence between Lie algebra homomorphisms $\rho : \mathcal{P}_n \to End(V)$ and flat connections ∇ on $\mathbb{C}^n_* \times V$ of the above form. Suppose we are given ρ as above. Since $\pi_1(\mathbb{C}^n_*, z) \cong P_n$ (z is a base-point), the monodromy representation of ∇ gives an induced representation of P_n to $Aut(V)$.

Let $F : \mathbb{C}^n_* \to V$ be a smooth map. Then F induces a parallel section of ∇ if and only if F satisfies the equation (of the V-valued 1-forms on $\mathbb{C}^n_*$)

$$dF = \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} \otimes A_{ij}(F).$$

4 A Hamiltonian action of $\mathcal{P}_n$ on M_r.

We define the function f_{ij} on $\tilde{N}_r$ by

$$f_{ij}(e) = \|e_i + e_j\|^2.$$

The next proposition was proved in [Kly]. Since it is central to our paper we give a proof here.

Proposition 4.1 1. $f_{ij} = f_{ji}$.

2. $\{f_{ij}, f_{kl}\} = 0$, *if* $\{i,j\} \cap \{k,l\} = \emptyset$.

3. $\{f_{ij}, f_{ij} + f_{jk} + f_{ki}\} = 0$*, if* i, j, k *are distinct.*

Proof: The assertions (1) and (2) are obvious. The third assertion will be a consequence of the following discussion. Since $\tilde{N}_r$ is a symplectic leaf of the Lie algebra $(\mathbb{R}^3, \times)$ equipped with the Lie Poisson structure it suffices to prove (3) for the functions f_{ij} extended to $(\mathbb{R}^3)^n$ using the same formula. Let $g_{ij} : (\mathbb{R}^3)^n \to \mathbb{R}$ be given by $g_{ij}(e) = e_i \cdot e_j$ and $h_{ijk} : (\mathbb{R}^3)^n \to \mathbb{R}$ be given by $h_{ijk}(e) = e_i \cdot (e_j \times e_k)$.

Lemma 4.2 $\{g_{ij}, g_{jk}\} = -h_{ijk}$.

Proof: It suffices to prove the lemma for $i = 1, j = 2, k = 3$. We use coordinates $(x_i, y_i, z_i), 1 \le i \le n$, on $(\mathbb{R}^3)^n$. Then

$$\{x_i, y_i\} = z_i, \{y_i, z_i\} = x_i, \{z_i, x_i\} = y_i, 1 \le i \le n \quad .$$

We have

$$\{g_{12}, g_{23}\} = \{x_1x_2 + y_1y_2 + z_1z_2, x_2x_3 + y_2y_3 + z_2z_3\} =$$

$$= \{x_1x_2, y_2y_3\} + \{x_1x_2, z_2z_3\} + \{y_1y_2, x_2x_3\} + \{y_1y_2, z_2z_3\} + \{z_1z_2, x_2x_3\} + \{z_1z_2, y_2y_3\} =$$

$$= x_1y_3z_2 - x_1y_2z_3 - x_3y_1z_2 + x_2y_1z_3 + x_3y_2z_1 - x_2y_3z_1 = -e_1 \cdot (e_2 \times e_3). \qquad \square$$

Corollary 4.3 $\{f_{ij}, f_{jk}\} = -4e_i \cdot (e_j \times e_k)$.

Proof: $f_{ij} = f_{ii} + f_{jj} + 2g_{ij}$. But f_{ii} and f_{jj} are Casimirs. $\square$

We now prove the 3-rd assertion.

$$\{f_{ij}, f_{ij} + f_{jk} + f_{ki}\} = \{f_{ij}, f_{jk}\} + \{f_{ij}, f_{ki}\} = \{f_{ij}, f_{jk}\} + \{f_{ji}, f_{ik}\} =$$

$$\{f_{ij}, f_{jk}\} + \{f_{ji}, f_{ik}\} = -4e_i \cdot e_j \times e_k - 4e_j \cdot e_i \times e_k = -4e_i \cdot e_j \times e_k + 4e_i \cdot (e_j \times e_k) = 0.$$

$\square$

Since the function f_{ij} is $SO(3)$-invariant it induces a function (which is again denoted by f_{ij}) on M_r. The Poisson bracket of these functions remain the same and we obtain

Theorem 4.4 *There exists a Hamiltonian action of the Lie algebra* $\mathcal{P}_n$ *on the symplectic manifold* $\tilde{N}_r$*. This action induces an action on* M_r*.*

From Lemma 3.1 and Proposition 4.1 we see that if we can find a finite-dimensional representation of the Lie subalgebra of $C^\infty(M_r)$ generated by $\{f_{ij}, 1 \le i < j \le n\}$ then we will get a representation of P_n. As explained in the introduction we obtain such a representation on $T_e(M_r)$ for a degenerate n-gon e.

5 Linearization of the bending fields at degenerate polygons.

This section is the heart of the paper. We compute $A_{ij} \in End(T_e(M_r))$, the linearization of the bending field B_{ij} at a degenerate polygon $e \in M_r$. Now assume that e is degenerate, so we may write

$$e = (r_1\epsilon_1 u, ..., r_n\epsilon_n u)$$

for some vector $u \in S^2$ and $\epsilon_i = \pm 1$.

Let M be a manifold, $m \in M$. We recall the definition of the linearization $A_X \in End(T_m(M))$ of a vector field X at a point m where $X(m) = 0$. Choose a connection ∇ on $T(M)$. Let $u \in T_m(M)$, then

$$A_X(u) := (\nabla_u X)(m)$$

Since $X(m) = 0$, A_X is independent of the choice of connection.

For the case in hand the above definition must be modified since M_r is singular at e. There is a commutative algebra version of the above construction that goes as follows. Assume M is a real affine variety, $m \in M$ and X is a vector field on M satisfying $X(m) = 0$. Let $\mathfrak{m}$ be the maximal ideal of m. Then (since $X(m) = 0$) we have $X\mathfrak{m} \subset \mathfrak{m}$ whence $X\mathfrak{m}^2 \subset \mathfrak{m}^2$ and X induces an element of $End(\mathfrak{m}/\mathfrak{m}^2) = End(T^*_m(M))$. By duality we obtain $A_X \in End(T_m(M))$. The reader will verify that if m is a smooth point of M then the two definitions coincide.

We now compute the linearization of B_{ij} at e in M_r. Recall that we have a diagram

$$\begin{tikzcd} \tilde{M}_r \arrow[r] \arrow[d] & \tilde{N}_r \arrow[d] \\ M_r \arrow[r] & N_r \end{tikzcd}$$

where $\tilde{N}_r = S^2(r_1) \times ... \times S^2(r_n)$ and $\tilde{M}_r = \{e \in \tilde{N}_r : \sum_{i=1}^n e_i = 0\}$. Define $g_{ij} : \tilde{N}_r \to \mathbb{R}$ by $g_{ij}(e) = \|e_i + e_j\|^2$. Hence $g_{ij}|\tilde{M}_r$ is $SO(3)$-invariant and descends to the function f_{ij} on M_r. Let $\tilde{B}_{ij}$ be the Hamiltonian vector field of g_{ij}. Then

$$\tilde{B}_{ij}(e) = (0, \dots , e_j \times e_i, 0, \dots , e_i \times e_j, 0, \dots)$$

and hence $\tilde{B}_{ij}$ vanishes at e and is tangent to $\tilde{M}_r$. The induced field on $\tilde{M}_r$ will be denoted B'_{ij}. Then B'_{ij} projects to B_{ij} on M_r. We note $dimT_e(\tilde{N}_r) = 2n$, $dimT_e(\tilde{M}_r) = 2n - 2$ and $dimT_e(M_r) = 2n - 4$.

