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The relative deformation theory of representations and flat connections and deformations of linkages in constant curvature spaces

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Abstract. In this paper we develop the relative deformation theory of representations and flat connections and apply our theory to the local deformation theory of linkages in spaces of constant curvature.

Key words: relative deformations, differential graded Lie algebras, mechanical linkages, representation varieties.

In this paper we develop the relative deformation theory of representations and flat connections and apply our theory to the local deformation theory of linkages in X where X is one of the three model spaces of constant curvature S^m, $\mathbb{E}^m$ and $\mathbb{H}^m$. By the relative deformation theory of a representation ρ_0 we mean the following. Let Γ be a finitely generated group, G the group of $\mathbf{k}$-points of an affine algebraic group defined over $\mathbf{k}(\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ in what follows) which will also be denoted G and $R = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_r\}$ a collection of subgroups of Γ. Let $\rho_0 : \Gamma \to G$ be a homomorphism such that the $\mathrm{Ad}\, G$ orbit $\mathcal{O}_j$ of $\rho_0|\Gamma_j$ in $\mathrm{Hom}(\Gamma_j, G)$ is closed, $1 \leqslant j \leqslant r$. We then define the affine variety of relative deformations of ρ_0; to be denoted $\mathrm{Hom}(\Gamma, R; G)$, to be the inverse image of $\prod_{j=1}^r \mathcal{O}_j$ under the natural map of $\mathbf{k}$-varieties $\mathrm{Hom}(\Gamma, G) \to \prod_{j=1}^r \mathrm{Hom}(\Gamma_j, G)$. One can regard $\mathrm{Hom}(\Gamma, R; G)/G$ as the analogue of a relative first cohomology group and $\mathrm{Hom}(\Gamma, R; G)$ as the analogue of the relative one-cocycles.

We next suppose that Γ is the fundamental group of a smooth connected manifold M (possibly with boundary) containing *disjoint* domains $U_1, U_2, \ldots, U_r$ in M such that the image of $\pi_1(U_j)$ under the natural map is conjugate to Γ_j in Γ. We let $U = \bigcup_{j=1} U_j$ and P be the flat principal G-bundle over M associated to ρ_0. We then construct a controlling differential graded Lie algebra $\mathcal{B}^{\cdot}(M, U; \mathrm{ad}\, P)_0$ of $\mathrm{ad}\, P$-valued differential forms on M. Roughly speaking this means we can calculate the deformation space of ρ_0 by solving the equation $d\,\eta + \frac{1}{2}[\eta, \eta] = 0$ in

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$\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)_0$. Precisely, this means that the complete local $\mathbf{k}$-algebra $R_{\mathcal{B}_0^{\cdot}}$ associated to the differential graded Lie algebra $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)_0$ by the procedure of [M1] is isomorphic to the complete local ring $\hat{\mathcal{O}}$ of the germ $(\operatorname{Hom}(\Gamma, R; G), \rho_0)$. We can also express this in terms of functors on the category of Artin local $\mathbf{k}$-algebras. By [BM], Theorem 1.3, $R_{\mathcal{B}_0^{\cdot}}$ is a hull ([Sc]) for the functor $\operatorname{Iso}\mathcal{C}(\mathcal{B}_0^{\cdot}, A)$ described in Chapter 2. Since hulls are unique, $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)_0$ controls the relative deformation theory of ρ_0 if and only if $\hat{\mathcal{O}}$ is a hull for $\operatorname{Iso}\mathcal{C}(\mathcal{B}_0^{\cdot}, A)$. The differential graded Lie algebra $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)_0$ is the augmentation ideal in an augmented differential graded Lie algebra $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)$ which we now describe. The Lie algebra $\mathcal{B}^0(M, U; \operatorname{ad} P)$ is defined to be the subalgebra of smooth sections of $\operatorname{ad} P$ whose restrictions to U_j are parallel, $1 \leqslant j \leqslant r$. Also, for $i > 0$, $\mathcal{B}^i(M, U; \operatorname{ad} P)$ is defined to be the subspace of smooth $\operatorname{ad} P$-valued i-forms that vanish on U_j, $1 \leqslant j \leqslant r$. The augmentation is obtained by evaluation at a point not in U. In Chapter 2 we define the relative deformation theory of the flat bundle P, prove that $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)$ is a controlling differential graded Lie algebra for that deformation theory and then prove that the holonomy map induces an isomorphism of relative deformation theories from the relative deformations of ρ_0 to the relative deformations of P. The letter $\mathcal{B}$ is chosen because of the connection with 'bending', see [KM2].

Now let Λ be a linkage with n vertices in a two-point homogeneous Riemannian manifold M. Let G be the isometry group of M. The n vertices of Λ determine n elements of G, the Cartan involutions associated to these points, whence a representation $\rho_0 \colon \Phi_n \to G$, where Φ_n is the free product of n copies of $\mathbb{Z}/2$. If two vertices are joined by an edge of Λ, then fixing the conjugacy class in G of the restriction of ρ_0 to the infinite dihedral group generated by the two $\mathbb{Z}/2$-factors corresponding to the two vertices corresponds to fixing the distance between the vertices (in order that this statement be true in the positively-curved case we must use $\mathbb{RP}^m$ instead of S^m – however, it is true infinitesimally, see Lemma 3.9). Thus the set of edges $\mathcal{E}$ of Λ determines a collection $R = \{D_2^{i,j}; (i,j) \in \mathcal{E}\}$ of dihedral subgroups of Φ_n. We let $C(\Lambda)$ denote the configuration space of Λ (we do not divide out by G). Then $C(\Lambda)$ is easily seen to be an affine variety and we obtain a (local) isomorphism from $C(\Lambda)$ to $\operatorname{Hom}(\Phi_n, R; G)$. We then use Schottky groups to construct a controlling differential graded Lie algebra $\mathcal{B}^{\cdot}(N, U; \operatorname{ad} P)_0^{\mathbb{Z}/2}$ for the deformations of a linkage Λ with n vertices and e edges. Here N is the connected sum of n copies of $S^1 \times S^{n-1}$ and U is a tubular neighborhood of e disjoint circles in N.

The theory developed in this paper may be applied in two directions. In [KM1] and Chapter 5 of this paper we use our theory to compute the singularities in the deformation space of a polygonal linkage in one of the model spaces of constant curvature using techniques coming from the theory of deformations of flat connections. The singular points of these spaces coincide with the degenerate polygons (i.e. the polygonal linkages that are contained in a geodesic of the model space).

Each such singularity is isolated (modulo the action of the isometry group) and has a neighborhood locally analytically equivalent to the product of the isometry group divided by the isotropy subgroup of the geodesic and a quadratic cone defined by a single non-degenerate quadratic equation whose signature is determined by the number of 'back-tracks' in the linkage. In order to determine this signature, in Theorem 5.8 we give an explicit formula for the cup-product on $H^1(\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)^{\mathbb{Z}/2})$ as a tri-diagonal symmetric matrix.

In [KM3] and Chapter 6 of this paper we give applications going in the other direction. In [KM3] we use linkages related to those of Example 3.2 of [Co], carried over from $\mathbb{R}^2$ to S^2, to construct an example of a fundamental group of a compact hyperbolic three manifold whose representation variety in SO(3) has a nonquadratic singularity at an irreducible representation. In Chapter 6 of this paper we use Theorem 3.2 of this paper to give a physical interpretation of the nilpotent in the deformation space of the representation of a (3, 3, 3) triangle group into $\mathrm{GL}_2(\mathbb{C})$ described in [LM], 2.10.4 and [GM1], 9.3.

We remark that using the ideas set forth in this paper, it should be possible to carry over any deformation problem concerning configurations of totally-geodesic subspaces of a model space of constant curvature to a relative deformation problem of reflection groups. From there it should be possible to arrive at a controlling differential graded Lie algebra of differential forms.

We conclude this introduction with a problem. When does the differential graded Lie algebra $\mathcal{L}^{\cdot} = \mathcal{B}^{\cdot}(N, U, \operatorname{ad} P)_0^{\mathbb{Z}/2}$ of Chapter 4 admit the structure of a (real) mixed Hodge complex (see Definition 1.8 of [BZ]) such that the graded bracket is compatible with the weight and Hodge filtrations? In this case Richard Hain, [H], has shown that the associated ring $R_{\mathcal{L}^{\cdot}}$ is defined by weighted homogenous equations. Thus in this case the deformation space of the planar linkage Λ is a weighted homogenous cone. The Hodge theoretic calculations of [KM1] are a prototype for such an approach to the singularities of the deformation space of Λ.

1. Linkages in constant curvature spaces

Let X be as above. By a linkage Λ in X we mean a piecewise-geodesic mapping $\varphi\colon Y \to X$ from a finite, connected, metrized graph Y into X which is an isometry on edges. Thus Λ is determined by a finite number of geodesic arcs of fixed length in X which are allowed to overlap. We let $V = \{y_1, y_2, \dots, y_n\}$ be the vertices of Y and let $\mathcal{E}$ be the set of unordered pairs $\{i, j\}$ such that y_i and y_j are joined by an edge of Y. We say Λ is admissible if all of the above geodesic arcs are minimizing and that the points $\varphi(y_i)$ and $\varphi(y_j)$, $\{i, j\} \in \mathcal{E}$, can be joined by a unique minimizing geodesic in X (i.e., we do not allow linked points to be antipodal in S^m). If Λ is admissible then Λ is determined by its vertices $u_i = \varphi(y_i), 1 \leqslant i \leqslant n$, and the combinatorial structure of Y. We will consider only admissible linkages and usually write $\Lambda = (u_1, u_2, \dots u_n)$, assuming Y is fixed. A morphism of linkages $\Lambda = (u_1, u_2, \dots, u_n)$ and $\Lambda' = (u'_1, u'_2, \dots, u'_n)$ will be given by an isometry g of

X such that $gu_1 = u'_1, \ldots, gu_n = u'_n$ or equivalently $g \circ \varphi = \varphi'$. We let $C(\Lambda)$ be the set of all deformations of Λ and $M(\Lambda)$ the set of isomorphism classes of elements of $C(\Lambda)$. We now show that $C(\Lambda)$ has the structure of an affine variety.

We realize hyperbolic space $\mathbb{H}^m$ as the upper sheet of the hyperboloid of two sheets in Minkowski space $\mathbb{R}^{m,1}$. Then S^m and $\mathbb{H}^m$ are both realized in $V = \mathbb{R}^{m+1}$ with the geometry induced by a quadratic form (,) which we take to be diagonal relative to the standard basis with diagonal entries $(1, 1, \ldots, \varepsilon)$ where $\varepsilon = +1$ for S^m and $\varepsilon = -1$ for $\mathbb{H}^m$. We let α be the linear functional on V satisfying $\alpha(e_i) = 0, 1 \leqslant i \leqslant m, \alpha(e_{m+1}) = 1$ and we let W be the kernel of α. We may realize $\mathbb{E}^m$ in $\mathbb{R}^{m+1}$ as the affine plane with equation $\alpha(v) = 1$. Then the isometry groups of S^m, $\mathbb{H}^m$ and $\mathbb{E}^m$ are realized as affine algebraic subgroups of $\mathrm{GL}_{m+1}(\mathbb{R})$. The reader will note that α is invariant under the isometries of $\mathbb{E}^m$.

We can give $C(\Lambda)$ the structure of an affine subvariety of X^n as follows. We let $f(u, u')$ be the quadratic polynomial on $V \times V$ given by (u, u') for the spherical and hyperbolic cases and $\|u - u'\|^2$ for the Euclidean case. For each pair $\{i, j\} \in \mathcal{E}$ we let $c_{ij} = f(u_i, u_j)$ and let $c = (c_{ij})$ be the corresponding vector in $\mathbb{R}^e$ where e is the number of edges of Y. We define $\nu \colon X^n \to \mathbb{R}^e$ by

$$\nu((u_i)) = (f(u_i, u_j)).$$

Then $C(\Lambda) = \nu^{-1}(c)$ and we have represented $C(\Lambda)$ as the zero locus of a system of polynomial equations.

We will not go into much detail here concerning the quotient $M(\Lambda)$. We check that it is Hausdorff in the quotient topology.

LEMMA 1.1 *G acts properly on $C(\Lambda)$.*

Proof. The action of G on X is easily seen to be proper. Hence the diagonal action on X^n is proper and consequently the induced action on $C(\Lambda) \subset X^n$ is proper. □

COROLLARY 1.2 *The space of orbits $M(\Lambda) = C(\Lambda)/G$ is Hausdorff in the quotient topology.*

2. Relative deformations of representations and flat connections

In this chapter we will continue with the notation of the introduction. Our goal will be to construct a controlling differential graded Lie algebra for $\mathrm{Hom}(\Gamma, R; G)$. We will assume that the closures $\overline{U}_j$ of the domains $U_j, 1 \leqslant j \leqslant r$ have disjoint neighborhoods $N(\overline{U}_j), 1 \leqslant j \leqslant r$, which induce collar neighborhoods of $\partial U_j = \overline{U}_j - U_j$. The idea behind our definition of $\mathcal{B}^{\cdot}(M, U; \mathrm{ad}\, P)$ is a simple one. We want the perturbations of the basic connection ω_0 to be trivial on U_j. So we replace $\mathcal{A}^1(M, \mathrm{ad}\, P)$ in the usual deformation theory of ω_0 by the one-forms that vanish on U_j. This still allows a non-constant trivial deformation of $\rho_0|\pi_1(U_j)$

because the perturbing forms can be nonzero along an approach path from the base-point of M to U_j. In order to get a complex we replace $\mathcal{A}^0(M, \operatorname{ad} P)$ by the sections of $\operatorname{ad} P$ that are parallel on $U_j, 1 \leqslant j \leqslant r$. Finally in order that we don't have irrelevant obstructions on U_j we replace $\mathcal{A}^2(M, \operatorname{ad} P)$ by the two-forms that vanish on $U_j, 1 \leqslant j \leqslant r$. The verification that the resulting differential graded Lie algebra does in fact control the relative deformations of ρ_0 proceeds along the lines of [GM1]. We begin with a review of that paper.