Remark 5.1 *Since e is a singular point of M_r we have*

$$dimT_e(M_r) = 2n - 4 > dimM_r = 2n - 6.$$

We will first compute the linearization of $\tilde{B}_{ij}$ at e in $\tilde{N}_r$ (e is a smooth point on $\tilde{N}_r$ so we use the first procedure) to obtain $\tilde{A}_{ij} \in End(T_e(\tilde{N}_r))$. Then $\tilde{A}_{ij}$ will preserve the subspace $T_e(\tilde{M}_r) \subset T_e(\tilde{N}_r)$ whence we obtain an induced element $A'_{ij} \in End(T_e(\tilde{M}_r))$. But there is an exact sequence

$$V_e \to T_e(\tilde{M}_r) \to T_e(M_r)$$

where $V_e = \{\delta : \exists v \in \mathbb{R}^3 \text{ such that } \delta_i = e_i \times v, 1 \leq i \leq n\}$, we note that $dimV_e = 2$. We will verify that $A'_{ij}(V_e) \subset V_e$ (in fact $A_{ij}(V_e) = 0$). Hence A'_{ij} will descend to $T_e(M_r)$. The resulting element of $End(T_e(M_r))$ will be A_{ij}, the linearization of B_{ij} at e.

Accordingly we begin by computing the linearization $\tilde{A}_{ij}$ of $\tilde{B}_{ij}$ on $T_e(\tilde{N}_r)$. Thus $\tilde{A}_{ij}$ will be $2n \times 2n$ matrix (instead of a $2n - 4 \times 2n - 4$ matrix).

Another advantage in passing to $\tilde{N}_r$ is that $T_e(\tilde{N}_r)$ is now a direct sum of the tangent bundles of the factors

$$T_e(\tilde{N}_r) = \oplus_{i=1}^n T_{e_i}(S^2(r_i)).$$

The Riemannian connection on $\tilde{N}_r$ is a direct sum of the Riemannian connections on the summands. Thus we may write (for $\delta \in T_e(\tilde{N}_r)$)

$$\tilde{A}_{ij}(\delta) = (0, ..., \nabla_\delta(e_j \times e_i), 0, ..., \nabla_\delta(e_i \times e_j), 0, ...).$$

We will suppress the zeroes in the above row vectors henceforth.

Lemma 5.2

$$\tilde{A}_{ij}(\delta) = (u \times \delta_i, u \times \delta_j) \begin{bmatrix} \epsilon_j r_j & -\epsilon_j r_j \\ -\epsilon_i r_i & \epsilon_i r_i \end{bmatrix}$$

Proof: In the above formula for $\tilde{A}_{ij}(\delta)$ we use the Riemannian connection ∇ on S^2. We will compute using the flat connection $\bar{\nabla}$ on $T(\mathbb{R}^3)|S^2$ and then project back into $T(S^2)$ to get ∇. We have

$$\bar{\nabla}_\delta(e_j \times e_i) = \delta_j \times e_i + e_j \times \delta_i$$
$$\bar{\nabla}_\delta(e_i \times e_j) = \delta_i \times e_j + e_i \times \delta_j.$$

Evaluating at e we obtain

$$\bar{\nabla}_\delta(e_j \times e_i)|_e = \epsilon_i r_i \delta_j \times u + \epsilon_j r_j u \times \delta_i = \epsilon_j r_j u \times \delta_i - \epsilon_i r_i u \times \delta_j.$$

Since the right-hand side is in $T_e(S^2)$ we have also

$$\nabla_\delta(e_j \times e_i)|_e = \epsilon_j r_j u \times \delta_i - \epsilon_i r_i u \times \delta_j.$$

Finally $\nabla_\delta(e_i \times e_j)|_e = -\nabla_\delta(e_j \times e_i)|_e$ and the lemma follows. $\square$

We now relate the action of $\mathcal{P}_n$ on $T_e(\tilde{N}_r)$ we have just computed to the action on $T_e(M_r)$. We recall that $\tilde{M}_r = \{e \in \tilde{N}_r : \sum_{i=1}^n e_i = 0\}$ whence $T_e(\tilde{M}_r) = \{\delta \in T_e(\tilde{N}_r) : \sum_{i=1}^n \delta_i = 0\}$. We have the 2-dimensional subspace V_e of tangents to the $SO(3)$-orbit through e described above. Thus we have a filtration $F_\bullet$ given by

$$V_e \subset T_e(\tilde{M}_r) \subset T_e(\tilde{N}_r)$$

and a canonical isomorphism

$$T_e(\tilde{M}_r)/V_e \cong T_e(M_r).$$

We now show that $\mathcal{P}_n$ preserves the above filtration.

Lemma 5.3 *1.* $\mathcal{P}_n T_e(\tilde{N}_r) \subset T_e(\tilde{M}_r)$.

2. $\mathcal{P}_n V_e = 0$.

Proof: (1) is immediate. We prove (2). Suppose $\delta \in V_e$. We claim

$$\delta_j \times e_i + e_j \times \delta_i = 0, \quad 1 \le i < j \le n.$$

Indeed,

$$\delta_j \times e_i + e_j \times \delta_i = (e_j \times v) \times e_i + e_j \times (e_i \times v) = (e_j \times v) \times e_i + (e_j \times e_i) \times v + e_i \times (e_j \times v).$$

But e is degenerate, so $e_i \times e_j = 0$. $\square$

We collect our results in

Theorem 5.4 *1. There is a $\mathcal{P}_n$-stable filtration*

$$V_e \subset T_e(\tilde{M}_r) \subset T_e(\tilde{N}_r).$$

2. $T_e(M_r) \cong T_e(\tilde{M}_r)/V_e$.

3. There is an isomorphism

$$\phi : T_e(\tilde{N}_r) \to T_u(S^2) \otimes \mathbb{R}^n$$

such that $\phi\rho\phi^{-1}(X_{ij}) = adu \otimes J_{ij}(\epsilon_i r_i, \epsilon_j r_j)$.

4. $\phi(T_e(\tilde{M}_r)) = T_u(S^2) \otimes \mathbb{R}_0^n$ *and* $\phi(V_e) = T_u(S^2) \otimes \mathbb{R}v(\epsilon, r)$. *Here* $\mathbb{R}_0^n = \{(x_1, ..., x_n) : \sum_{i=1}^n x_i = 0\}$ *and* $v(\epsilon, r) = (\epsilon_1 r_1, ..., \epsilon_n r_n)$.

Here $\mathbb{R}^n$ is realized as the space of row vectors with n components.

6 The action on the holomorphic tangent space.

The point of this section is that $T_e(\tilde{N}_r)$ has a $\mathcal{P}_n$-invariant almost complex structure that descends to $T_e(M_r)$. We will compute the corresponding action of $\mathcal{P}_n$ on the holomorphic tangent space.

Define an almost complex structure $J \in End(T_e(\tilde{N}_r))$ by

$$J(\delta) = \eta \text{ such that } \eta_i = u \times \delta_i, 1 \leq i \leq n \quad .$$

Lemma 6.1 *1. J is $\mathcal{P}_n$-invariant.*

2. The filtration $F_\bullet$ is invariant under J.

Proof: The first assertion is immediate. It is also clear that $T_t(\tilde{M}_r)$ is invariant under J. It remains to check that V_e is invariant under J. Suppose $\delta \in V_e$. Hence there exists $v \in \mathbb{R}^3$ such that $\delta_i = \epsilon_i r_i u \times v$, $1 \leq i \leq n$. Then $J\delta_i = u \times (\epsilon_i r_i u \times v) = \epsilon_i r_i u \times (u \times v)$. Hence if we put $w = u \times v$ then

$$J\delta_i = \epsilon_i r_r u \times w, 1 \leq i \leq n \quad .$$

Therefore $J\delta \in V_e$. $\square$

Remark 6.2 *The almost complex structure J is* **not** *the one induced by the complex structure on $\tilde{N}_r = \prod_{i=1}^n S^2(r_i)$. We have changed the complex structure on $S^2(r_i)$ to its conjugate for each i such that e_i is a back-track (i.e. $\epsilon_i = -1$).*

We can decompose $T_e(\tilde{N}_r) \otimes \mathbb{C}$ into the $+i$-eigenspace of J denoted by $T_e^\epsilon(\tilde{N}_r)$ and the $-i$-eigenspace denoted by $T_e^{-\epsilon}(\tilde{N}_r)$. Accordingly we have

$$T_e^\epsilon(\tilde{N}_r) = \{\delta \in T_e(\tilde{N}_r) \otimes \mathbb{C} : u \times \delta_j = \sqrt{-1}\delta_j\}$$

Similarly we denote the $+i$-eigenspaces of J acting on $T_e(\tilde{M}_r) \otimes \mathbb{C}$ and $V_e \otimes \mathbb{C}$ by $T_e^\epsilon(\tilde{M}_r)$ and V_e^ϵ respectively. We denote the quotient $T_e^\epsilon(\tilde{M}_r)/V_e^\epsilon$ by $T_e^\epsilon(M_r)$. Clearly the latter space is the $+i$-eigenspace of J acting on $T_e(M_r) \otimes \mathbb{C}$.

Now we recall that we have an isomorphism

$$\phi : T_e(\tilde{N}_r) \to T_u(S^2) \otimes \mathbb{R}^n$$

complexifying we obtain

$$\phi : T_e(\tilde{N}_r) \otimes \mathbb{C} \to T_u(S^2) \otimes_{\mathbb{R}} \mathbb{C}^n.$$

We see that ϕ conjugates J to $adu \otimes 1$ and we have an induced isomorphism (again denoted by ϕ)

$$\phi : T_e^\epsilon(\tilde{N}_r) \to T_u^{1,0}(S^2) \otimes_{\mathbb{C}} \mathbb{C}^n.$$

Under ϕ the action of X_{ij} transforms to $\sqrt{-1}I \otimes J_{ij}(\epsilon_i r_i, \epsilon_j r_j)$. We note that $dim_{\mathbb{C}} T_u^{1,0}(S^2) = 1$ and we obtain a canonical isomorphism

$$\psi : T_e^\epsilon(\tilde{N}_r) \to \mathbb{C}^n.$$

This isomorphism has the property:

$$\psi(T_e^\epsilon(\tilde{M}_r)) = \mathbb{C}_0^n, \quad \psi(V_e^\epsilon) = \mathbb{C}v(\epsilon, r).$$

We have completed our computation of the action of $\mathcal{P}_n$.

Theorem 6.3 1. *There is a canonical isomorphism* $\psi : T_e^\epsilon(\tilde{N}_r) \to \mathbb{C}^n$

2. ψ *induces the action of* $X_{ij} \in \mathcal{P}_n$ *on* $\mathbb{C}^n$ *by* $\sqrt{-1}J_{ij}(\epsilon_i r_i, \epsilon_j r_j)$.

3. $\mathbb{C}^n$ *admits a* $\mathcal{P}_n$*-invariant filtration by* $\psi(T_e^\epsilon(\tilde{M}_r)) = \mathbb{C}_0^n$, $\psi(V_e^\epsilon) = \mathbb{C}v(\epsilon, r)$.

4. *There is an* $\mathcal{P}_n$*-invariant complex structure* J *on* $T_e(M_r)$. *The induced action of* $\mathcal{P}_n$ *on the* $+i$*-eigenspace of* J *in* $T_e(M_r) \otimes \mathbb{C}$ *corresponds to the action of* $\mathcal{P}_n$ *on the quotient* $\mathbb{C}_0^n/\mathbb{C}v(\epsilon, r)$.

Here $\mathbb{C}^n$ is realized as the space of row vectors with n components.

7 The associated hypergeometric equation.

As discussed in the introduction we use the linear operators $A_{ij} \in End(\mathbb{C}^n)$ to obtain a flat holomorphic connection ∇ on the trivial $T_e^\epsilon(\tilde{N}_r)$-bundle $\mathcal{E}$ over $\mathcal{M} = \mathbb{C}_*^n$. The connection form ω of ∇ is

$$\omega = \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} \otimes A_{ij}.$$

A (multivalued) holomorphic section of $\mathcal{E}$ corresponds to a *row* vector $F = (F_1, ..., F_n)$ of (multivalued) holomorphic functions. The hypergeometric equation comes from the condition that F be parallel for the connection ∇:

$$dF = F\omega$$

or equivalently

$$dF_i = \sum_{j, j \neq i} (\lambda_j F_i - \lambda_i F_j) \frac{dz_i - dz_j}{z_i - z_j} \tag{1}$$

with $\lambda_j = \sqrt{-1}\epsilon_j r_j$. We will refer to (1) as the *hypergeometric equation.*

We observe that the operators A_{ij} leave invariant the subspace $\mathbb{C}_0^n$ and annihilate the line $V_\lambda = \mathbb{C}(\lambda_1, ..., \lambda_n)$. We obtain a diagram of flat bundles over $\mathbb{C}_*^n$:

$$\begin{tikzcd}
\mathbb{C}_*^n \times \mathbb{C}_0^n \arrow[r] \arrow[d] & \mathbb{C}_*^n \times \mathbb{C}^n \\
\mathbb{C}_*^n \times \mathbb{C}_0^n/V_\lambda &
\end{tikzcd}$$

The monodromies of these bundles will be the representations of P_n corresponding to the actions of $\mathcal{P}_n$ on $T_e(\tilde{N}_r)$, $T_e(\tilde{M}_r)$, $T_e(M_r)$.

8 Solving the hypergeometric equation by hypergeometric integrals.

Let $\lambda_1, ..., \lambda_n$ be complex numbers with $\lambda_j \notin \mathbb{Z}, 1 \leq j \leq n$. Let $(\xi, z_1, ..., z_n) \in (\mathbb{C}^{n+1})_*$ and $\Phi(\xi, z_1, ..., z_n)$ be the hypergeometric integrand

$$\Phi(\xi, z_1, ..., z_n) := (\xi - z_1)^{\lambda_1} \ldots (\xi - z_n)^{\lambda_n}.$$

Let $\chi := \chi_\lambda : \mathbb{F}_n \to \mathbb{C}^*$ be the character defined by $\chi(\gamma_j) = \exp(2\pi\sqrt{-1}\lambda_j)$, $1 \leq j \leq n$. Recall that $\{\gamma_1, ..., \gamma_n\}$ is a generating set for $\mathbb{F}_n$, the free group of rank n. Here we identify $\mathbb{F}_n$ with the fundamental group $\pi_1(M, b)$, where $M = \mathbb{C} - \{z_1, ..., z_n\}$, so that the conjugacy class of γ_j is represented by a sufficiently small loop which goes once around z_j in the counterclockwise direction. Note that $\chi(\gamma_j) \neq 1$, $1 \leq j \leq n$. For any character $\chi : \mathbb{F}_n \to \mathbb{C}^*$ we let L_χ be the local system over M given by

$$L_\chi = \tilde{M} \times \mathbb{C}/((x,z) \sim (\gamma x, \chi(\gamma) z)).$$

We define a multivalued parallel section σ of L_χ by $\sigma(x) = [x,1]$ (where $[x,z]$ denotes the equivalence class of (x,z)). Note that the lift of σ to the universal cover satisfies

$$\sigma(\gamma x) = [\gamma x, 1] = [x, \chi(\gamma)^{-1}] = \chi(\gamma)^{-1}\sigma(x).$$

The following lemma is obvious:

Lemma 8.1 *The L_χ-valued 1-forms ζ_j, $1 \leq j \leq n$, defined by*

$$\zeta_j(\xi) = (\xi - z_1)^{\lambda_1} \dots (\xi - z_n)^{\lambda_n} \frac{d\xi}{\xi - z_j} \otimes \sigma$$

are single-valued on M.

Hence ζ_j gives rise to a class $[\zeta_j]$ in the de Rham cohomology group $\mathrm{H}^1_{dR}(M, L_\chi)$.