Let A be an Artin local $\mathbf{k}$-algebra with maximal ideal $\mathcal{M}$. Then G_A is the algebraic group over $\mathbf{k}$ such that for any $\mathbf{k}$-algebra B we have $G_A(B) = G(A \otimes B)$. We will abuse notation henceforth and indentify G_A with its group of $\mathbf{k}$-points $G(A)$. The inclusion map $i : \mathbf{k} \to A$ and the projection map $q : A \to \mathbf{k}$ induce maps $G \xrightarrow{i} G_A \xrightarrow{q} G$ whose composition is the identity map. We form the extended principal bundle $P_A = P \times_G G_A$. We have induced maps $P \xrightarrow{i} P_A \xrightarrow{q} P$ whose composition is the identity. We observe that $i : P \to P_A$ is the homomorphism of principal bundles associated to the homomorphism $i : G \to G_A$ of the structure groups. Moreover, if G_A^0 denotes the kernel of $q : G_A \to G$ then the extension $G_A^0 \to G_A \to G$ is split and $q : P_A \to P$ is obtained by quotienting P_A by the normal subgroup G_A^0 of G_A.

We now consider the action of the maps i and q on connections. Let $\mathcal{G}$ be the Lie algebra of G. We observe that if $\tilde{\omega}$ is a G_A-connection on P_A, then $i^*\tilde{\omega}$ is a $\mathcal{G}_A$-valued G-equivariant 1-form on P hence $q \circ i^*\tilde{\omega}$ is a G-connection on P. Here we have used q to denote the map from $\mathcal{G}_A$ to $\mathcal{G}$ induced by q. We again abuse notation and use $q(\tilde{\omega})$ to denote $qoi^*\tilde{\omega}$. We note that $\tilde{\omega}$ is a deformation of ω_0 if and only if $q(\tilde{\omega}) = \omega_0$. We may map connections in the opposite direction by observing that $P \to P_A$ is a homomorphism of principal bundles with corresponding homomorphism of structure groups $i : G \to G_A$. Thus if ω is a G-connection on P then there is a unique induced G_A-connection $i(\omega)$ on P_A such that di carries the horizontal distribution of ω to the horizontal distribution of $i(\omega)$. Thus $i(\omega)$ is flat if and only if ω is. We have $i^*i(\omega) = io\omega$ whence $qi(\omega) = \omega$. By definition $i(\omega_0)$ is the trivial deformation of ω_0. We will often use $\tilde{\omega}_0$ to denote $i(\omega_0)$.

Finally we consider the action of i and q on gauge transformations. We let $\mathbf{G}(P_A)$ (resp. $\mathbf{G}(P)$) denote the group of bundle automorphisms of P_A (resp. P). We define $i : \mathbf{G}(P) \to \mathbf{G}(P_A))$ by base change whence

$$i(F)[p, g] = [F(p), g], p \in P, g \in G_A.$$

Here $[p, g]$ denotes the class of (p, g) in $P \times_G G_A$. We define $q : \mathbf{G}(P_A) \to \mathbf{G}(P)$ by $q(F) = q \circ F \circ i$. Again we have an induced sequence $\mathbf{G}(P) \xrightarrow{i} \mathbf{G}(P_A) \xrightarrow{q} \mathbf{G}(P)$ with $q \circ i$ equal to the identity. We put $\mathbf{G}^0(P_A) = \ker q$. Then $\mathbf{G}^0(P_A)$ is a nilpotent (infinite-dimensional) Lie group with Lie algebra cannonically isomorphic to $\Gamma(M, \operatorname{ad} P) \otimes \mathcal{M}$. This isomorphism is realized as follows. Given $F \in \mathbf{G}(P_A)$ we define $f : P_A \to G_A$ by $F(p) = pf(p)$ whence $f(pg) = g^{-1}f(p)g$. Thus f

is a section of $\operatorname{Ad} P_A = P_A \times_{\operatorname{Ad}} G_A$. If $F \in \mathbf{G}^0(P_A)$ then f takes values in the nilpotent Lie group $\mathbf{G}^0(P_A)$. Thus f may be written uniquely as $f = \exp \lambda$ with $\lambda \in \Gamma(M, \operatorname{ad} P) \otimes \mathcal{M}$. We observe that the extension $\mathbf{G}^0(P_A) \to \mathbf{G}(P_A) \to \mathbf{G}(P)$ is split by i. Also we observe that for $F \in \mathbf{G}(P_A)$ and ω a connection on P_A

$$q(F^*\omega) = q(F)^*q(\omega).$$

We let $\mathbf{F}(P_A)$ denote the set of flat G_A-connections on P_A. We leave the proof of the following lemma to the reader.

LEMMA 2.1 *The previous formulas for i and q define functors between the categories of principal G-bundles with connection over M and principal G_A-bundles with connection over M.*

This concludes the review of [GM1].

We now define the groupoid $\mathcal{F}^r_A(\omega_0)$ of relative deformations of ω_0. We define the set of objects of $\mathcal{F}^r_A(\omega_0)$ to be the flat $\mathcal{G}_A$-valued connections $\tilde{\omega}$ on P_A such that $q(\tilde{\omega}) = \omega_0$. Further, we require that there exist $F_j \in \mathbf{G}^0(P^j_A), 1 \leqslant j \leqslant m$, such that $\tilde{\omega}|P^j_A = F^*_j\tilde{\omega}_0|P^j_A$. The group of morphisms of $\mathcal{F}^r_A(\omega_0)$ is then defined to $\mathbf{G}^0(P_A)$. The next lemma shows that it makes no difference if we allow F_j above to be in $\mathbf{G}(P^j_A)$ instead of $\mathbf{G}^0(P^j_A)$. In it we drop the superscript j on P^j_A.

LEMMA 2.2 *Suppose $\tilde{\omega}$ is an object of $\mathcal{F}^r_A(\omega_0)$. Suppose further there exists $F \in \mathbf{G}(P_A)$ such that $\tilde{\omega} = F^*\tilde{\omega}_0$. Then there $\overline{F} \in \mathbf{G}^0(P_A)$ such that $\tilde{\omega} = \overline{F}^*\tilde{\omega}_0$.*

Proof. By assumption $F^*\tilde{\omega}_0$ is a deformation of ω_0. Hence $q(F^*\tilde{\omega}_0) = \omega_0$ so $q(F)^*\omega_0 = \omega_0$ and $q(F) \in \operatorname{Aut}(\omega_0)$. By Lemma 2.1, $H = iq(F) \in \operatorname{Aut}(\tilde{\omega}_0)$. Thus $\tilde{\omega} = (H^{-1}F)^*\omega_0$ and $\overline{F} = H^{-1}F \in \mathbf{G}^0(P_A)$. □

We now describe a deformation theory equivalent to that of ω_0 using representations. We define the groupoid $\mathcal{R}^r_A(\rho_0)$ of relative deformations of ρ_0 by defining an object of $\mathcal{R}^r_A(\rho_0)$ to be an element $\rho \in \operatorname{Hom}(\Gamma, G_A)$ such that $q(\rho) = \rho_0$ and $\rho|\Gamma_j$ is conjugate to $\rho_0|\Gamma_j$ by an element $g_j \in G^0_A, 1 \leqslant j \leqslant r$. We define the morphisms of $\mathcal{R}^r_A(\rho_0)$ to be G^0_A acting in the usual way. In what follows we consider the set of objects $\operatorname{Obj}\mathcal{R}^r_A(\rho_0)$ as a functor of A. We will also need the groupoids $\mathcal{R}_A(\rho_0)$ of of [GM1], Section 4.2. The objects of $\mathcal{R}_A(\rho_0)$ are the elements of $\operatorname{Hom}(\Gamma, G_A)$ such that $q(\rho) = \rho_0$ and the morphisms are again the elements of G^0_A acting in the usual way. Let $\mathcal{O}_j, 1 \leqslant j \leqslant r$, be the $\mathbf{k}$-variety which is the orbit of $\rho_0|\Gamma_j$ under G. We define $\mathcal{O}^0_j(A) \subset \mathcal{O}_j(A)$ by

$$\mathcal{O}^0_j(A) = \{\rho \in \mathcal{O}_j(A) \colon q(\rho) = \rho_0\}.$$

We have a square S_1

$$\begin{array}{ccc} \mathcal{R}^r_A(\rho_0) & \longrightarrow & \mathcal{R}_A(\rho) \\ \downarrow & & \downarrow \\ \prod_{j=1}^r \mathcal{O}^0_j(A) & \longrightarrow & \prod_{j=1}^r \mathcal{R}_A(\rho_0|\Gamma_j) \end{array}$$

LEMMA 2.3 *Let* $\rho \in \mathrm{Hom}(\Gamma, G_A)$ *with* $\rho \equiv \rho_0 \operatorname{mod} \mathcal{M}$. *Suppose that there exists* $g \in G_A$ *such that* $\rho = \mathrm{Ad}(g) \circ \tilde{\rho}_0$. *Then there exists* $\bar{g} \in G_A^0$ *such that* $\rho = \mathrm{Ad}(\bar{g}) \circ \tilde{\rho}_0$.

Proof. Upon reducing modulo $\mathcal{M}$ we find $q(g)\rho_0 q(g)^{-1} = \rho_0$. Hence $h = iq(g)$ satisfies $h^{-1}\tilde{\rho}_0 h = \tilde{\rho}_0$ and $q(h) = q(g)$. Hence $\rho = \mathrm{Ad}\,(gh^{-1})\,\tilde{\rho}_0$. But $gh^{-1} \in G_A^0$. Put $\bar{g} = gh^{-1}$. □

We now relate the previous groupoid to the relative representation variety.

PROPOSITION 2.4 *The functor* Obj $\mathcal{R}_A^r(\rho_0)$ *is pro-represented by the germ of the variety* $\mathrm{Hom}(\Gamma, R; G)$ *at* ρ_0.

Proof. By definition we have a fiber square of analytic germs

$$\begin{array}{ccc} (\mathrm{Hom}(\Gamma, R; G), \rho_0) & \longrightarrow & (\mathrm{Hom}(\Gamma, G), \rho_0) \\ \downarrow & & \downarrow \\ \prod_{j=1}^r (\mathcal{O}_j, \rho_0|\Gamma_j) & \longrightarrow & \prod_{j=1}^r (\mathrm{Hom}(\Gamma_j, G), \rho_0|\Gamma_j) \end{array}$$

We obtain a fiber square of sets by taking A-points of these germs. The previous square S_1 maps to the resulting square S_2. We place S_1 above S_2 to obtain a cubical diagram. The vertical (in space) arrows corresponding to the two right-hand corners and the lower left-hand corners of the squares are clearly bijections. We wish to prove that vertical arrow corresponding to the upper left-hand corners is a bijection. Since the lower square S_2 is a fiber square it suffices to prove that S_1 is also a fiber square; that is, that the induced map

$$\mathcal{R}_A^r(\rho_0) \to \prod_{j=1}^r \mathcal{O}_j^0(A) \times_P \mathcal{R}_A(\rho_0),$$

is a bijection. Here, we have abbreviated $\prod_{j=1}^r \mathcal{R}_A(\rho_0|\Gamma_j)$ to P. It is clearly an injection. Thus, we see that it suffices $\rho|\Gamma_j \in \mathcal{O}_j^0(A), 1 \leqslant i \leqslant r$, then there exists $g \in G_A^0$ with $\mathrm{Ad}\, g(\rho_0) = \rho$. But evaluation at $\rho_0, ev_{\rho_0} : G \to \mathcal{O}_j$ is a smooth map. It is then immediate, [GM2], Lemma 4.7, that the induced map $G_A^0 \to \mathcal{O}_j^0(A)$ is onto. □

We now define a functor hol: $\mathcal{F}_A^r(\omega_0) \to \mathcal{R}_A^r(\rho_0)$. We choose a point $p \in P \subset P_A$ such that the image of P in M does not lie in U. We define hol on objects by defining hol(ω) to be the holonomy representation of Γ associated to ω and p (see [GM1], Section 5.9). We define $\varepsilon_p : \mathbf{G}(P_A) \to G_A$ by $F(p) = p\varepsilon_p(F)$. Then $\varepsilon_p(\mathbf{G}^0(P_A)) \subset G_A^0$. We then define hol on morphisms by hol$(F) = \varepsilon_p(F)$. Thus we obtain the required functor.

The next proposition follows from [GM1], Proposition 6.3. Note that the morphisms of the above (relative) groupoids are the same as the corresponding groupoids of [GM1].

PROPOSITION 2.5 *The functor* hol *is an equivalence of groupoids.*

We now define a differential graded Lie algebra $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)_0$ which will be a controlling differential graded Lie algebra for the relative deformations of ρ_0. We associate to ρ_0 the principal bundle $P = \tilde{M} \times_{\pi_1(M)} G$ and the Lie algebra bundle $\operatorname{ad} P = \tilde{M} \times_{\pi_1(M)} \mathcal{G}$. Since $\operatorname{ad} P$ is a flat bundle of Lie algebras, we obtain the differential graded Lie algebra $(\mathcal{A}^{\cdot}(M, \operatorname{ad} P), d)$ of smooth forms with values in $\operatorname{ad} P$. We define $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P) \subset \mathcal{A}^{\cdot}(M, \operatorname{ad} P)$ as follows

$$\begin{aligned}\mathcal{B}^0(M, U; \operatorname{ad} P) &= \{\lambda \in \mathcal{A}^0(M, \operatorname{ad} P) : \lambda | U_j \quad \textit{is parallel}, \\ &\qquad 1 \leqslant j \leqslant r\} \quad \textit{and for} \quad i \geqslant 1 \\ \mathcal{B}^i(M, U; \operatorname{ad} P) &= \{\eta \in \mathcal{A}^i(M, \operatorname{ad} P) : \eta | U_j = 0, 1 \leqslant j \leqslant r\}.\end{aligned}$$

We will often abbreviate $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)$ to $\mathcal{B}^{\cdot}$.

REMARK. Let V be any flat bundle over M. Then we may define a complex $\mathcal{B}^{\cdot}(M, U; V)$ by replacing $\operatorname{ad} P$ in the above definition by V. We will use this notation throughout this paper without further comment.

$\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)$ is a sub differential graded Lie algebra of $\mathcal{A}^{\cdot}(M, \operatorname{ad} P)$. Since $\mathcal{B}^{\cdot}$ is a differential graded Lie algebra, there is an associated transformation groupoid $\mathcal{C}(\mathcal{B}^{\cdot}; A)$, [GM1], pg. 45–46. We recall its definition. The objects of $\mathcal{C}(\mathcal{B}^{\cdot}, A)$ are the elements η of $\mathcal{B}^1 \otimes \mathcal{M}$ satisfying the deformation equation

$$d\eta + \tfrac{1}{2}[\eta, \eta] = 0.$$

The group of morphisms of $\mathcal{C}(\mathcal{B}^{\cdot}, A)$ is the nilpotent Lie group $\exp \mathcal{B}^0 \otimes \mathcal{M}$ associated to the nilpotent Lie algebra $\mathcal{B}^0 \otimes \mathcal{M}$. The morphisms act on the objects by exponentiating the infinitesimal action $d\alpha\colon \mathcal{B}^0 \times \mathit{Obj}\, \mathcal{C}(\mathcal{B}^{\cdot}, A) \to \mathit{Obj}\, \mathcal{C}(\mathcal{B}^{\cdot}, A)$ given by

$$d\alpha(\lambda) \cdot \eta = [\lambda, \eta] - d\lambda.$$

A formula for the action of $\exp(\lambda)$ is given in [GM1], pg. 45. We let $\operatorname{Iso} \mathcal{C}(\mathcal{B}^{\cdot}, A)$ denote the quotient set of the objects by the morphisms.

We next note that there is a functor $\iota : \mathcal{C}(\mathcal{B}^{\cdot}, A) \to \mathcal{F}_A^r(\omega_0)$. The functor ι is defined on an object η by

$$\iota(\eta) = \tilde{\omega}_0 + \eta.$$

Here we have identified η with a horizontal, G_A-equivariant, $\mathcal{G}_A$-valued 1-form on P_A. The definition of ι on morphisms is as follows. An element $\exp(\lambda) \in \exp \mathcal{B}^0 \otimes \mathcal{M}$ may be identified with a cross-section of the Lie group bundle $\operatorname{Ad} P_A$

associated to P_A and the adjoint action of G_A on itself. The section $\exp(\lambda)$ may in turn be identified with an $\mathrm{Ad}\, G_A$-equivariant map $f_\lambda \colon P_A \to G_A$. Then f_λ may be identified with the bundle automorphism $F_\lambda \colon P_A \to P_A$ given by

$$F_\lambda(p) = pf_\lambda(p).$$

We define $\eta(\exp(\lambda)) = (F_\lambda^{-1})^*$. We observe that

$$F_\lambda^* \tilde{\omega}_0 | P_A^j = \tilde{\omega}_0 | P_A^j,$$

since λ is parallel on U_j, $1 \leqslant j \leqslant r$. Clearly $F_\lambda \equiv I \bmod \mathcal{M}$ whence $F_\lambda \in \mathbf{G}^0(P_A)$.

We will now prove that ι is an equivalence of groupoids. We note that ι is not an *isomorphism* of groupoids, it is not surjective on objects unlike the absolute case, [GM1], Proposition 6.6. We first prove a lemma.

LEMMA 2.6 *Let $F_j \in \mathbf{G}^0(P_A^j)$. Then there is an extension $\tilde{F}_j$ of F_j to P_A with $\tilde{F}_j \in \mathbf{G}^0(P_A)$, which is the identity outside $P_A|N(\overline{U}_j)$.*

Proof. We define $f_j \in \Gamma(U_j, \mathrm{Ad}\, P_A^j)$ by $F_j(p) = pf_j(p)$. By assumption $f_j = \exp \lambda_j$ for $\lambda_j \in \Gamma(U_j, \mathrm{ad}\, P_A^j) \otimes \mathcal{M}$. We let $\varphi(x)$ be a smooth function with $\varphi(x) = 1$ on U_j and $\varphi(x) \equiv 0$ on the complement of $N(\overline{U}_j)$ in M. Then define $\tilde{f}_j(p) = \exp \varphi(\pi(p))\lambda_j(p))$ for $p \in P_A|U_j$ and $f_j(p) = I$ otherwise. Then $\tilde{F}_j$ defined by $\tilde{F}_j(p) = p\tilde{f}_j(p)$ is the required extension. □

PROPOSITION 2.7 *The natural transformation ι is an equivalence of groupoids.*

Proof. Surjective on isomorphism classes. Suppose $\omega \in \mathrm{Obj}\, \mathcal{F}_A^r(\omega_0)$. Put $\omega_j = \omega | P_A^j$, $1 \leqslant j \leqslant r$. By assumption there exists $F_j \in \mathbf{G}^0(P_A)$, $1 \leqslant j \leqslant r$, such that $F_j^* \omega_j = \tilde{\omega}_0$. Let $\tilde{F}_j$ be an extension of F_j in $\mathbf{G}^0(P_A)$ which is the identity outside $P_A|N(\overline{U}_j)$, $1 \leqslant j \leqslant r$. Put $F = \tilde{F}_1 \circ \tilde{F}_2 \circ \cdots \circ \tilde{F}_r$. Then $F^*\omega | P_A^j = \tilde{\omega}_0$, $1 \leqslant j \leqslant r$, whence $F^*\omega = \iota(\eta)$ some $\eta \in \mathcal{B}^1(M, U; \mathrm{ad}\, P) \otimes \mathcal{M}$. But $\omega = (F^{-1})^* F^* \omega$.

Faithful. Clear.

Full. Suppose $\iota(\eta_1)$ and $\iota(\eta_2)$ are equivalent in $\mathcal{F}_A^r(\omega_0)$. Hence there exists $F \in \mathbf{G}^0(P_A)$ such that $F^*(\tilde{\omega}_0 + \eta_1) = \tilde{\omega}_0 + \eta_2$. We restrict this equation to U_j, $1 \leqslant j \leqslant r$, to deduce $(F^*\tilde{\omega}_0)|P_A^j = \tilde{\omega}_0 | P_A^j$, $1 \leqslant j \leqslant r$. Hence $F|P_A^j \in \mathrm{Aut}(\tilde{\omega}_0)$ and consequently $\lambda = \log f \in \mathcal{B}^0(M, U; \mathrm{ad}\, \rho_0) \otimes \mathcal{M}$ where $F(p) = pf(p)$. Hence $F = \iota(\exp(-\lambda))$. □

In what follows we will need to know the cohomology groups $H^\cdot(\mathcal{B}^\cdot(M, U; \mathrm{ad}\, P))$. We let $\mathcal{A}^\cdot(M, U; \mathrm{ad}\, P) \subset \mathcal{A}^\cdot(M, \mathrm{ad}\, P)$ denote the subalgebra of forms vanishing on $U = \bigcup_{j=1}^r U_j$. We have an exact sequence $0 \to \mathcal{A}^\cdot(M, U; \mathrm{ad}\, P) \to \mathcal{B}^\cdot(M, U; \mathrm{ad}\, P) \to \prod_{j=1}^m H^0(U_j, \mathrm{ad}\, P) \to 0$.

LEMMA 2.8 (i) $H^0(\mathcal{B}^{\cdot}(M,U;ad\,P)) = H^0(\mathcal{A}^{\cdot}(M,ad\,P))$.

(ii) $H^1(\mathcal{B}^{\cdot}(M,U;\mathrm{ad}\,P)) = \ker(H^1(M,\mathrm{ad}\,P) \to H^1(U,\mathrm{ad}\,P))$
$= im(H^1(M,U;\mathrm{ad}\,P) \to H^1(M,\mathrm{ad}\,P))$

(iii) $H^i(\mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P)) = H^i(M,U;\mathrm{ad}\,P), i \geqslant 2$.

Proof. Equations (i) and (iii) are obvious. To prove (ii) observe that the long exact sequence of cohomology associated to the above short exact sequence give

$$H^1(\mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P)) = coker(H^0(U,\mathrm{ad}\,P) \to H^1(M,U;\mathrm{ad}\,P)).$$

Then (ii) follows from the exact sequence of the pair (M,U). □

We now construct a controlling differential graded Lie algebra for the germ of the relative representation variety $\mathrm{Hom}(\Gamma, R; G)$ at ρ_0. We choose a base-point $x_0 \in M$ such that $x_0 \notin \bigcup_{j=1}^{r} N(\overline{U}_j)$ (we assume that the point $p \in P$ lies over x_0). We obtain an augmentation $\varepsilon : \mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P) \to g$ by evaluation at x_0. Clearly, ε is surjective and $\varepsilon|H^0(\mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P)$ is injective. We let $\mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P)_0$ be the kernel of the augmentation ε. We then have the main result of this section. The proof is analogous to that of [GM1], Theorem 6.8. We put $L^{\cdot} = \mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P)_0$.

THEOREM 2.9 $\mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P)_0$ *is a controlling differential graded Lie algebra for the relative deformations of* ρ_0*; that is, the analytic germ* $(\mathcal{X},\rho_0)$ *of* $\mathrm{Hom}(\Gamma, R; G)$ *at* ρ_0 *pro-represents the functor* $\mathrm{Iso}\,\mathcal{C}(L^{\cdot}, A)$.

Proof. Following [GM1], pg. 81, we define groupoids $(\mathcal{F}^r_A)'(\omega_0)$ and $(\mathcal{R}^r_A)'(\rho_0)$ as follows. The objects of $(\mathcal{F}^r_A)(\omega_0)$ coincide with those of $(\mathcal{F}^r_A)'(\omega_0)$ but the morphisms of $(\mathcal{F}^r_A)'(\omega_0)$ are defined to be those $F \in \mathrm{Aut}(P_A)$ satisfying $\varepsilon_p(F) = 1$. The objects of $(\mathcal{R}^r_A)(\rho_0)$ coincide with those of $(\mathcal{R}^r_A)'(\rho_0)$; however, the group of morphisms of $(\mathcal{R}^r_A)'(\rho_0)$ is defined to be the trivial group. By Proposition 2.4 $(\mathcal{R}^r_A)'(\rho_0)$ is pro-represented by $(\mathcal{X},\rho_0)$. It follows from Section 6.4 of [GM1] and Proposition 2.5, that hol induces on equivalence of groupoids $(\mathcal{F}^r_A)'(\omega_0) \to (\mathcal{R}^r_A)'(\rho_0)$. We claim that ι induces an equivalence of groupoids from $\mathcal{C}(L^{\cdot}, A)$ to $(\mathcal{F}^r_A)^{\cdot}(\omega_0)$. It is clear that ι is faithful and full. To see that ι is surjective on isomorphism classes we have only to examine the proof of Proposition 2.7 and observe that since $x_0 \notin \bigcup_{i=1}^{r} N(\overline{U}_j)$ the gauge transformation F constructed there satisfies $\varepsilon_p(F) = I$. The claim follows. We obtain natural isomorphisms of functors $\mathrm{Iso}\,\mathcal{C}(L^{\cdot}, A) \cong \mathrm{Iso}\,(\mathcal{F}^r_A)'(\omega_0) \cong \mathrm{Iso}\,(\mathcal{R}^r_A)'(\rho_0)$. □

REMARK. Let $\hat{\mathcal{O}}$ be the complete local ring of the above analytic germ. We have observed in the introduction that $\mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P)_0$ controls the relative deformation theory of ρ_0 if and only if $\hat{\mathcal{O}}$ is a hull for $\mathrm{Iso}\,\mathcal{C}(L^{\cdot}, A)$. Since $H^0(\mathcal{B}^{\cdot}(M, U;\mathrm{ad}\,P)_0) = \{0\}$ the functor $\mathrm{Iso}\,\mathcal{C}(L^{\cdot}, A)$ is pro-representable. This is why we can prove the stronger statement that $\hat{\mathcal{O}}$ pro-represents $\mathrm{Iso}\,\mathcal{C}(L^{\cdot}, A)$.

We leave the proof of the next lemma to the reader. We define $Z^1(\Gamma, R; \mathcal{G})$ to be the subspace of the 1-cocycles $Z^1(\Gamma, \mathcal{G})$ whose restrictions to each Γ_j are *exact.*

LEMMA 2.10 (i) $H^0(\mathcal{B}^\cdot(M, U; \operatorname{ad} P)_0) = \{0\}$.
(ii) $H^1(\mathcal{B}^\cdot(M, U; \operatorname{ad} P)_0) = Z^1(\Gamma, R; \mathcal{G})$.
(iii) $H^i(\mathcal{B}^\cdot(M, U; \operatorname{ad} P)_0) = H^i(M, U; \operatorname{ad} P), i \geqslant 2$.

Finally, we will need to extend the above result to the case in which Γ is the fundamental group of the orbifold obtained by quotienting M as above by a finite group H of diffeomorphisms of M such that each U_j is carried into itself by H. We let Λ be the orbifold fundamental group of M/H so we have an extension $\Gamma \to \Lambda \to H$. We let Λ_j be the orbifold fundamental group of U_j/H. We assume ρ_0 extends to Λ and denote this extension by ρ'. We let $R' = \{\Lambda_1, \ldots, \Lambda_r\}$ and consider $\operatorname{Hom}(\Lambda, R'; G)$, the space of relative deformations of ρ_0'. The proof of the following theorem is analogous to that of [GM1], Theorem 9.3.