Let $\gamma \in \mathrm{H}_1(M, L_{\chi^{-1}})$. Let G_j be the Kronecker pairing $\langle \zeta_j, \gamma \rangle$ considered as a function of $z_1, ..., z_n$. This Kronecker pairing is traditionally represented as an integral. To make this precise let $\gamma = \sum_{i=1}^k a_i \otimes \tau_i$, where each a_i, $1 \leq i \leq k$, is a 1-simplex and τ_i is a parallel section of $\mathcal{L}^{-1}|a_i$. Then $\langle \zeta_j, \gamma \rangle$ is given by

$$G_j(z_1, .., z_n) = \sum_{i=1}^k \int_{a_i} (\xi - z_1)^{\lambda_1} \dots (\xi - z_n)^{\lambda_n} \langle \sigma, \tau_i \rangle \frac{d\xi}{\xi - z_j} \quad .$$

We will use the following more economical notation:

$$G_j(z_1, .., z_n) = \int_\gamma (\xi - z_1)^{\lambda_1} \dots (\xi - z_n)^{\lambda_n} \frac{d\xi}{\xi - z_j} \otimes \sigma \quad .$$

Now we let $z = (z_1, ..., z_n)$ vary. Let $\pi : \mathbb{C}_*^{n+1} \to \mathbb{C}_*^n$ be the map that forgets the first component. Then $\pi^{-1}(z)$ is isomorphic to $\mathbb{C} - \{z_1, ..., z_n\}$. By [DM, 3.13], the flat line bundle L_χ on $\pi^{-1}(z)$ is the restriction of a flat line bundle $\tilde{L}_\chi$ on $\mathbb{C}_*^{n+1}$. As z varies, the forms $\zeta_1, ..., \zeta_n$ give rise to *relative holomorphic 1-forms* on $\mathbb{C}_*^{n+1}$ with coefficients in $\tilde{L}_\chi$. We recall that a relative holomorphic form on the total space E of a holomorphic fiber bundle $p : E \to B$ is an element of the quotient differential graded algebra

$$\Omega^\bullet(E)/(p^*\Omega^\bullet(B)^+).$$

Here Ω^q denotes the holomorphic q-forms and $(p^*\Omega^\bullet(B)^+)$ denotes the differential ideal in $\Omega^\bullet(E)$ generated by the pull-backs to E of holomorphic forms on B of positive degree. A relative holomorphic q-form η is relatively closed if $d\eta$ is in the above ideal. The forms $\zeta_1, ..., \zeta_n$ are relatively closed, hence they induce holomorphic sections $[\zeta_1], ..., [\zeta_n]$ of the vector bundle $\mathcal{H}^1$ over $\mathbb{C}_*^n$ with fiber over z given by

$$\mathrm{H}^1(\pi^{-1}(z), \tilde{L}_\chi | \pi^{-1}(z)).$$

Precisely, $[\zeta_i](z)$ is the class of the 1-form $\zeta_i(z)$ on $\pi^{-1}(z)$ in the above cohomology group. The bundle $\mathcal{H}^1$ has a flat connection, the *Gauss-Manin* connection, whose definition we now recall. Note first that a local trivialization of π induces a local trivialization of $\mathcal{H}^1$.

Then a smooth section of $\mathcal{H}^1$ is parallel for the Gauss-Manin connection if it is constant when expressed in terms of all such induced local trivializations. The bundle $\mathcal{H}_1$ of the first homology groups with coefficients in $\tilde{L}_{\chi^{-1}}$ admits an analogous flat connection. Now let $p : \tilde{\mathbb{C}}^n_* \to \mathbb{C}^n_*$ denote the universal cover of $\mathbb{C}^n_*$. We obtain a pull-back fiber bundle $\tilde{\pi} : E \to \tilde{\mathbb{C}}^n_*$ of n-punctured complex lines over $\tilde{\mathbb{C}}^n_*$ and pull-back flat vector bundles $\tilde{\mathcal{H}}^1$ and $\tilde{\mathcal{H}}_1$. Choose a base-point $z^0 = (z^0_1, ..., z^0_n)$ in $\mathbb{C}^n_*$. We use M to denote $\mathbb{C} - \{z^0_1, ..., z^0_n\}$ henceforth. Choose a base-point $\tilde{z}^0$ in $\tilde{\mathbb{C}}^n_*$ lying over z^0. We may identify the fiber of $\tilde{\mathcal{H}}_1$ over $\tilde{z}^0$ with $\mathrm{H}_1(M, L_{\chi^{-1}})$. Hence given $\gamma \in \mathrm{H}^1(M, L_{\chi^{-1}})$ there is a unique parallel section $\tilde{\gamma}$ of $\tilde{\mathcal{H}}_1$ such that $\tilde{\gamma}(\tilde{z}^0) = \gamma$. We can now define a global holomorphic function $G_j(z)$ on $\tilde{\mathbb{C}}^n_*$ by

$$G_j(z) = \int_{\tilde{\gamma}} (\xi - z_1)^{\lambda_1} \dots (\xi - z_n)^{\lambda_n} \frac{d\xi}{\xi - z_j} \otimes \sigma.$$

Here we have used the same notation for corresponding (under pull-back) objects on $\mathbb{C}^n_*$ and $\tilde{\mathbb{C}}^n_*$. We may also write

$$G_j(z) = \langle [\zeta_j(z)], \tilde{\gamma} \rangle$$

where $\langle , \rangle$ is the fiberwise pairing between $\tilde{\mathcal{H}}^1$ and $\tilde{\mathcal{H}}_1$. We have

Lemma 8.2

$$dG_i(z) = \sum_{j=1}^{n} \Big(\int_{\gamma} \frac{\partial}{\partial z_j} \Big(\frac{\Phi}{\xi - z_i} \Big) d\xi \otimes \sigma \Big) dz_j.$$

Proof: We have

$$dG_i(z) = \langle \nabla [\zeta_i(z)], \tilde{\gamma} \rangle$$

where ∇ is the Gauss-Manin connection. We will need another formula for the Gauss-Manin connection, see [KO] or Remark 8.3 below. Before stating the formula we need more notation. Let $F^q\Omega^q(E)$ denote the subspace of holomorphic q-forms on E that are multiples of pull-backs of q-forms from the base $\tilde{\mathbb{C}}^n_*$ by elements of $\mathcal{O}(E)$. Then we have a canonical isomorphism (because the fibers of $\tilde{\pi}$ have complex dimension 1)

$$\frac{\Omega^2(E)}{dF^1\Omega^1(E) + F^2\Omega^2(E)} \cong \Omega^1(\tilde{\mathbb{C}}^n_*, \tilde{\mathcal{H}}^1).$$

Now the formula for ∇ is

$$\nabla[\zeta_i] = [d\zeta_i].$$

Here $d\zeta_i$ denotes the exterior differential of ζ_i where ζ_i is considered as a 1-form on E (modulo $F^1\Omega^1(E)$) with values in the line bundle $p^*\tilde{L}_\chi$. The symbol $[d\zeta_i]$ denotes the class of $d\zeta_i$ modulo $dF^1\Omega^1(E) + F^2\Omega^2(E)$. The lemma follows from the formula

$$d\zeta_i \equiv \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \Big(\frac{\Phi}{\xi - z_i} \Big) dz_j \wedge d\xi \otimes \sigma$$

together with the observation that integration over $\tilde{\gamma}$ factors through [ ]. □

Remark 8.3 *The above formula for ∇ can be proved as follows. First note that the formula does indeed define a connection, to be denoted ∇' on $\mathcal{H}^1$. To show that ∇ and ∇' agree it suffices to show they agree locally. Since they are both invariantly defined it suffices to prove that they agree on trivial bundles. But it is clear that in this case a section of $\mathcal{H}^1$ is parallel for ∇' if and only if it is constant.*

The proof of the next lemma is a modification of [K1, Proposition 2.2.2].

Lemma 8.4 *The functions* $G = (G_1, ..., G_n)$ *satisfy*

$$dG_i = \sum_{j,j\neq i} (\lambda_j G_i - \lambda_j G_j) \frac{dz_i - dz_j}{z_i - z_j} \otimes \sigma \quad or \quad dG = \omega G \quad .$$

Proof: We will drop the $\otimes\sigma$ for the course of the proof:

$$G_i(z) = \int_\gamma \Phi \frac{d\xi}{\xi - z_i}.$$

whence by Lemma 8.2

$$dG_i = -\sum_{j=1}^{n} [\int_\gamma \lambda_j \Phi (\xi - z_j)^{-1} (\xi - z_i)^{-1} d\xi] dz_j - [\int_\gamma \Phi (\xi - z_i)^{-2} d\xi] dz_i$$

$$= -\sum_{j\neq i} [\int_\gamma \lambda_j \Phi (\xi - z_j)^{-1} (\xi - z_i)^{-1} d\xi] dz_j - [\int_\gamma (\lambda_i - 1) \Phi (\xi - z_i)^{-2} d\xi] dz_i.$$

We simplify the first term using

$$\frac{1}{\xi - z_i} \cdot \frac{1}{\xi - z_j} = \frac{1}{z_i - z_j} (\frac{1}{\xi - z_i} - \frac{1}{\xi - z_j})$$

to obtain

$$= -\sum_{j\neq i} \frac{\lambda_j}{z_i - z_j} [\int_\gamma \Phi \frac{d\xi}{\xi - z_i}] dz_j - \int_\gamma \Phi \frac{d\xi}{\xi - z_j} - [\int_\gamma (\lambda_i - 1) \Phi (\xi - z_i)^{-2} d\xi] dz_i =$$

$$= -\sum_{j\neq i} \frac{\lambda_j G_i}{z_i - z_j} dz_j + \sum_{j\neq i} \frac{\lambda_j G_j}{z_i - z_j} dz_j - [\int_\gamma (\lambda_i - 1) \Phi (\xi - z_i)^{-2} d\xi] dz_i.$$

Now we have

$$d(\Phi(\xi - z_i)^{-1}) = (\lambda_i - 1) \Phi (\xi - z_i)^{-2} d\xi + \sum_{j\neq i} \lambda_j \Phi (\xi - z_i)^{-1} (\xi - z_j)^{-1} d\xi.$$

Thus by Stokes' Theorem

$$-\int_\gamma (\lambda_i - 1) \Phi (\xi - z_i)^{-2} d\xi = \int_\gamma \sum_{j\neq i} \lambda_j \Phi (\xi - z_i)^{-1} (\xi - z_j)^{-1} d\xi =$$

$$\int_\gamma \sum_{j\neq i} \lambda_j \Phi \frac{1}{z_i - z_j} (\frac{1}{\xi - z_i} - \frac{1}{\xi - z_j}) d\xi =$$

$$\sum_{j\neq i} \frac{\lambda_j}{z_i - z_j} G_i - \sum_{j\neq i} \frac{\lambda_j}{z_i - z_j} G_j$$

hence

$$-[\int_\gamma (\lambda_i - 1) \Phi (\xi - z_i)^{-2} d\xi] dz_i = \sum_{j\neq i} \frac{dz_i}{z_i - z_j} (\lambda_j G_i - \lambda_j G_j) \quad .$$

We obtain

$$dG_i = \sum_{j\neq i} \frac{dz_i - dz_j}{z_i - z_j} (\lambda_j G_i - \lambda_j G_j) \ . \quad \Box$$

Remark 8.5 *The simplification using Stokes' Theorem above is equivalent to observing that*

$$\Phi(\xi - z_i)^{-1} dz_i \otimes \sigma \in F^1\Omega^1(E), 1 \leq i \leq n,$$

and we work modulo $dF^1\Omega^1(E)$ *in computing* ∇.

We now define $F_i := \lambda_i G_i$, $1 \leq i \leq n$.

Lemma 8.6 $F = (F_1, ..., F_n)$ *is a solution of the hypergeometric equation (1).*

Proof:

$$dF_i = \lambda_i dG_i = \sum_{j \neq i} \frac{dz_i - dz_j}{z_i - z_j}(\lambda_i \lambda_j G_i - \lambda_i \lambda_j G_j) =$$

$$= \sum_{j \neq i} \lambda_j(\lambda_i G_i) - \lambda_i(\lambda_j G_j) \frac{dz_i - dz_j}{z_i - z_j} =$$

$$= \sum_{j \neq i} (\lambda_j F_i - \lambda_i F_j) \frac{dz_i - dz_j}{z_i - z_j} . \quad \Box$$

We have proved

Theorem 8.7 *Let* γ *be an element of* $\mathrm{H}_1(M, L_{\chi^{-1}})$ *and* σ *a flat multivalued section of* L_χ. *For* $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ *define a holomorphic function on* $\tilde{\mathbb{C}}^n_*$ *by*

$$F_i := \lambda_i \int_{\tilde{\gamma}} (\xi - z_1)^{\lambda_1} \dots (\xi - z_n)^{\lambda_n} \frac{d\xi}{\xi - z_i} \otimes \sigma.$$

Then $F = (F_1, ..., F_n)$ *is a solution of the hypergeometric equation.*

9 The monodromy representation of the hypergeometric equation and the action on homology.

We have seen that for $\gamma \in \mathrm{H}_1(M, L_{\chi^{-1}})$ we obtain a solution $S = (F_1, ..., F_n)$ of the hypergeometric equation by the formula

$$F_i := \lambda_i \int_{\tilde{\gamma}} (\xi - z_1)^{\lambda_1} \dots (\xi - z_n)^{\lambda_n} \frac{d\xi}{\xi - z_i} \otimes \sigma.$$

It is important to recall that $\sum_{j=1}^n \lambda_j = 0$. The differential forms

$$\eta_j = \lambda_j (\xi - z_1)^{\lambda_1} \dots (\xi - z_n)^{\lambda_n} \frac{d\xi}{\xi - z_i} \otimes \sigma$$

are de Rham representatives of the cohomology classes $[\eta_j], 1 \leq j \leq n$, in $\mathrm{H}^1(M, L_{\chi^{-1}})$. Note that

$$d((\xi - z_1)^{\lambda_1} \dots (\xi - z_n)^{\lambda_n} \otimes \sigma) = \eta_1 + \dots \eta_n$$

hence we have the relation

$$[\eta_1] + \dots + [\eta_n] = 0 \tag{2}$$

Lemma 9.1 *The span of the cohomology classes* $[\eta_j], 1 \leq j \leq n$, *has dimension* $n - 1$.

Proof: First since $\sum_{j=1}^{n}\lambda_j = 0$ we have $\chi(\gamma_1\gamma_2\ldots\gamma_n) = 1$. Thus L_χ extends to a flat line bundle over $\mathbb{CP}^1 - \{z_1, \ldots, z_n\}$. Hence L_χ extends trivially to $\mathbb{CP}^1 - \{z_1, ..., z_n\}$. Also, η_j extends meromorphically over infinity with a simple pole at infinity.

Next we extend the flat line bundle L_χ to a holomorphic line bundle $\mathcal{L}^{hol}$ on $\mathbb{CP}^1$ so that $(\xi - z_j)^{\lambda_j} \otimes \sigma$ is a local basis around z_j. Then $(\xi - z_1)^{\lambda_1}\ldots(\xi - z_n)^{\lambda_n} \otimes \sigma$ is a holomorphic section of $\mathcal{L}^{hol}$ which has no zeroes or poles.

We can now prove the lemma. We have a flat line bundle L_χ over M (with trivial monodromy around ∞). The argument of [DM, §2.7] proves that we can compute the group $\mathrm{H}^1(M, L_\chi)$ as the 1-st cohomology group of the complex $(\Omega^\bullet(\mathbb{CP}^1, *D, L_\chi), d)$ of holomorphic L_χ-valued forms on M which have at worst poles at $z_1, ..., z_n, \infty$. Here the (additive) divisor D is defined by $D = z_1 + ... + z_n + \infty$. Now $\eta_j \in \Omega^1(\mathbb{CP}^1, *D, L_\chi)$ and

$$\Omega^0(\mathbb{CP}^1, *D, L_\chi) = \{f\Phi \otimes \sigma : \text{ so that } f \text{ has at worst poles at } D\}.$$

First note that $Span(\eta_1, ..., \eta_n) \subset \Omega^1(\mathbb{CP}^1, *D, L_\chi)$ has dimension n since the forms η_j have singularities at distinct points of $\mathbb{C}$.