THEOREM 2.11 *The subalgebra $\mathcal{B}^\cdot(M, U; \operatorname{ad} P)_0^H$ of H-invariants is a controlling differential graded Lie algebra for the relative deformations of ρ_0'.*

Proof. We indicate the modifications required in the proof of [GM1], Theorem 9.3, using the notation of that proof. Define $\tilde{U}_j \subset M \times K, 1 \leqslant j \leqslant r$, by $\tilde{U}_j = U_j \times K$, whence the diagonal action of H on $\tilde{U}_j$ is free and $\pi_1(\tilde{U}_j/H) = \Lambda_j$. We put $U_j' = \tilde{U}_j/H$ and $U' = U_{j=1}^r U_j'$. We define $(\mathcal{B}^\cdot)' = \mathcal{B}^\cdot(M \times_H K, U'; \operatorname{ad} P')$ using the domains U_j' in $M \times_H K$ and P' the principal G-bundle over $M \times_H K$ associated to $\rho_0' \colon \Lambda \to G$. Note that $\pi_1(M \times_H K) \cong \Lambda$. We put $x_0' = [x_0, 1]$ (the equivalence class of $(x_0, 1)$) and define an augmentation of $(\mathcal{B}^\cdot)'$ by evaluation at x_0'. By Theorem 2.9, the augmentation ideal of $(\mathcal{B}^\cdot)'$ is a controlling differential graded Lie algebra for the relative deformations of ρ_0'. The projection map $\pi : M \times K \to M$ induces a homomorphism of augmented differential graded Lie algebras $\mathcal{B}^\cdot(M, U; \operatorname{ad} P)^H$ into $(\mathcal{B}^\cdot)'$. By [GM1], Theorem 2.4, it suffices to prove that π induces isomorphisms on H^0 and H^1 and an injection on H^2. Since the functor of H-invariants is exact, it suffices in turn to prove that the natural map from $\mathcal{B}^\cdot(M, U; \operatorname{ad} P)$ into $\mathcal{B}^\cdot(M \times K, \tilde{U}; \operatorname{ad} P)$ has the same property (here $\tilde{U}$ is the union of the domains $\tilde{U}_j, 1 \leqslant j \leqslant r$). We may replace the second algebra by the quasi-isomorphic subalgebra $\mathcal{B}^\cdot(M, U; \operatorname{ad} P) \otimes \mathcal{A}^\cdot(K)$ and the theorem follows from the Kunneth formula. □

In Chapter 5 we will need to compare the complete local ring $\mathcal{R}_0$ associated to the augmentation ideal $\mathcal{B}^0(M, U, \operatorname{ad} P)_0^H$ to the complete local ring $\mathcal{R}$ associated to $\mathcal{B}^\cdot(M, U; \operatorname{ad} P)^H$. We do this in the general context of augmented differential graded Lie algebras. The following theorem generalizes Theorem 3.5 of [GM1].

THEOREM 2.12 *Suppose $(L^\cdot, d, \varepsilon)$ is a $\mathcal{G}$-augmented differential graded Lie algebra. Suppose that the augmentation $\varepsilon : L^0 \to \mathcal{G}$ is surjective and the restriction of ε to $H^0(L) \subset L^0$ is injective. Let $L_0 = \ker \varepsilon$ be the augmentation ideal and S be the complete local ring associated to the smooth germ $(\mathcal{G}/\varepsilon(H^0(L)), 0)$. Then R_{L_0} is isomorphic to the completed tensor product $R_L \hat{\otimes} S$.*

We apply the theorem to the case $L^{\cdot} = \mathcal{B}^{\cdot}(M, U; \mathrm{ad}\, P)^H$ to obtain the following corollary.

COROLLARY 2.13 *Let $\mathcal{Z}$ be the subalgebra of* $\mathrm{Ad}\,\rho_0$ *invariants in $\mathcal{G}$. Let $\mathcal{S}$ be the complete local ring of the smooth germ $(\mathcal{G}/\mathcal{Z}, 0)$. Then*

$$\mathcal{R}_0 \cong \mathcal{R}\hat{\otimes}\mathcal{S}.$$

The theorem will be a consequence of the next two lemmas. The next lemma is Lemma 1.7 of [BM]. It is an immediate consequence of the obstruction theory of [GM1], Section 2.6.

LEMMA 2.14 *Suppose $\phi\colon L \to \bar{L}$ is a homomorphism of differential graded Lie algebras such that $H^1(\phi)$ is surjective and $H^2(\phi)$ is injective. Then the induced natural transformation $\phi\colon \mathrm{Iso}\,\mathcal{C}(L,\cdot) \to \mathrm{Iso}\,\mathcal{C}(\bar{L},\cdot)$ is smooth.*

We owe the next lemma to Mike Schlessinger. We will adopt the following notation of [Sc]. Suppose R is a complete local **k**-algebra. Then h_R will denote the functor on the category of Artin local **k**-algebras given by $h_R(A) = \mathrm{Hom}(R, A)$ where Hom denotes **k**-algebra homomorphisms. Also if F is a functor on the category of Artin local **k**-algebras, we extend F to the category of complete local **k**-algebras by $F(R) = \mathrm{proj\,lim}\, F(R/\mathcal{M}^n)$ where $\mathcal{M}$ is the maximal of R. We note that the extension of h_R is represented by R.

LEMMA 2.15 *Suppose F and G are functors on the category of Artin local* **k**-*algebras and $\eta\colon F \to G$ is smooth. Suppose R is a hull for F and S is a hull for G. Then η induces a natural transformation $\tilde{\eta}\colon h_R \to h_S$ and $\tilde{\eta}$ is smooth.*

Proof. We have a diagram

$$\begin{tikzcd} h_R \arrow[r, "\alpha"] & F \arrow[d, "\eta"] \\ h_R \arrow[r, "\beta"'] & G \end{tikzcd}$$

We apply the above diagram to R.

Let $I \in h_R(R)$ be the identity map and let $g \in G(R)$ be given by $g = \eta(R) \circ \alpha(R)(I)$. We claim that $\beta(R)$ is onto. Indeed since $h_S \to G$ is smooth $\mathrm{proj\,lim}\, h_S(R/\mathcal{M}^n) \to \mathrm{proj\,lim}\, G(R/\mathcal{M}^n)$ is onto. But $h_S(R) \to \mathrm{proj\,lim}\, h_S(R/\mathcal{M}^n)$ is an isomorphism. The claim follows.

Choose $f \in h_S(R)$ satisfying $\beta(R)f = g$. Then $f : S \to R$ is a **k**-algebra homomorphism and induces a natural transformation $f^* : h_R \to h_S$. We obtain a diagram

$$\begin{tikzcd} h_R \arrow[r, "\alpha"] \arrow[d, "f^*"'] & F \arrow[d, "\eta"] \\ h_S \arrow[r, "\beta"'] & G \end{tikzcd}$$

of functors on the category of Artin local **k**-algebras. We claim this diagram is commutative.

To see this let A be an Artin local **k**-algebra. We obtain a square by applying the above diagram to A. Let $\phi \in h_R(A)$. We want to show that the results of following ϕ around the two pairs of consecutive edges of the square coincide. To see this we form a second square by applying the above diagram to R. We use ϕ to map this second square to the first square. We place the second square over the first square in space and obtain a cubical diagram where the vertical arrows are induced by ϕ. Since α, β, η and f^* are natural transformations all the vertical squares commute. But ϕ is the image of I from the vertex $h_R(R)$ and by construction the results of following I around the two pairs of consecutive edges of the top square coincide. It is then immediate that the results of following ϕ around the two pairs of consecutive edges of the bottom square coincide.

We put $\tilde{\eta} = f^*$. It remains to prove that $\tilde{\eta}$ is smooth. Suppose we are given an extension $I \to A \to \bar{A}$ of Artin local **k**-algebras with $I\mathcal{M} = 0$. Hence $\mathcal{M}$ is the maximal ideal of A. We wish to prove that the induced map

$$w\colon h_R(A) \to h_R(\bar{A}) \times_{h_S(\bar{A})} h_S(A),$$

is onto. To this end we consider the diagram

$$\begin{tikzcd} & h_R(\bar{A}) \times_{G(\bar{A})} G(A) \\ h_R(A) \arrow[ur, "a"] \arrow[dr, "c"'] & \\ & h_R(\bar{A}) \times_{h_S(\bar{A})} h_S(A) \arrow[uu, "b"'] \end{tikzcd}$$

It is immediate that a and b are surjective. Let $(x,z) \in h_R(\bar{A}) \times_{h_S(\bar{A})} h_S(A)$. Let (x,w) be the image of (x,z) in $h_R(\bar{A}) \times_{G(\bar{A})}] B(A)$. Let $\tilde{x} \in h_R(A)$ satisfy $a(\tilde{x}) = (x,w)$. Let $c(\tilde{x}) = (x,\tilde{z}) \in h_R(\bar{A}) \times_{h_S(\bar{A})} h_S(A)$. Now the image of $\tilde{z}$ in $h_S(\bar{A})$, coincides with the image of x hence is equal to the image of z. Hence z and $\tilde{z}$ both lie over the same point of $h_S(\bar{A})$. By [Sc], Remarks, pg. 213, there exists an element i of the tangent space $t_{h_S} \otimes I$ whose action carries $\tilde{z}$ to z. Since the induced map on tangent spaces $t_{h_R} \to t_{h_S}$ is onto there exists $\tilde{i} \in t_{h_R} \otimes I$ whose image is $i \in t_{h_S} \otimes I$, then

$$c(\tilde{x} + \tilde{i}) = (x, \tilde{z} + i) = (x,z). \qquad \square$$

Theorem 2.12 now follows. Indeed by Lemma 2.14, the induced natural transformation $\operatorname{Iso} \mathcal{C}(L_0, \cdot) \to \operatorname{Iso} \mathcal{C}(L, \cdot)$ is smooth. Hence by Lemma 2.15, the induced natural transformation $h_{R_{L_0}} \to h_{R_L}$ is smooth. Hence by [Sc], Proposition 2.5(i), R_{L_0} is a power series ring over R_L. A calculation of tangent spaces completes the proof.

Let $\tilde{V}$ be a flat bundle over M acted on by H. We will abbreviate $\mathcal{B}^{\cdot}(M,U;\tilde{V})^H$ to $\mathcal{C}^{\cdot}(M,\tilde{V})$ henceforth. We conclude this chapter with two duality results we will need later.

LEMMA 2.16 *Assume M is compact orientable of dimension n and let ε be the character of H obtained by the action of H on the orientation of M. Let $\tilde{V}$ be the flat bundle over M corresponding to the $\pi_1(M)$-module V. Then $H^n(\mathcal{C}^{\cdot}(M,\tilde{V}))$ is dually paired with $H^0(\Lambda, V^* \otimes \varepsilon)$.*

Proof. The lemma follows immediately from the fact that the integral of the wedge product gives a perfect pairing

$$H^n(M,U;\tilde{V}) \otimes H^0(M,\tilde{V}^*) \to \mathbb{R}.$$

But the relative fundamental class of M transforms by ε under H. $\square$

We will also need duality for the first cohomology in the case that M is the complement of n points, $p_1, p_2, \ldots, p_n$ in a compact surface $\overline{M}$ and U_i is the complement of p_i in a coordinate disk $D_i \subset \overline{M}$ with center p_i, $1 \leqslant i \leqslant n$. We assume $\tilde{V}$ is a flat bundle over M such that $\tilde{V}|U_i$ does not admit a parallel section. We assume further that there is a nonsingular parallel bilinear form b on the bundle $\tilde{V}$. We combine the bilinear form b on $\tilde{V}$ with the wedge product of forms and integration to obtain a real-valued pairing B on $H^1(\mathcal{B}^{\cdot}(M,U;\tilde{V}))$.

LEMMA 2.17 *The pairing B is non-singular.*

Proof. Poincaré duality.

3. Deformations of linkages in constant curvature spaces and relative deformations of reflection groups

In this chapter we describe the deformation theory of a linkage $\Lambda = (u_1, u_2, \dots, u_n)$ in $X = S^m, \mathbb{E}^m$ or $\mathbb{H}^m$ in terms of the relative deformation theory of a representation of Φ_n, the free product of n copies of $\mathbb{Z}/2$, into G, the isometry group of X.

A point u in X determines an element of order 2 in G, the Cartan involution at u. For the cases $X = S^m$ and $X = \mathbb{H}^m$ the involution s_u is the restriction to X of the element $s_u \in \mathrm{GL}_{m+1}(\mathbb{R})$ given by

$$s_u(v) = -\left(v - 2\frac{(u,v)}{(u,u)}u\right), v \in V.$$

In the Euclidean case we replace the previous formula by

$$s_u(v) = -(v - 2\alpha(v)u), v \in V.$$

The reader will observe that since $\alpha(u) = 1$ we have $s_u(u) = u$. Also $s_u(w) = -w$, all $w \in W$.

We note that in the Euclidean case we have $V = U + W$ and in the other two cases $V = U + U^\perp$ where U is the line through u. Also we note that for the Euclidean and hyperbolic cases the fixed-point set of s_u is $\{u\}$ whereas for the spherical case it consists of $\{\pm u\}$. Finally, we note that $gs_ug^{-1} = s_{gu}, g \in G$. We now describe how to associate a representation $\rho\colon \Phi_n \to G$ to Λ.