Suppose that there exists $f\Phi \otimes \sigma \in \Omega^0(\mathbb{CP}^1, *D, L_\chi)$ and $c_1, ..., c_n$ such that

$$d(f\Phi \otimes \sigma) = c_1\eta_1 + ... + c_n\eta_n \quad .$$

We claim that f cannot have any poles. Indeed, assume f has a pole of order $k \geq 1$ at z. Then

$$f(\xi) = \frac{c}{(\xi - z_i)^k} + \ldots$$

We are assuming

$$df\Phi + fd\Phi = \sum_{i=1}^{n} c_i\eta_i$$

or

$$df\Phi + (f\sum_{i=1}^{n}\frac{\lambda_i}{\xi - z_i}d\xi)\Phi = \sum_{i=1}^{n} c_i\eta_i. \tag{3}$$

Equating the coefficients of $(\xi - z_i)^{-k-1}$ in the equation (3) from each side we obtain $-kc + \lambda_i c = 0$, or $\lambda_i = k$. This contradicts the assumption that each λ_i is pure imaginary. It remains to check that f is not a polynomial. Assume f has a pole of order $k \geq 1$ at ∞, whence $f(\xi) = a_0 + a_1\xi + ... + a_k\xi^k$. We equate the coefficients at $\xi^{k-1}d\xi$ on each side of (3) to obtain $ka_k + (\sum_{i=1}^{n}\lambda_i)a_k = 0$ or $ka_k = 0$. This contradiction proves the claim. Hence $f \equiv c$ and hence

$$df = c\sum_{i=1}^{n}\eta_i$$

which means that the dimension of the subspace of coboundaries in $Span(\eta_1, ..., \eta_n)$ is 1. □

In the group cohomology computations that follow $\gamma_1, ..., \gamma_n$ will be a generating set of $\mathbb{F}_n$ and $b_1, ..., b_n$ will be its image under abelianization in $\mathbb{Z}^n$. Here the loop representing γ_i is obtained by connecting the small circle a_i going around z_i to the base-point $* \in \mathbb{C} - \{z_1, ..., z_n\}$. We recall that P_n acts on $\mathbb{F}_n$ preserving the conjugacy classes of the generators γ_j. Hence the induced action on $\mathbb{Z}^n$ is trivial and P_n fixes any character $\chi : \mathbb{F}_n \to \mathbb{C}^*$. Hence P_n acts on $\mathrm{H}^1(\mathbb{F}_n, \mathbb{C}_\chi)$. Here we let $\mathbb{C}_\chi$ denote the 1-dimensional space on which $\mathbb{F}_n$ acts via χ. We next need

Lemma 9.2 *Suppose that* $\chi : \mathbb{F}_n \to \mathbb{C}^*$ *satisfies* $\chi(\gamma_i) \neq 1$ *for all* i. *Then* $dim_{\mathbb{C}}\mathrm{H}^1(\mathbb{F}_n, \mathbb{C}_\chi) = n - 1$.

Proof: The Euler characteristic $E(\mathbb{F}_n, \mathbb{C}_1) = 1 - n$. Hence $E(\mathbb{F}_n, \mathbb{C}_\chi) = 1 - n$. On the other hand, $\mathrm{H}^0(\mathbb{F}_n, \mathbb{C}_\chi) = 0$. $\square$

Corollary 9.3 *$dim_{\mathbb{C}}\mathrm{H}_1(M, L_{\chi^{-1}}) = n - 1$ and the classes $[\eta_1], ..., [\eta_{n-1}]$ form a basis for $\mathrm{H}^1(M, L_\chi)$*

We can construct an explicit basis $w_1, ..., w_{n-1}$ for $\mathrm{H}_1(M, \mathcal{L}^{-1})$ following [DM, §2] as follows. We write $w_i = \gamma_i \otimes \sigma_i + \gamma_{i+1} \otimes \sigma_{i+1}$, where σ_i, σ_{i+1} are multivalued flat sections along γ_i, γ_{i+1} respectively and the jump experienced by σ_i (at the base-point) after parallel translating along γ_i cancels that of σ_{i+1} along γ_{i+1}.

Define flat sections S_i, $1 \leq i \leq n-1$, of $\tilde{\mathbb{C}}^n_* \times \mathbb{C}^n_0$ by

$$S_i := (S_{i1}, ..., S_{in}), \quad \text{where} \quad S_{ij} = \lambda_j \int_{\tilde{w}_i} \eta_j.$$

We see then that $S_1, ..., S_{n-1}$ are multivalued parallel sections of $\mathbb{C}^n_* \times \mathbb{C}^n_0$.

The desired representation $\rho : P_n \to Aut(\mathbb{C}^n_0)$ is obtained by parallel translation of $S_1, ..., S_{n-1}$ along loops in $\mathbb{C}^n_*$. The resulting automorphisms leave invariant the line $\mathbb{C}\lambda$ where $\lambda = (\lambda_1, ..., \lambda_n)$.

Before stating the main result of this section we need to define a special class w_∞ in $\mathrm{H}_1(M, L^{-1}_\chi)$. Let $a_\infty \subset \mathbb{C}$ be a circle whose interior contains all the punctures $z_1, ..., z_n$. Since $\lambda_1 + \ldots \lambda_n = 0$, the monodromy of L^{-1}_χ around a_∞ is trivial. Hence there is a nonzero parallel section $\sigma^\vee$ of $L^{-1}_\chi|a_\infty$. We let w_∞ be the homology class represented by $a_\infty \otimes \sigma^\vee$.

Let $\tau : P_n \to Aut\mathrm{H}_1(M, L^{-1}_\chi)$ be the homomorphism induced by the inclusion $P_n \subset Aut(\mathbb{F}_n)$ (recall that P_n acts trivially on the sheaf of parallel sections of L^{-1}_χ).

Lemma 9.4 *(1) $\int_{w_\infty} \eta_i = -\lambda_i$, in particular $w_\infty \neq 0$.*

(2) The class w_∞ is fixed by P_n.

Proof: To prove (1) we apply the residue theorem and note that

$$\Phi(\xi, z)|_{\xi=\infty} = 1$$

and the residue of $(\xi - z_i)^{-1}d\xi$ at $\xi = \infty$ is -1. To verify (2) we identify P_n with the subgroup of the mapping class group of M. Then choose representatives for the elements of P_n so that they act by the identity on the closure of the exterior of the circle a_∞. $\square$

We now have

Theorem 9.5 *(i) The monodromy representation of the flat bundle $\mathbb{C}^n_* \times \mathbb{C}^n_0$ is equivalent to τ.*

(ii) Under the above equivalence the invariant line $V_\lambda \subset \mathbb{C}^n_0$ corresponds to the line $\mathbb{C}w_\infty \subset \mathrm{H}_1(M, \mathcal{L}^{-1}_\chi)$.

(iii) We obtain an induced equivalence of the monodromy representation of $\mathbb{C}^n_* \times \mathbb{C}^n_0/V_\lambda$ and the induced action of P_n on $\mathrm{H}_1(\mathbb{CP}^1 - \{z_1, ..., z_n\}, L^{-1}_\chi)$.

Proof: We have an isomorphism Ψ from $\mathrm{H}_1(M, L^{-1}_\chi)$ onto the space of parallel sections on $\tilde{\mathbb{C}}^n_* \times \mathbb{C}^n_0$ given by $\Psi(w) = S_w$ where

$$S_w = (\int_{\tilde{w}_1} \eta_1, ..., \int_{\tilde{w}_n} \eta_n) = (\langle [\eta_1], \tilde{w} \rangle, \ldots, \langle [\eta_n], \tilde{w} \rangle).$$