We let $\tau_1, \tau_2, \dots, \tau_n$ be the generators of the $\mathbb{Z}/2$ factors of Φ_n and $D_2^{i,j} = \langle \tau_i, \tau_j \rangle$ be the (infinite dihedral) subgroup of Φ_n generated by $\{\tau_i, \tau_j\}$. An n-tuple of points $u_1, u_2, \dots, u_n$ determines an n-tuple $s_{u_1}, s_{u_2}, \dots, s_{u_n}$ in G which we identify with the homomorphism $\rho\colon \Phi_n \to G$ satisfying $\rho(\tau_i) = s_{u_i}, 1 \leqslant i \leqslant n$. We define $\Psi\colon X^n \to \mathrm{Hom}(\Phi_n, G)$ by $\Psi(u_1, u_2, \dots, u_n) = \rho$ where ρ is as above. It is immediate from the above formulas for s_u that the map Ψ is a regular map of real algebraic varieties. Moreover, it is easy to see that the differential of Ψ at the point $(u_1, u_2, \dots, u_n)$ is invertible. Thus the map Ψ induces an analytic equivalence from a neighborhood of u to a neighborhood of ρ. Finally, the condition $f(u_i, u_j) = f(v_i, v_j)$ is easily seen to be equivalent to the condition that the dihedral groups $\langle s_{u_i}, s_{u_j} \rangle$ and $\langle s_{v_i}, s_{v_j} \rangle$ are conjugate in G. Thus if we let $R = \{D_2^{i,j} : (i,j) \in \mathcal{E}\}$ the following theorem is clear.

THEOREM 3.1 *Let Λ_0 be an admissible linkage in X and put $\rho_0 = \Psi(\Lambda_0)$. Then the map Ψ induces a analytic equivalence of the germ of $C(\Lambda_0)$ at Λ_0 and the germ of* $\mathrm{Hom}(\Phi_n, R; G)$ *at ρ_0 where both germs are given their reduced structures.*

REMARK. The above argument generalizes immediately to an admissible linkage Λ in any two-point homogenous space.

However, the deformation spaces of linkages are frequently not reduced and this failure to be reduced leads to interesting geometric consequences, e.g. nth order deformations that cannot be lifted to $(n+1)$-st order deformations, see [Co] for examples. For this reason we will prove the following more precise version of Theorem 3.1. The proof will be a consequence of the next five lemmas.

THEOREM 3.2 *The map Ψ above is the map of reduced germs underlying an isomorphism*

$$\Psi\colon (C(\Lambda_0), \Lambda_0) \to (\mathrm{Hom}(\Phi_n, R; G), \rho_0),$$

of analytic germs.

COROLLARY 3.3 *In case $X = \mathbb{E}^m$ or $\mathbb{H}^n$ the induced map $\Psi : C(\Lambda_0) \to \mathrm{Hom}(\Phi, R; G)$ is an isomorphism of analytic varieties.*

By [GM1], Theorem 3.1, an isomorphism of germs is equivalent to compatible isomorphisms of the associated sets of points over Artin local $\mathbb{R}$-algebras. We recall that an Artin local $\mathbb{R}$-algebra A is a Noetherian $\mathbb{R}$-algebra such that the maximal ideal $\mathcal{M}$ of A is nilpotent. Then to prove the above theorem it is necessary and sufficient to extend Ψ to a family of isomorphisms Ψ_A (one for each A) which are compatible with homomorphisms of the A's.

We now construct Ψ_A. We define $V(A) = V \otimes_{\mathbb{R}} A$ and $W(A) = W \otimes_{\mathbb{R}} A$. In the spherical (resp. hyperbolic) cases we define $X(A)$ to be the set of $u \in V(A)$ such that $(u,u) = 1$, (resp. $(u,u) = -1$). In the Euclidean case we define $X(A) = \{u \in V(A) : \alpha(u) = 1\}$. Here we extend α to $V \otimes A$ as $\alpha \otimes 1$ and $(\,,)$ to $V \otimes A$ by the formula

$$(u_1 \otimes t_1, u_2 \otimes t_2) = (u_1, u_2) \otimes t_1 t_2.$$

We recall $G(A)$ is the subset of the $m+1$ by $m+1$ matrices over A satisfying the defining equations for G and define $C(\Lambda_0)_{\Lambda_0}(A)$ by $C(\Lambda_0)(A) = \{(\tilde{u}_1, \dots, \tilde{u}_n) \in X(A)^n : (\tilde{u}_i, \tilde{u}_j) = c_{ij}, \ \text{all} \ (i,j) \in \mathcal{E} \ \text{and} \ \tilde{u}_i \equiv u_i \mod \mathcal{M}, 1 \leqslant i \leqslant n\}$.

The following lemma is clear.

LEMMA 3.4 *The functor (of A) $C(\Lambda_0)_{\Lambda_0}(A)$ is pro-represented by the germ $(C(\Lambda_0), \Lambda_0)$.*

In order to construct the map Ψ_A on A-points we will need to classify pairs of elements of order 2 in $G(A)$, which are deformations of symmetries s_u in G, up to conjugacy in $G(A)$.

We begin with the following elementary result concerning the equation $x^2 = a$ in an Artin local $\mathbf{k}$-algebra – in fact this is a special case of Hensel's Lemma for complete local rings, [Bo], III, Section 4.5.

LEMMA 3.5 *Suppose a is a unit. Then the above equation has a solution if and only if a is a square modulo $\mathcal{M}$. If a is a square modulo $\mathcal{M}$ there are exactly two solutions which are negatives of each other.*

Proof. It is evident that if $x^2 = a$ has a solution, then a is a square mod $\mathcal{M}$. Suppose then that a is a square modulo $\mathcal{M}$. Let $y \in \mathbf{k}$ satisfy $a \equiv y^2 \bmod \mathcal{M}$. We claim there exists a unique solution x to $x^2 = a$ such that $x \equiv y \bmod \mathcal{M}$. By Artinian induction, [GM1], Section 2.5, we may assume there is an ideal $\mathcal{I} \subset \mathcal{M}$ such that $\mathcal{I}\mathcal{M} = 0$ and such that there is a unique solution $\overline{x}$ to $x^2 = a$ in $\overline{A} = A/\mathcal{I}$ with the property $\overline{x} \equiv y \bmod \mathcal{M}$. We choose $x' \in A$ such that $x' \equiv \overline{x} \bmod \mathcal{I}$. We may write $a = z + m$ and $x' = y + m'$ where $z, y \in \mathbf{k} - \{0\}$ and $m, m' \in \mathcal{M}$. Then there exists $i_0 \in \mathcal{I}$ such that $(y + m')^2 = z + m + i_0$. Hence $x = y + m' - \frac{1}{2}y^{-1}i_0$ solves $x^2 = a$. If $\tilde{x}$ is any other solution, then by induction $\tilde{x} = x + i$ and we obtain $2xi = 0$. Since x is a unit, $i = 0$ and the lemma follows. □

We can now begin our study of elements of order 2 of $G(A)$. Note that A is local ring in which 2 is a unit. We observe that $G(A)$ acts transitively on $X(A)$. For $X = \mathbb{E}^n$ this is obvious. For $X = S^n$ or $X = \mathbb{H}^n$ see [Sch], Ch. 1, Lemma 6.5. For $u \in X(A)$ we define s_u by the above formulas. Then if U denotes the line through u, we have $V = U + U^{\perp}$, [Sch], Ch. 1, Lemma 6.1 in the spherical and hyperbolic cases. In the Euclidean case we have $V = U + W$ by Nakayama's Lemma, [AM], pg. 21, clearly $U \cap W = \{0\}$. From the defining formula for s_u, we see that s_u coincides with I on U and $-I$ on $U^{\perp}$ (or W). We claim that the fixed-point set of $s_u|X(A)$ consists of $\{\pm u\}$ if $X = S^m$ or $\{u\}$ if $X = \mathbb{H}^m$ or $X = \mathbb{E}^m$. Indeed any fixed-point lies in U and consequently is of the form xu with $x \in A$. Since $xu \in X(A)$ we have $x^2 = 1$ in case $X = S^m$ and $\mathbb{H}^m$ and $x = 1$ in case $X = \mathbb{E}^m$. Hence $x = \pm 1$ by Lemma 3.5. In case $X = \mathbb{H}^m$ we have $x \equiv 1$ mod$\mathcal{M}$ whence $x = 1$ in case $X = \mathbb{H}^m$ or $\mathbb{E}^m$ and the claim follows.

Now let $X(A)^n_{\Lambda_0}$ denote the set of n-tuples $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n) \in X(A)^n$ such that $\tilde{u}_i \equiv u_i$, mod $\mathcal{M}, 1 \leqslant i \leqslant n$, and $\mathrm{Hom}(\Phi_n, G(A))_{\rho_0}$ denote the subset of representations satisfying $\rho \equiv \rho_0$ mod$\mathcal{M}$. We then define $\Psi'_A \colon X(A)^n_{\Lambda_0} \to \mathrm{Hom}(\Phi_n, G(A))_{\rho_0}$ by

$$\Psi'_A(\tilde{u}_1, \dots, \tilde{u}_n) = \rho \quad \text{where} \quad \rho(\tau_i) = s_{\tilde{u}_i}, 1 \leqslant i \leqslant n.$$

LEMMA 3.6 *Ψ'_A is invertible.*

Proof. We first show that Ψ'_A is onto. It suffices to prove that if $s \in G(A)$ has order 2 and satisfies $s \equiv s_u$ mod$\mathcal{M}$ for some $u \in X$ then there exists $\tilde{u} \in X(A)$ with $\tilde{u} \equiv u \bmod \mathcal{M}$ such that $s = s_{\tilde{u}}$. To prove this let V^+ and V^- be the $+1$ and -1 eigenspaces in $V(A)$ of s. Then V^+ and V^- are orthogonal for $(\,,)$ in the spherical and hyperbolic cases. In the Euclidean case we have $\alpha(v) = 0$, all $v \in V^-$ (since $\alpha(v) = \alpha(sv) = -\alpha(v)$). In all three cases $V^+ \cap V^- = \{0\}$ and $V(A) = V^+ + V^-$ by Nakayama's Lemma. Also, by Nakayama's Lemma, V^+ is a free A-module of rank 1 generated by any $\tilde{u} \in V^+$ such that $\tilde{u} \equiv u \bmod \mathcal{M}$.

Choose such a $\tilde{u}$. Then in the spherical and hyperbolic cases $a = (\tilde{u}, \tilde{u})$ satisfies $a \equiv 1 \bmod \mathcal{M}$ and in the Euclidean case $a = \alpha(\tilde{u}) \equiv 1 \bmod \mathcal{M}$. In the latter case we replace $\tilde{u}$ by $a^{-1}\tilde{u}$ to get $\tilde{u} \in X(A)$. In the other two cases we apply Lemma 3.5 to find $x \in A$ such that $x^2 = a$. We then replace $\tilde{u}$ by $x^{-1}\tilde{u}$ to get $\tilde{u} \in X(A)$. In the spherical and hyperbolic cases it is clear that $s = s_{\tilde{u}}$. In the Euclidean case $s = s_{\tilde{u}}$ since $s_{\tilde{u}}(v) = -v$ for $v \in V^-$ because $\alpha(v) = 0$.

We next prove that Ψ_A is injective. Indeed suppose ρ is in the image of Ψ_A with $\rho(\tau_i) = s_{\tilde{u}_i}, 1 \leqslant i \leqslant n$. The vector $\tilde{u}_i$ is uniquely determined in the Euclidean case and determined up to multiplication by ± 1 in the spherical and hyperbolic cases. But since $\tilde{u}_i \equiv u_i \bmod \mathcal{M}$, the vector $\tilde{u}_i$ is uniquely determined in all cases. □

We now take account of the conditions $f(u_i, u_j) = c_{ij}, \{i, j\} \in \mathcal{E}$. We first need the Witt extension theorem for isometries.

LEMMA 3.7 *Suppose V is a real vector space equipped with a non-singular bilinear form $(\,,\,)$. Suppose $U(A) \subset V(A)$ is a submodule such that the restriction of $(\,,\,)$ to $V(A)$ is nonsingular. Suppose $h\colon U(A) \to V(A)$ is an isometric embedding. Then h extends to an isometry $h\colon V(A) \to V(A)$.*

Proof. Let $W(A)$ be the image of h. Then $V = U(A) \oplus U(A)^{\perp} = W(A) \oplus W(A)^{\perp}$. By the Witt Cancellation Theorem [Sch], Ch. 1, Lemma 6.6, there is an isometry $k : U(A)^{\perp} \to W(A)^{\perp}$. Hence $h \oplus k$ is an isometry of V that extends h. □

COROLLARY 3.8 *Let $V, (\,,\,)$ be as above. Suppose $u_1, u_2, \ldots, u_k$ and $v_1, v_2, \ldots, v_k$ are elements of $V(A)$ such that $(u_i, u_j) = (v_i, v_j), 1 \leqslant i, j \leqslant k$ and the k by k matrix $((u_i, u_j)) = ((v_i, v_j))$ with entries in A is invertible. Then there exists an element $g \in G(A)$ such that $gu_i = v_i, 1 \leqslant k \leqslant k$.*

Proof. Use the above lemma to extend the isometry sending u_i to $v_i, 1 \leqslant i \leqslant k$ to all of $V(A)$. □

LEMMA 3.9 *Suppose u_1, u_2 and v_1, v_2 are in $X(A)^2$ and assume further that $u_i \equiv v_i \bmod \mathcal{M}, i = 1, 2$. Assume that the inner products (u_1, u_2) and (v_1, v_2) are not congruent to $\pm 1 \bmod \mathcal{M}$ in the spherical and hyperbolic cases. Then $f(u_1, u_2) = f(v_1, v_2)$ if and only if the dihedral groups $\langle s_{u_1}, s_{u_2} \rangle$ and $\langle s_{v_1}, s_{v_2} \rangle$ are conjugate in G_A^0.*

Proof. We first treat the spherical and hyperbolic cases. Assume first $(u_1, u_2) = (v_1, v_2)$. If $(u_1, u_2) = (v_1, v_2) \neq \pm 1 \bmod \mathcal{M}$ then the 2 by 2 matrices $((u_i, u_j))$ and $((v_i, v_j))$ are equal and invertible. Hence there exists $g \in G(A)$ with $gu_1 = v_1, gu_2 = v_2$ by Corollary 3.8, and $\langle s_{u_1}, s_{u_2} \rangle$ and $\langle s_{v_1}, s_{v_2} \rangle$ are conjugate. Let $U = \operatorname{span}\{u_1, u_2\}$. Then $V(A) = U + U^{\perp}$. In what follows we use a bar over an object to denote its reduction modulo $\mathcal{M}$. We have $\bar{g}|\bar{U} = I$. Let $\bar{h} = \bar{g}|\bar{U}^{\perp}$ and h be any element of the special orthogonal group of $U^{\perp}, (\,,\,)$ which reduces to $\bar{h}$ modulo $\mathcal{M}$. Then $goh^{-1}(u_1) = v_1, goh^{-1}(u_2) = v_2$ and goh^{-1} conjugates $\langle s_{u_1}, s_{u_2} \rangle$ to $\langle s_{u_1}, s_{u_2} \rangle$. But $goh^{-1} \in G_A^0$.

Conversely if $\langle s_{u_1}, s_{u_2}\rangle$ are conjugate by $g \in G_A^0$ then $gu_1 = \pm v_1$ and $gu_2 = \pm v_2$. Reducing modulo $\mathcal{M}$ (and using $g \equiv I \bmod \mathcal{M}$) we see both signs are positive whence $f(u_1, u_2) = f(v_1, v_2)$.

We now treat the Euclidean case. We first assume $f(u_1, u_2) = f(v_1, v_2)$. By applying congruent translations we may assume $u_1 = v_1 = e_{m+1}$. Then the vectors $u_2 - e_{m+1}$ and $v_2 - e_{m+1}$ in $W(A)$ have the same inner product. Hence by Corollary 3.8 there exists an element g of $\mathrm{SO}(W(A))$ which carries $u_2 - e_{m+1}$ to $v_2 - e_{m+1}$. Then the image of g in $G(A)$ carries u_2 to v_2 and consequently $\langle s_{u_1}, s_{u_2}\rangle$ and $\langle s_{v_1}, s_{v_2}\rangle$ are conjugate. By modifying g on the orthogonal complement to $u_2 - e_{m+1}$ as above we may assume $g \in G_A^0$. Conversely, since s_u determines u, if $\langle s_{u_1}, s_{u_2}\rangle$ is conjugate to $\langle s_{v_1}, s_{v_2}\rangle$ by g then $gu_1 = v_1, gu_2 = v_2$. $\square$

Thus Ψ'_A carries $\mathcal{C}(\Lambda_0)_{\Lambda_0}(A)$ onto $\mathrm{Obj}\,\mathcal{R}_A^r(\rho_0)$ and Theorem 3.2 is an immediate consequence of Lemma 3.4 and Proposition 2.4.

REMARK. Since $\Lambda = (u_1, u_2, \dots, u_n)$ is admissible $(u_i, u_j) \neq \pm 1$ in the spherical and hyperbolic cases.

4. Deformations of linkages and local systems over Schottky quotients of B^3.

Let Λ be a linkage with n vertices. Let Y be the graph underlying Λ. Since the complete graph on n-vertices embeds totally-geodesically into S^n, we have an induced totally-geodesic embedding of Y into S^n. We do not assume it is an isometry. We identify Y with its image in S^n and choose disjoint spheres around the vertices of Y. Let $\tau_1, \tau_2, \dots, \tau_n$ be the reflections in these spheres. We may extend $\tau_1, \tau_2, \dots, \tau_n$ to reflections in the totally-geodesic hyperplanes in $\mathbb{H}^{n+1}$ bounded by the spheres.

The group generated by $\tau_1, \tau_2, \dots, \tau_n$ is a discrete subgroup of $\mathrm{SO}(n+1, 1)$ which we may identify with Φ_n. The group Φ_n acts properly discontinuously on $\mathbb{H}^{n+1} \cup \Omega$ where Ω is the complement of the limit set of Ψ_n in S^n. The hyperbolic quotient $W = \mathbb{H}^{n+1} \cup \Omega/\Phi_n$ is an orbifold which may be visualized as follows. Each of the reflector hyperplanes H_i bounds a half $n+1$-ball which intersects S^n in an n-ball B_i bounded by the corresponding reflector sphere $\sum_i$. Remove each of the open half-balls from $\mathbb{H}^{n+1}$. We obtain an orbifold with boundary which is isomorphic to the closed $(n+1)$-ball with n reflector n-balls in the boundary S^n. The orbifold boundary ∂W of W (which will be our primary concern here) is the orbifold quotient of S^n by the group Φ_n. It has underlying topological space the complement of n n-balls (bounded by $\sum_1, \dots, \sum_n$) in S^n. Thus as a topological space it is a manifold with boundary components $\sum_1, \dots, \sum_n$. Each of the spheres $\sum_i$ corresponds to a vertex of the graph Y. If the vertices corresponding to $\sum_i$ and $\sum_j$ are joined by an edge of Y, then the edge is realized by a circular arc β_{ij} orthogonal to $\sum_i$ and $\sum_j$.

In case the graph Y is a planar graph with n vertices, we can obtain a 'more efficient' realization of Λ as follows. By Andreev's Theorem [Th], Ch. 13, Corollary 13.6.2, we may find a planar (hence spherical) circle packing with nerve equal to Y. By shrinking the circles a little we may realize Y as above with S^n replaced by S^2 – so $\sum_1, \sum_2, \ldots, \sum_n$ are reflector circles.

We will need the following algebraic results concerning the group Φ_n. Let $\varepsilon : \Phi_n \to \{\pm 1\}$ be the homomorphism that takes value -1 on each generator $\tau_i, 1 \leqslant i \leqslant n$. Let Γ_n denote the kernel of ε. Thus Γ_n is the subgroup of Φ_n consisting of the words of even length in $\tau_1, \tau_2, \ldots \tau_n$. The following lemma will be very useful to us.

LEMMA 4.1 *The group Γ_n is isomorphic to the fundamental group of the complement of n points in S^2.*

Proof. The group Γ_n is generated by $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ where $\gamma_1 = \tau_1\tau_2, \gamma_2 = \tau_2\tau_3, \ldots, \gamma_n = \tau_n\tau_1$. It is immediate that the only relation among the above generators is the obvious one $\gamma_1\gamma_2 \ldots \gamma_n = 1$. □

We observe that the extension $\Gamma_n \to \Phi_n \to \mathbb{Z}/2$ is split but not canonically split. We choose the splitting obtained by sending the generator τ of $\mathbb{Z}/2$ to τ_1. We then obtain a semi-direct product decomposition

$$\Psi_n = \frac{\mathbb{Z}}{2} \propto \Gamma_n,$$

where τ acts as follows

$$\tau\gamma_k\tau = (\gamma_1 \cdots \gamma_{k-1})\gamma_k^{-1}(\gamma_1 \cdots \gamma_{k-1})^{-1}.$$

We now return to the orbifold ∂W. We recall that Ω denotes the domain of discontinuity for Φ_n operating on S^n. We let N denote the quotient Ω/Γ_n. To visualize N note that a fundamental domain for Γ_n in Ω is obtained as the union of a fundamental domain for Φ_n and its image under τ. Thus a fundamental domain for Γ_n is given by the exteriors of the $2n$ spheres $\sum_1, \tau\sum_1, \ldots, \sum_n, \tau\sum_n$. The spheres $\sum_i$ and $\tau\sum_i$ are identified with the result that N is the connected sum of n copies of $S^1 \times S^{n-1}$. The union of each are $\beta_{ij}, \{i,j\} \in \mathcal{E}$ with its image $\tau\beta_{ij}$ is a round circle w_{ij}. The various w_{ij}'s are disjoint.

We can now construct a controlling differential graded Lie algebra of bundle-valued forms on N. Indeed let $\rho\colon \Gamma_n \to G$ be the representation associated to Λ by Chapter 3. Since $\Gamma_n \cong \pi_1(N)$ we obtain a flat bundle ad P on N. Choose disjoint $\mathbb{Z}/2$-invariant neighborhoods $U_{ij}, \{i,j\} \in \mathcal{E}$, of the w_{ij}'s in N. Then $\pi_1(U_{ij})$ is generated by w_{ij} and we may apply the theory of Chapter 2 and Chapter 3 with $U = \bigcup U_{ij}$ to obtain the following theorem.

THEOREM 4.2 *Let Λ be a linkage whose underlying graph Y has n vertices and e edges. Let ρ_0 be the representation of Φ_n associated to Λ. Then there is a controlling* *differential graded Lie algebra of the form* $\mathcal{B}^{\cdot}(N, U; \mathrm{ad}\, P)_0^{\mathbb{Z}/2}$ *for the deformations of* Λ, *where* N *is the connected sum of* n *copies of* $S^1 \times S^{n-1}$ *and* U *is a collection of tubular neighborhoods in* N *of a collection of* e *disjoint circles determined by* Λ.

We now specialize to the case of polygon linkages. Thus we now assume that the graph Y is a combinatorial n-gon. In this case we may realize the above circle packing as a 'necklace' of circles with centers on a circle C orthogonal to the n circles in the packing. Let $p_1, p_2, \ldots, p_n$ be the points of tangency of the circles in the circle packing. Then $p_1, p_2, \ldots, p_n$ lie on C. Let $\mathbb{H}^2$ be the hyperbolic plane bounded by C. The hyperbolic planes bounded by the circles in the packing intersect $\mathbb{H}^2$ in an ideal n-gon Ω with vertices at $p_1, p_2, \ldots, p_n$. We then realize Φ_n in $\mathrm{PSL}_2(\mathbb{R})$ by reflections in the sides of Ω. Choose a side e of Ω. Let τ be the reflection in e and put $R = \Omega \cup \tau\Omega$. Then R is a fundamental domain for Γ_n. Moreover $M = \Gamma_n \setminus \mathbb{H}^2$ is a complete hyperbolic manifold of finite volume diffeomorphic to the n-times punctured sphere. The involution τ is realized by an orientation reversing isometry of M with fixed-point set diffeomorphic to an n-times punctured circle. The orbifold quotient of $B^2 = \mathbb{H}^2 \cup S^1$ is an n-gon with reflector edges and vertices $p_1, p_2, \ldots, p_n$ which are the 'cusps' of M. Let Σ be the compactification of M obtained by adding $p_1, p_2, \ldots, p_n$ and $\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_n$ be disjoint coordinate disk neighborhoods of $p_1, p_2, \ldots, p_n$. Let $U_i = \overline{U}_i - \{p_i\}, U = \bigcup_{i=1}^n U_i$, and γ_i be a loop in U_i which generates $\pi_1(U_i)$. We choose a base-point p near p_1 in such a way that we may identify Γ_n with $\pi_1(M, p)$ and Φ_n with the semi-direct product $\mathbb{Z}/2 \propto \pi_1(M, p)$ where the generator of $\mathbb{Z}/2$ acts by τ. Then for $1 \leqslant i \leqslant j \leqslant n$ the dihedral group $D_2^{i,j}$ is identified with the group $\langle \tau_i, \gamma_i\gamma_{i+1} \cdots \gamma_{j-1}\rangle$ because

$$\tau_i\tau_j = \tau_i\tau_{i+1} \cdots \tau_{j-1}\tau_j = \gamma_i \cdots \gamma_{j-1}.$$

Now assume we have an admissible linkage $\prod$ in X with underlying graph the n-gon Y. In this case that we have $\mathcal{E} = \{(i, i+1) : 1 \leqslant i \leqslant n\}$ where the indices are taken modulo n. Thus the linkage conditions correspond to the dihedral groups $D_2^{i,i+1}, 1 \leqslant i \leqslant n$, where $D_2^{i,i+1} = \langle \tau, \gamma_i \rangle$. From the considerations of Chapter 2 we obtain the representation $\rho\colon \Phi_n \to G$ associated to $\prod$. From Chapter 3, taking $R = U$ we then obtain a controlling augmented differential graded Lie algebra $\mathcal{B}^{\cdot}(M,\, U; \mathrm{ad}\, P)^{\mathbb{Z}/2}$ for the deformations of $\prod$ which consists of ad P-valued forms on M. We will abbreviate $\mathcal{B}^{\cdot}(M,\, U; \mathrm{ad}\, P)^{\mathbb{Z}/2}$ to $\mathcal{C}^{\cdot}(M, \mathrm{ad}\, P)$ henceforth.

Let V be a Φ_n-module. We define the parabolic cohomology groups $H^1_{\mathrm{par}}(\Gamma_n, V)$ and $H^1_{\mathrm{par}}(\Phi_n, V)$ by

$$H^1_{\mathrm{par}}(\Gamma_n, V) = H^1(\mathcal{B}^{\cdot}(M, U; \tilde{V})), \tag{1}$$

$$H^1_{\mathrm{par}}(\Phi_n, V) = H^1(\mathcal{B}^{\cdot}(M, U; \tilde{V})^{\mathbb{Z}/2}). \tag{2}$$

We note that if V is trivial as a Γ_n-module, then $H^1_{\mathrm{par}}(\Gamma_n, V) = 0$ and so $H^1_{\mathrm{par}}(\Phi_n, V) = H^1_{\mathrm{par}}(\Gamma_n, V)^{\mathbb{Z}/2} = 0$. We obtain the following lemma.

LEMMA 4.3 $H^1_{\text{par}}(\Phi_n, \varepsilon) = 0.$

5. Polygonal linkages in constant curvature spaces

In this chapter we apply our theory to determine the local analytic structure of the space $C(\Pi)$ where Π is an n-gon linkage in one of the model spaces of constant curvature.