We claim that Ψ intertwines the representations τ and ρ (see above) of P_n. The monodromy representation $\rho : P_n \to Aut(\mathbb{C}_0^n)$ is defined by

$$S_w(g^{-1}z) = S_w(z)\rho(g).$$

In order to go further we will need to lift the P_n action on $\tilde{\mathbb{C}}_*^n$ to the total space of $\tilde{\pi} : E \to \tilde{\mathbb{C}}_*^n$. We note that from the fiber bundle $\pi : \mathbb{C}_*^{n+1} \to \mathbb{C}_*^n$ we get an exact sequence $\mathbb{F}_n \to P_{n+1} \to P_n$. We may split this sequence by mapping P_n to the subgroup of P_{n+1} which consists of those elements that do not involve the first string of a braid – recall that π forgets the first point. Let $\tilde{\mathbb{C}}_*^{n+1}$ be the universal cover of $\mathbb{C}_*^{n+1}$. Then P_{n+1} acts on $\tilde{\mathbb{C}}_*^{n+1}$. But $E = \tilde{\mathbb{C}}_*^{n+1}/\mathbb{F}_n$, whence $P_n = P_{n+1}/\mathbb{F}_n$ acts on E as the group of deck transformations of the cover $E \to \mathbb{C}_*^{n+1}$, and the obtained the required lift $\tilde{g}$ of elements $g \in P_n$ to E. We now can give a formula for the monodromy representation τ, namely

$$\tilde{w}(gz) = \tilde{g}_*\tau(g)^{-1}w(z)$$

or

$$\tilde{w}(g^{-1}z) = \tilde{g}_*^{-1}\tau(g)w(z).$$

We can now prove the claim. Observe that since η_i is an invariantly defined 1-form with values in L_χ on $\mathbb{C}_*^{n+1}$ we have

$$\eta_i(gz) = (\tilde{g}^{-1})^*\eta_i(z)$$

or

$$\eta_i(g^{-1}z) = (\tilde{g})^*\eta_i(z).$$

Hence

$$S_w(z)\rho(g) = S_w(g^{-1}z) = (\int_{\tilde{w}(g^{-1}z)} \eta_1(g^{-1}z), ..., \int_{\tilde{w}(g^{-1}z)} \eta_n(g^{-1}z)) =$$

$$(\int_{\tilde{w}(\tilde{g}^{-1}\tau(g)wz)} \tilde{g}^*\eta_1(z), ..., \int_{\tilde{w}(\tilde{g}^{-1}\tau(g)wz)} \tilde{g}^*\eta_n(z)) =$$

$$(\int_{\tau(g)w(z)} \eta_1(z), ..., \int_{\tau(g)w(z)} \eta_n(z))$$

and the claim is proved. Hence (i) follows.

To verify (ii) it suffices to observe that $S_{w_\infty} = (-\lambda_1, \ldots, -\lambda_n)$, which follows from Lemma 9.4. From (i) and (ii) we deduce that the monodromy representation of ∇ on $\mathbb{C}^n/V_\lambda$ is equivalent to the action of P_n on $\mathrm{H}_1(M, \mathcal{L}_\chi^{-1})/\mathbb{C}w_\infty$. But it is clear from the exact sequence of the pair $(M, \mathbb{CP}^1 - \{z_1, ..., z_n\})$ that we have a natural isomorphism $\mathrm{H}_1(M, \mathcal{L}_\chi^{-1})/\mathbb{C}w_\infty \cong \mathrm{H}_1(\mathbb{CP}^1 - \{z_1, ..., z_n\}, \mathcal{L}_\chi^{-1})$. □

Remark 9.6 *Since we have seen that* $T_e(\tilde{M}_r)$ *contains an invariant line, the corresponding representation of* P_n *must be on* $\mathrm{H}_1(M, \mathcal{L}^{-1})$*, not on* $\mathrm{H}^1(M, \mathcal{L})$ *(the latter has an invariant hyperplane).*

10 The Gassner Representation.

We will follow [Bi] and [Mo] for our treatment of the Gassner representation. We begin with a quick review of the Fox calculus.

Let G be a finitely generated group and M a G-module. Let $\mathbb{C}[G]$ be the group ring.

Definition 10.1 *A* **derivation** $D : \mathbb{C}[G] \to M$ *is a* $\mathbb{C}$*-linear map satisfying*

$$D(fh) = (D(f))\epsilon(h) + fD(h)$$

where $\epsilon : \mathbb{C}[G] \to \mathbb{C}$ *is the augmentation. We let* $Der(G, M)$ *denote the space of derivations.*

Remark 10.2 *The restriction of each derivation* D *to* G *is a 1-cocycle* $\delta \in Z^1(G, M)$. *Conversely, given a 1-cocycle* $\delta \in Z^1(G, M)$ *we define a derivation* D *by*

$$D(\sum_{i=1}^{n} c_i g_i) = \sum_{i=1}^{n} c_i \delta(g_i).$$

Thus $Der(G, M)$ *and* $Z^1(G, M)$ *are canonically isomorphic. We will identify them henceforth.*

In the case G is the free group $\mathbb{F}_n$ on the generators $\{x_1, .., x_n\}$ there is a unique derivation $\frac{\partial}{\partial x_i} \in Der(\mathbb{F}_n, \mathbb{C}[F_n])$ given by

$$\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}, 1 \leq i, j \leq n.$$

Then $Der(\mathbb{F}_n, \mathbb{C}[F_n])$ is free over $\mathbb{C}[F_n]$ with the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Note that the projection $p : \mathbb{F}_n \to \mathrm{H}_1(\mathbb{F}_n) \cong \mathbb{Z}^n$ induces a ring-homomorphism $p : \mathbb{C}[\mathbb{F}_n] \to \mathbb{C}[\mathrm{H}_1(\mathbb{F}_n)]$ and a push-forward map on derivations

$$p_* : Der(\mathbb{F}_n, \mathbb{C}[\mathbb{F}_n]) \to Der(\mathbb{F}_n, \mathbb{C}[\mathrm{H}_1(\mathbb{F}_n)]).$$

We may identify $\mathbb{C}[\mathrm{H}_1(\mathbb{F}_n)]$ with the $\mathbb{C}$-algebra $\mathcal{L}$ of Laurant polynomials in $t_1, ..., t_n$. The space $Der(\mathbb{F}_n, \mathcal{L})$ is free over $\mathcal{L}$ with the basis $p_* \frac{\partial}{\partial x_1}, \ldots, p_* \frac{\partial}{\partial x_n}$. We will drop p_* henceforth.

The main point in the construction of the Gassner representation is that there is a homomorphism $\sigma : P_n \hookrightarrow Aut(\mathbb{F}_n)$. This homomorphism is described in terms of formulas in [Bi, Corollary 1.8.3]. There is an elementary description of σ in terms of "pushing a loop along the braid", see [Mo, Page 87]. In both cases the action of P_n on $\mathbb{F}_n$ is a *right* action, i.e. there is $\bar{\sigma}$ such that $\bar{\sigma}(p_1 p_2) = \bar{\sigma}(p_2)\bar{\sigma}(p_1)$. Therefore, the homomorphism σ is actually given by $\sigma(p) := \bar{\sigma}(p^{-1})$. Next we note that we have an action of P_n on $Der(\mathbb{F}_n, \mathcal{L})$:

$$g \cdot D(x) = D(\sigma(g)^{-1} x).$$

Since P_n acts trivially on $\mathcal{L}$, $g \cdot D$ is still a derivation and the operator $g\cdot$ is $\mathcal{L}$-linear.

Remark 10.3 *In [Bi] and [Mo] the action of* P_n *on* $Der(\mathbb{F}_n, \mathcal{L})$ *is defined by* $g * D(x) = D(\bar{\sigma}(g)x)$. *But* $\bar{\sigma}(g) = \sigma(g)^{-1}$ *and hence* $g \cdot D = g * D$. *The composition of two right actions is a homomorphism!*

We can now define the Gassner representation.

Definition 10.4 *The Gassner representation* $\rho : P_n \to Aut_{\mathcal{L}}(Der(\mathbb{F}_n, \mathcal{L}))$ *assigns to each* $g \in P_n$ *the operator* $g\cdot$ *on* $Der(\mathbb{F}_n, \mathcal{L})$, *where* $Der(\mathbb{F}_n, \mathcal{L})$ *is considered as a free* $\mathcal{L}$*-module of rank* n.

It is traditional to represent $\rho(g)$ as an element (a_{ij}) of $GL_n(\mathcal{L})$ using the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, see [Bi, Page 119], [Mo, Page 194]:

$$a_{ij} = \frac{\partial}{\partial x_j} \bar{\sigma}(g) x_i |_{x_i = t_i}.$$

The Gassner representation is reducible. We will see shortly that $Der(\mathbb{F}_n, \mathcal{L})$ contains the P_n-fixed line $B^1(\mathbb{F}_n, \mathcal{L})$ and the P_n-invariant hyperplane $Der(\Gamma_n, \mathcal{L})$. The line does not intersect the hyperplane, nor it is complementary to it ($\mathcal{L}$ is not a field). We begin by describing the line.