Assume Π is degenerate. Then Π is contained in a geodesic γ of X. We orient γ so that its orientation agrees with that of the first edge of Π. We let $a_i = r_i$ if the ith edge is oriented compatibly with γ and $a_i = -r_i$ otherwise. We define b, the number of back-tracks of Π, to be the number of negative a_i's and f, the number of forward-tracks of Π, to be the number of positive a_i's. In the spherical case, we will assume that Π is contained in a geodesic segment (i.e., it does not go 'all the way around' γ). The general spherical case when Π is allowed to go 'all the way around' is treated (for $m = 2$) in [KM1] (again the singularity is quadratic but the following formula for the signature must be modified).

THEOREM 5.1 (i) $\dim C(\Pi) = (m-1)n$.

(ii) *If $\Pi' \in C(\Pi)$ is nondegenerate, then $C(\Pi)$ is smooth at Π'.*

(iii) *If $\Pi' \in C(\Pi)$ is degenerate, let b and f be as above. Then Π' has a neighborhood in $C(\Pi)$ analytically equivalent to a neighborhood of 0 in the quadratic cone defined by a quadratic form of nullity $2m-1$ and signature $((m-1)(f-1), (m-1)(b-1))$.*

The remainder of Section 5 will constitute a proof of the theorem. By the discussion at the end of Chapter 4 we may construct a controlling differential graded Lie algebra $\mathcal{C}^{\cdot}(M, \operatorname{ad} P)_0$ for the deformations of Π consisting of bundle-valued differential forms on M, the n-times punctured two sphere. If then follows immediately from Lemma 2.16 that $H^2(\mathcal{C}^{\cdot}(M, \operatorname{ad} P)_0)$ is zero unless Π is degenerate and in this case $H^2(\mathcal{C}^{\cdot}(M, \operatorname{ad} P)) = \mathbb{R}$. From the general theory of controlling differential graded Lie algebras, see the proof of Lemma 5.2, it follows that Π is a smooth point of $C(\Pi)$ unless Π is degenerate. Also we have

$$\dim C(\Pi) = \dim H^1(\mathcal{C}^{\cdot}(M, \operatorname{ad} P)_0),$$

where the dimension on the right is calculated for a representation ρ not corresponding to a degenerate polygon. It is easy to see that this dimension is $n(m-1)$, and (i) and (ii) of Theorem 5.1 follow. It remains to prove (iii).

Assume now that Π is degenerate and f and b are as above. In the spherical case we can deduce Theorem 5.1 immediately by the methods of [KM1]. Indeed the flat bundle $\operatorname{ad} P$ admits a parallel metric. We use this metric to define the complex $\mathcal{A}^{\cdot}_{(2)}(M, \operatorname{ad} P)$ of square-integrable forms. The inclusion $\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P)) \to \mathcal{A}^{\cdot}_{(2)}(M, \operatorname{ad} P))$ is a quasi-isomorphism as in Theorem 4.1 of [KM1]. We deduce that $C^{\cdot}(M, \operatorname{ad} P)$ is formal as in Theorem 4.3 of [KM1]. Then by [GM1], Theorem 3.5, it follows that the deformation space of Π in $C(\Pi)$ is locally analytically equivalent to the product of the germ at 0 of $Q^{-1}(0) \subset H^1(C^{\cdot}(M, \operatorname{ad} P)_0)$, with $Q(\eta)$ equal to the cup-product $[\eta, \eta]$, and the (smooth) germ of G/H at the identity coset where H is the isotropy of Π. We then calculate the signature of Q by the Hodge–Riemann bilinear relations as in Section 7 of [KM1].

Unfortunately, in the hyperbolic and Euclidean cases the analogue of the above proof runs into analytic difficulties. Rather than deal with these problems here we give an ad hoc proof of these two cases.

By Theorem 2.11, the augmentation ideal $C^{\cdot}(M, \operatorname{ad} P)_0$ is a controlling differential graded Lie algebra for the relative deformations of ρ_0. Thus it suffices to prove that complete local ring associated to $C^{\cdot}(M, \operatorname{ad} P)_0$ is the quotient of a formal power series ring in $(m-1)(n-2)+2m-1$ variables by a quadratic polynomial with the above signature and nullity. We note that $\dim \mathcal{G}/\varepsilon(H^0(C^{\cdot}(M, \operatorname{ad} P)_0))) = 2m-1$. Hence by Corollary 2.13 it suffices to prove that the complete local ring associated to $C^{\cdot}(M, \operatorname{ad} P))$ is defined by a non-singular quadratic form of signature $((m-1)(f-1), (m-1)(b-1)$. The following criterion for the complete local $\mathbf{k}$-algebra R_L associated to a differential graded Lie algebra L to be defined by a single quadratic relation was pointed out to us by Eric Klassen.

LEMMA 5.2 *Suppose L is a differential graded Lie algebra with* $\dim H^1(L) < \infty$ *and* $\dim H^2(L^{\cdot}) = 1$. *Suppose the cup-product* $Q: H^1(L^{\cdot}) \to H^2(L^{\cdot}) \cong \mathrm{k}$ *is a non-degenerate quadratic form. These* $R_L \cong \mathrm{k}[[H^1(L^{\cdot})]]/(Q)$.

Proof. By [GM2], Theorem 3.9, there is a formal map (the Kuranishi map) $F\colon H^1(L^{\cdot}) \to H^2(L^{\cdot})$ such that R_L is isomorphic to the complete local ring $\mathbf{k}[[H^1(L^{\cdot})]]/(F)$.

Moreover the map F satisfies

$$\text{(i)}\quad F(0)=0; \quad \text{(ii)}\quad dF(0)=0; \quad \text{(iii)}\ D^2F(0)=Q.$$

Thus by assumption the Hessian $D^2F(0)$ is nondegenerate. By the formal Morse Lemma, F is formally equivalent to $D^2F(0) = Q$. □

Thus the proof of Theorem 5.1 has been reduced to showing Q is nondegenerate and computing its signature.

LEMMA 5.3 *The quadratic form Q on* $H^1(C^{\cdot}(M, \operatorname{ad} P))$ *is the orthogonal direct sum of* $m-1$ *copies of the form Q for the case* $m = 2$.

Proof. We first consider the spherical and hyperbolic cases. Let U be the span of Π (so $\dim U = 2$). Then the decomposition $V = U + U^{\perp}$ is invariant under Φ_n and Φ_n acts by the signum representation ε on $U^{\perp}$. Using the isomorphism $\mathcal{G} \cong \Lambda^2 U^{\perp} \oplus U \otimes U^{\perp}$ and $H^1_{\mathrm{par}}(\Gamma_n, \Lambda^2 U^{\perp}) = H^1_{\mathrm{par}}(\Gamma_n, \Lambda^2 U) = H^1_{\mathrm{par}}(\Gamma_n, U^{\perp}) = 0$, we obtain

$$H^1(C^{\cdot}(M, \operatorname{ad} P)) = H^1_{\mathrm{par}}(\Gamma_n, U)^{\varepsilon} \otimes U^{\perp}.$$

Here the superscript ε denotes the subspace transforming by ε under Φ_n. The form Q on $H^1(\mathcal{C}^{\cdot}(M, \operatorname{ad} P))$ corresponds up to a scalar multiple with the bilinear form obtained by tensoring the exterior product of 1-forms with a symplectic form on U and the form $(\,,)$ on $U^{\perp}$.

In the Euclidean case we proceed as follows. We assume γ is 'the x-axis', more precisely, γ is the line through e_{m+1} tangent to e_1, see Section 1. We break up $\mathcal{G} = e(m)$ into a direct sum of $m-1$ isomorphic Φ_n-modules each isomorphic to $e(2)$. Let E_i denote infinitesimal translation in the direction of the ith standard basis vector e_i and V_{ij} denote infinitesimal rotation in the plane spanned by e_i and e_j. We note the bracket relations

$$[V_{ij}, E_k] = \begin{cases} 0, & k \neq i, j \\ E_j, & k = i, \\ -E_i, & k = j. \end{cases}$$

We let $\mathcal{G}_j$ be the Lie subalgebra of $\mathcal{G}$ spanned by $\{E_1, E_j, V_{1j}\}$ and $\tilde{\mathcal{G}}_j$ denote the corresponding local system. Also we let W_j be the sub Φ_n-module spanned by $\{E_j, V_{1j}\}$ and $\tilde{W}_j$ be the corresponding local system. We claim that the natural maps $\bigoplus_{j=2}^m H^1(\mathcal{C}^{\cdot}(M, \tilde{W}_j)) \to H^1(\mathcal{C}^{\cdot}(M, \operatorname{ad} P))$ and $\bigoplus_{j=2}^m H^1(\mathcal{C}^{\cdot}(M, \tilde{W}_j)) \to \bigoplus_{j=2}^m H^1(\mathcal{C}^{\cdot}(M, \tilde{\mathcal{G}}_j))$ are isomorphisms. Put $W = \bigoplus_{j=2}^m W_j$ and $W' = \operatorname{span}\{E_1, V_{ij}, 2 \leqslant i, j \leqslant m\}$. Then $e(m) = W + W'$ and Φ_n acts trivially on W' whence $H^1_{\text{par}}(\Phi_n, W') = 0$. This proves the first claim. But applying the first claim in the special case $m = 2$ we find that the natural map $H^1_{\text{par}}(\Phi_n, W_j) \to H^1_{\text{par}}(\Phi_n, \mathcal{G}_j), \epsilon \leqslant | \leqslant m$, is an isomorphism and the second claim follows. Finally, we claim the cup-product Q on $H^1(\mathcal{C}^{\cdot}(M, \operatorname{ad} P))$ carries over to the orthogonal direct sum of the corresponding forms Q on $H^1(\mathcal{C}^{\cdot}(M, \tilde{\mathcal{G}}_j))$. In order to prove the claim we observe that scalar form Q is obtained by evaluating the cup-product on $H^1(\mathcal{C}^{\cdot}(M, \operatorname{ad} P)$ on the relative fundamental class with coefficients $[\mathrm{M}] \otimes \theta_1$. Here $\{\theta_i, \omega_{ij} : 1 \leqslant i < j \leqslant m\}$ is the basis for $e(m)^*$ dual to $\{E_i, V_{ij} : 1 \leqslant i < j \leqslant m\}$. We may define a corresponding form Q_j on $H^1(\mathcal{C}^{\cdot}(M, \tilde{\mathcal{G}}_j))$ and under the above isomorphism Q and $\bigoplus_{j=2}^m Q_j$ correspond. The lemma follows. □

COROLLARY 5.4 *The cup-product Q is a non-degenerate quadratic form on $H^1(\mathcal{C}^{\cdot}(M, \operatorname{ad} P))$.*

Proof. By the lemma we may assume $m = 2$. In the spherical and hyperbolic cases the cup-product $\tilde{Q}$ on $H^1(\mathcal{B}^{\cdot}(M, U; \operatorname{ad} P))$ is non-degenerate by Lemma 2.17. Thus it suffices to prove that τ_1 is an isometry of $\tilde{Q}$, for then the restriction of $\tilde{Q}$ to the fixed space of τ_1 will be non-degenerate. Now τ_1 reverses the sign of the fundamental class of M but it also reverses the sign of the symplectic form on U. Thus τ_1 is an isometry of $\tilde{Q}$. The Euclidean case follows from Lemma 5.9. □

LEMMA 5.5 $\dim H^1(\mathcal{C}^\cdot(M, \operatorname{ad} P)) = (m-1)(n-2)$.

Proof. By Lemma 5.4 it suffices to consider the case $m = 2$. In the hyperbolic and spherical cases we have $H^1(\mathcal{C}^\cdot(M, \operatorname{ad} P)) \cong H^1_{\text{par}}(\Gamma_n, U)^\varepsilon$. Since $\rho(\gamma_i)$ has no fixed vectors, $i = 1, 2, \ldots, n$ we have $H^1(\langle\gamma_i\rangle, U) = \{0\}$ and $H^1_{\text{par}}(\Gamma_n, U) = H^1(\Gamma_n, U)$. Since Γ_n is a free group on $n-1$ generators we have $\dim H^1(\Gamma_n, U) = 2n-4$. Now by combining the cup-product on H^1 with the symmetric form $(\,,)$ on U and then evaluating on the relative fundamental class of M we obtain a symplectic form A on $H^1(\Gamma_n, U)$. We have

$$A(\rho(\tau_i)v, \rho(\tau_i)w) = -A(v, w), 1 \leqslant i \leqslant n,$$

since τ_i changes the sign of the relative fundamental class. Hence the $+1$ and -1 eigenspaces of $\rho(\tau_i)$ are dually-paired Lagrangians and the lemma is proved in the spherical and hyperbolic cases.

In the Euclidean case we have an exact sequence of Lie algebras $t \to e(2) \to e(2)/t$ where t is the Lie subalgebra of translations. The group Φ_n acts on t by the character ε and trivially on $e(2)/t$. Since Φ_n is a free product of $\mathbb{Z}/2$'s a cocycle $c \in Z^1(\Phi_n, e(2))$ corresponds to an n-tuple (c_i) with $c_i \in Z^1(\langle\tau_i\rangle, e(2))$. The cocycle condition $c(\tau_i^2) = 0$ is equivalent to $\rho(\tau_i)c(\tau_i) = -c(\tau_i)$ whence $c(\tau_i) \in t, 1 \leqslant i \leqslant n$. Hence $\dim Z^1(\Phi_n, e(2)) = 2n$. Let $c \in Z^1(\Phi_n, e(2))$. Now the reader will verify that the restriction $c|\langle\tau_i, \tau_{i+1}\rangle$ is exact if and only if $c(\tau_{i+1}) - c(\tau_i)$ is perpendicular to the edge joining u_i to u_{i+1}. In this case this means that $c(\tau_{i+1}) - c(\tau_i)$ lies in the y-axis. *Since* Π *is degenerate* we obtain $n-1$ independent conditions whence $\dim Z^1_{\text{par}}(\Phi_n, e(2)) = n+1$ (here the parabolic cocycles above are the cocycles whose associated classes are parabolic). Since $H^0(\Phi_n, e(2)) = \{0\}$ we have $\dim B^1(\Phi_n, e(2)) = 3$ whence $\dim H^1_{\text{par}}(\Phi_n, e(2)) = n-2$. □

REMARK. The argument in the Euclidean case may be extended to the other cases to realize an infinitesimal deformation of ρ (relative to R) as an assignment of an element $\delta_i \in T_{u_i}(X)$ to each vertex u_i of Π such that $\delta_{i+1} - \delta_i$ is orthogonal to the edge e_i joining u_i and u_{i+1}. Such an assignment is a first-order deformation of the linkage Π, see [Co1]. This is of course a special case of Theorem 3.2.