We have seen that $Der(\mathbb{F}_n, \mathcal{L}) \cong Z^1(\mathbb{F}_n, \mathcal{L})$. Consequently, $Der(\mathbb{F}_n, \mathcal{L})$ contains $B^1(\mathbb{F}_n, \mathcal{L})$, the Eilenberg-MacLane 1-coboundaries. Since $C^0(\mathbb{F}_n, \mathcal{L}) \cong \mathcal{L}$ and P_n acts trivially on $\mathcal{L}$, P_n will also act trivially on $B^1(\mathbb{F}_n, \mathcal{L})$.

Lemma 10.5 *$B^1(\mathbb{F}_n, \mathcal{L})$ is a free rank 1 submodule of $Z^1(\mathbb{F}_n, \mathcal{L})$ with the basis $\sum_{i=1}^n (1 - t_i)\frac{\partial}{\partial x_i}$.*

Proof: Recall that the coboundary $\delta : C^0(\mathbb{F}_n, \mathcal{L}) \to C^1(\mathbb{F}_n, \mathcal{L})$ is given by

$$\delta \ell(x_i) = \ell - x_i \ell = \ell - t_i \ell = (1 - t_i)\ell$$

But $(1-t_i)\ell = \ell \delta 1(x_i)$, thus δ is $\mathcal{L}$-linear and $B^1(\mathbb{F}_n, \mathcal{L}) = \mathcal{L}(\delta 1)$. We conclude by observing that

$$\delta 1 = \sum_{i=1}^{n} (1 - t_i) \frac{\partial}{\partial x_i} \qquad \square$$

We now describe the hyperplane. The element $x_\infty = x_1 \dots x_n \in \mathbb{F}_n$ is fixed by P_n. We define

$$Der(\mathbb{F}_n, \mathcal{L})^\infty := \{D \in Der(\mathbb{F}_n, \mathcal{L}) : Dx_\infty = 0\}$$

Lemma 10.6 *(i) $Der(\mathbb{F}_n, \mathcal{L})^\infty$ is a free summand of $Der(\mathbb{F}_n, \mathcal{L})$ of rank $n-1$.*

(ii) The quotient map $\mathbb{F}_n \to \Gamma_n$ induces an isomorphism $Der(\Gamma_n, \mathcal{L}) \to Der(\mathbb{F}_n, \mathcal{L})^\infty$ of P_n-modules.

Proof: Let $\{y_1, ..., y_n\}$ be the basis for $\mathbb{F}_n$ given by $y_i = x_1 \dots x_i$, $1 \le i \le n$. Then $Der(\mathbb{F}_n, \mathcal{L})$ is free on $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ and $Der(\mathbb{F}_n, \mathcal{L})^\infty$ is free on $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n-1}}$. The statement (ii) is clear. $\square$

Definition 10.7 *The reduced Gassner representation is the restriction of the action of P_n from $Der(\mathbb{F}_n, \mathcal{L})$ to $Der(\Gamma_n, \mathcal{L})$:*

$$\rho : P_n \to Aut_{\mathcal{L}}(Der(\Gamma_n, \mathcal{L})).$$

We may represent $\rho(g)$, $g \in P_n$ as elements of $GL_{n-1}(\mathcal{L})$ relative to the basis $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n-1}}$. Observe that $B^1(\mathbb{F}_n, \mathcal{L})$ does not intersect $Der(\Gamma_n, \mathcal{L})$, indeed

$$\ell \delta 1(x_\infty) = \ell(1 - t_1 \dots t_n) \neq 0.$$

Remark 10.8 *We will see below that there exist homomorphism images of $Der(\mathbb{F}_n, \mathcal{L})$ such that the image of $B^1(\mathbb{F}_n, \mathcal{L})$ is contained in the image of $Der(\Gamma_n, \mathcal{L})$. Hence $B^1(\mathbb{F}_n, \mathcal{L})$ is not a complement to $Der(\Gamma_n, \mathcal{L})$.*

Note also that there is a representation of P_n on $\mathrm{H}^1(\mathbb{F}_n, \mathcal{L}) = Z^1(\mathbb{F}_n, \mathcal{L})/B^1(\mathbb{F}_n, \mathcal{L})$. We do not know whether or not $\mathrm{H}^1(\mathbb{F}_n, \mathcal{L})$ is a free $\mathcal{L}$-module.

We now have

Definition 10.9 *Let $\alpha = (\alpha_1, ..., \alpha_n)$ with $\alpha_j \in \mathbb{C}^*$, $1 \le j \le n$ and $\mathcal{M}$ be an $\mathcal{L}$-module. Then the* **specialization** *$\mathcal{M}_\alpha$ of $\mathcal{M}$ at α is defined by $\mathcal{M}_\alpha = \mathcal{M} \otimes_{\mathcal{L}} \mathbb{C}_\alpha$. Here $\mathbb{C}_\alpha$ is the complex line equipped with the $\mathcal{L}$-module structure $t_i z = \alpha_i z$, $z \in \mathbb{C}$.*

More concretely, $\mathcal{M}_\alpha$ is the quotient of $\mathcal{M}$ by the submodule of elements $\{(t_j-\alpha_j)m, 1 \leq j \leq n, m \in \mathcal{M}\}$.

Suppose that $T \in End_{\mathcal{L}}(\mathcal{M})$. Then T induces an element $T_\alpha = T \otimes 1$ of $End(\mathcal{M}_\alpha)$. Now assume that $\mathcal{M}$ is free on $m_1, ..., m_n$. Then $m_1 \otimes 1, ..., m_n \otimes 1$ is a vector space basis for $\mathcal{M}_\alpha$. The matrix of T_α relative to this basis is obtained from a matrix of T relative to $m_1, ..., m_n$ by substituting α_j for t_j, $1 \leq j \leq n$.

Now we return to the case in hand. We have $\lambda_1, ..., \lambda_n$ with $\lambda_1 + ... + \lambda_n = 0$. Define $\alpha_j := e^{2\pi i \lambda_j}$, $1 \leq j \leq n$; whence $\alpha_1 ... \alpha_n = 1$.

Lemma 10.10 *Suppose that* $\alpha = (\alpha_1, ..., \alpha_n)$ *satisfies* $\alpha_1 ... \alpha_n = 1$. *Then in the specialization* $Der(\mathbb{F}_n, \mathcal{L})_\alpha$ *the image of the fixed line* $B^1(\mathbb{F}_n, \mathcal{L})$ *is contained in the image of the invariant hyperplane* $Der(\Gamma_n, \mathcal{L})$.

Proof: $\delta 1(x_\infty) = 1 - \alpha_1 ... \alpha_n = 0$. $\square$

Corollary 10.11 *The specialization* $Der(\Gamma_n, \mathcal{L})_\alpha$ *contains a* P_n*-fixed line* $B^1(\mathbb{F}_n, \mathcal{L})_\alpha$.

Now we observe that $Z^1(\mathbb{F}_n, \mathcal{L})_\alpha = Z^1(\mathbb{F}_n, \mathbb{C}_\chi)$, the group of 1-cocycles with values in the 1-dimensional module defined by $\chi(x_j) = \alpha_j, 1 \leq j \leq n$. Moreover

$$Z^1(\Gamma_n, \mathcal{L})_\alpha = Z^1(\Gamma_n, \mathbb{C}_\chi),$$

the group of $\mathbb{C}_\chi$-valued 1-cocycles that annihilate x_∞ and

$$B^1(\mathbb{F}_n, \mathcal{L})_\alpha = B^1(\Gamma_n, \mathbb{C}_\chi).$$

We obtain

Proposition 10.12 *Suppose* $\alpha = (\alpha_1, ..., \alpha_n)$ *satisfies* $\alpha_1 ... \alpha_n = 1$. *Then the specialization of the reduced Gassner representation at* α *contains a* P_n*-invariant line. The quotient of the representation of* P_n *by this line is* $\mathrm{H}^1(\Gamma_n, \mathbb{C}_\chi)$.

Theorem C follows.

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