It remains to compute the cup product Q on $H^1(\mathcal{C}^\cdot(M, \operatorname{ad} P))$. By Lemma 5.3 it suffices to compute Q for planar degenerate linkages (i.e., for the case $m = 2$). We use Poincaré duality and compute instead the intersection product B on $H_1(M, \operatorname{ad} P)^\varepsilon$. Hence the superscript ε denotes the subspace of $H_1(M, \operatorname{ad} P)$ transforming by the signum representation ε under Φ_n. We refer the reader to [JM], §4, for details on intersection products with local coefficients. We now describe a basis for $H_1(M, \operatorname{ad} P)^\varepsilon$. We choose a base-point of M and use it to identify elements of $\mathcal{G}$ with parallel sections of $\operatorname{ad} P$. Also, we must choose approach paths in what follows – such details we leave to the reader. Let $\alpha_i, 1 \leqslant i \leqslant n-1$, be a small loop in M going around p_i once in the counter-clockwise direction. We let c_i be the simplicial chain consisting of $-\alpha_i, \alpha_{i+1}$ and an arc $b_{i,i+1}$ joining α_i

and α_{i+1}. Let $x_i = \alpha_i \cap b_{i,i+1}$ and $y_i = \alpha_{i+1} \cap b_{i,i+1}$. We now wish to attach coefficients to c_i to create a 1-cycle with coefficients in $\operatorname{ad} P$ which transforms as a homology class by ε under Φ_n. We will find a parallel section σ_i of $\operatorname{ad} P$ along $\alpha_i - \{x_i\}$, a parallel section $\bar{\sigma}_{i+1}$ along $\alpha_{i+1} - \{y_i\}$ and a parallel section μ_i along $b_{i,i+1}$ such that the jump of σ_i at x_i is equal to $\mu_i(x_i)$ and the jump of $\bar{\sigma}_{i+1}$ at y_i is the negative of $\mu_i(y_i)$, see [DM], page 14. We will need the following explicit version of these equations. Choose a path from the base-point to $b_{i,i+1}$. Then the parallel sections $\sigma_i, \bar{\sigma}_{i+1}, \mu_i$ correspond to elements $w_i, \bar{w}_{i+1}, v_i$ in $\mathcal{G}$. Let γ_i and γ_{i+1} be the elements of $\pi_1(M)$ represented by α_i and α_{i+1}, then the previous equations become

$$\begin{aligned} \rho(\gamma_i)^{-1} w_i - w_i &= v_i, \\ \rho(\gamma_{i+1}) \bar{w}_{i+1} - \bar{w}_{i+1} &= -v_i. \end{aligned}$$

REMARK. In terms of group homology the cycle $\tilde{c}_i$ (below) corresponds to the 1-cycle $\gamma_i^{-1} \otimes w_i + \gamma_{i+1} \otimes \bar{w}_{i+1}$.

To ensure that the resulting homology class transforms correctly under Φ_n we require

$$\begin{aligned} \rho(\tau_{i+1}) v_{i+1} &= -v_{i+1}, \\ \rho(\tau_i) w_i &= w_i, \\ \rho(\tau_{i+1}) \bar{w}_{i+1} &= \bar{w}_{i+1}. \end{aligned}$$

We note that all τ_j act the same way on homology. We will then obtain a cycle with coefficients $\tilde{c}_i$ and a basis $\mathcal{B} = \{\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_{n-2}\}$ for $H_1(M, \operatorname{ad} P)^\varepsilon$. We describe the coefficients v_i, w_i in the three cases. In the spherical and hyperbolic cases we have

$$\operatorname{ad} \rho = \Lambda^2 V = V,$$

where the last equality holds because $m = 2$. Here V denotes the standard representation on $V = \mathbb{R}^3$, see Chapter 1. Thus we can describe our local coefficients by vectors in V. The symmetric space X is embedded in V, see Chapter 1. We assume henceforth that $\Pi \subset X_1$ where X_1 is the intersection of $U = \operatorname{span}\{e_1, e_3\}$ and X_1 is oriented in the direction of e_1.

We let $u_1, u_2, \ldots, u_n \in V$ be the vertices of Π and let $v_1, v_2, \ldots, v_n$ be the positively-directed unit tangent vectors to X_1 at $u_1, u_2, \ldots, u_n$, considered as elements in V. We then equip $b_{i,i+1}$ with the local coefficient v_{i+1} (the tangent vector to the last point on the edge $[u_i, u_{i+1}]$). It remains to solve the equations

$$\begin{aligned} \rho(\gamma_i)^{-1} w_i - w_i &= v_{i+1}, \\ \rho(\gamma_{i+1}) \bar{w}_{i+1} - \bar{w}_{i+1} &= -v_{i+1}, \end{aligned}$$

for w_i and w_{i+1}. We leave the verification of the following lemma to the reader.

LEMMA 5.6 (a) *In the spherical case we have*

(i) $w_i = \frac{1}{2\sin a_i} u_i$.
(ii) $\bar{w}_{i+1} = \frac{1}{2\sin a_{i+1}} u_{i+2}$.

(b) *In the hyperbolic case we have*

(i) $w_i = \frac{1}{2\sinh a_i} u_i$.
(ii) $\bar{w}_{i+1} = \frac{1}{2\sinh a_{i+1}} u_{i+2}$.

We next describe the cycles $\tilde{c}_i, 1 \leqslant i \leqslant n-2$, in the Euclidean case. We assume that Π is contained in the x-axis. We attach to each of the arcs $b_{i,i+1}$ the infinitesmal translation E_2 in the direction of the y-axis. Once again we have to solve the equations

$$\begin{aligned} \operatorname{Ad} \rho(\gamma_i)^{-1} w_i - w_i &= e_2, \\ \operatorname{Ad} \rho(\gamma_{i+1})^{-1} \bar{w}_{i+1} - \bar{w}_{i+1} &= -e_2. \end{aligned}$$

Let $V_{12} = e_1 \wedge e_2$ be infinitesmal rotation in the xy-plane.

LEMMA 5.7 *In the Euclidean case we have*

$$w_i = \frac{1}{a_i} V_{12} \quad \textit{and} \quad \bar{w}_{i+1} = \frac{1}{a_{i+1}} V_{12}.$$

We leave the computation of $\tilde{c}_i \cdot \tilde{c}_j$ to the reader remarking only that c_i and c_j do not intersect unless $j = i-1, i$ or $i+1$. We obtain the following theorem.

THEOREM 5.8 *Let Π be a degenerate n-gon with signed side-lengths $a_1, a_2, \ldots, a_n$. Then the matrix of B relative to the basis $\mathcal{B}$ is the following tridiagonal matrix $\beta = (\beta_{ij})$.*

(i) *The euclidean case*

(a) $\beta_{ii} = \frac{1}{a_i} + \frac{1}{a_{i+1}}, 1 \leqslant i \leqslant n-2$.
(b) $\beta_{i,i+1} = -\frac{1}{a_{i+1}}, 1 \leqslant i \leqslant n-3$.

(ii) *The spherical case*

(a) $\beta_{ii} = \cot\ a_i + \cot\ a_{i+1}, 1 \leqslant i \leqslant n-2$.
(b) $\beta_{i,i+1} = -\csc\ a_{i+1}, 1 \leqslant i \leqslant n-3$.

(iii) *The hyperbolic case*

(a) $\beta_{ii} = \coth\ a_i + \coth\ a_{i+1}, 1 \leqslant i \leqslant n-2$.
(b) $\beta_{i,i+1} = -\operatorname{csch}\ a_{i+1}, 1 \leqslant i \leqslant n-3$.

We now determine the signature of the form B. We first compute it in the Euclidean case where the calculation is easy then give a deformation argument to determine the other two cases.

LEMMA 5.9 *In the Euclidean case, the intersection form B is nonsingular of signature* $(b-1, f-1)$.

Proof. We consider the quadratic form $\tilde{B}$ on $\mathbb{R}^{n-1}$ which is diagonal relative to the standard basis of $\mathbb{R}^{n-1}$ with diagonal entries $(1/a_1, 1/a_2, \dots, 1/a_{n-1})$. The form B is obtained by restricting $\tilde{B}$ to the hyperplane $H \subset \mathbb{R}^{n-1}$ given by $H = \{(x_1, x_2, \dots, x_{n-1}) \in V : \sum_{i=1}^{n-1} x_i = 0\}$. Indeed note that the matrix representation of $\tilde{B}|H$ relative to the basis $\{e_i - e_{i+1} : 1 \leqslant i \leqslant n-2\}$ for H is the matrix β.

Let $L \subset H^{\perp}$ be the line orthogonal to H. Then $\tilde{B}|H$ is singular if and only if $L \subset H$. But clearly $L = \mathbb{R}\, a$ with $a = (a_1, a_2, \dots, a_{n-1})$. Since $\sum_{i=1}^{n-1} a_i = -a_n$ is non-zero by hypothesis, L is not contained in H and B is non-singular. It remains to compute the signature of B. We note that

$$\tilde{B}(a,a) = \sum_{i=1}^{n-1} a_i = -a_n \neq 0.$$

Let (p,q) be the signature of $\tilde{B}$. Since the decomposition $V = L + H$ is orthogonal for $\tilde{B}$ we see that the signature of B is $(p-1, q)$ if $\tilde{B}|L$ is positive definite $(a_n\langle 0)$ and $(p, q-1)$ if $\tilde{B}|L$ is negative definite $(a_n\rangle 0)$. In case we have $a_n\langle 0$ the segment e_n of Π is a back-track whence $p = f, q = b-1$. Thus in this case the signature of B is $(f-1, b-1)$. In case we have $a_n\rangle 0$ the segment e_n is a forward-track whence $p = f-1, q = b$ and the signature of B is again $(f-1, b-1)$. □

We now consider the one parameter family of matrices $k\beta(ka_1, \dots, ka_n)$ where $\beta(a_1, \dots, a_n)$ is the matrix β from Theorem 5.9 (either (i) or (ii)). These matrices are the intersection matrices (multiplied by k) for the local system corresponding to the shrunken n-gon with side-lengths $ka_1, ka_2, \dots, ka_n$. Hence these matrices are non-singular for all k by Poincaré duality. The reader will verify that the limit as k goes to zero of $k\beta(ka_1, \dots, ka_n)$ is the corresponding matrix in the Euclidean case. In the previous lemma we have seen that the signature of the limit matrix is $(b-1, f-1)$ and Theorem 5.1 follows.

6. An example of Lubotzky and Magid

In this section we give a geometric interpretation of the following result of [LM], 2.10.4 – sharpened in [GM1], Section 9.3. Let Γ be the $(3,3,3)$ triangle group. Thus Γ has a presentation $\Gamma = \langle a, b, c \,|\, a^2, b^2, c^2, (ab)^3, (bc)^3, (ca)^3\rangle$. The group Γ is a Bieberbach group – there is a short exact sequence $Z^2 \to \Gamma \to S_3$. Let $\rho_0 : S_3 \to \mathrm{GL}_2(\mathbb{C})$ be the unique two-dimensional irreducible representation of

S_3 and ρ be the induced representation of Γ. Lubotzky and Magid proved that the analytic germ of $\mathrm{Hom}(\Gamma, \mathrm{GL}_2(\mathbb{C}))$ at ρ was isomorphic to the germ at $(I, 0)$ of the non-reduced scheme $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{spec}(\mathbb{C}[t]/(t^n))$ for some $n \geqslant 2$. In [GM1], Section 9.3, it was proved that $n = 2$.

We first show that we can replace $\mathrm{GL}_2(\mathbb{C})$ by $\mathrm{PGL}_2(\mathbb{C})$ in the above paragraph. Let $\pi : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ be the projection. Then we have an induced map $\pi_* : \mathrm{Hom}(\Gamma, \mathrm{GL}_2(\mathbb{C})) \to \mathrm{Hom}(\Gamma, \mathrm{PGL}_2(\mathbb{C}))$ given by $\pi_*(\rho) = \pi \circ \rho$. In the lemma that follows we let $M = T^2 = \mathbb{C}/\mathbb{Z}^2$ and $H = S_3$.

LEMMA 6.1 *The morphism π_* induces an isomorphism of germs $\pi_* : (\mathrm{Hom}(\Gamma, \mathrm{GL}_2(\mathbb{C})), \rho) \to (\mathrm{Hom}(\Gamma, \mathrm{PGL}_2(\mathbb{C})), \pi \circ \rho)$.*

Proof. The exact sequence of Lie algebras $\mathbb{C} \to gl_2(\mathbb{C}) \to pgl_2(\mathbb{C})$ is split and induces an exact sequence of differential graded Lie algebras

$$0 \to \mathcal{A}^{\cdot}(M)_0^H \to (\mathcal{A}^{\cdot}(M) \otimes gl_2(\mathbb{C}))_0^H \to (\mathcal{A}^{\cdot}(M) \otimes pgl_2(\mathbb{C}))_0^H \to 0.$$

By Theorem 2.11 above, the second and third differential graded Lie algebras control the deformations of ρ and $\pi \circ \rho$ respectively. But $H^0(\mathcal{A}^{\cdot}(M)_0^H) = H^1(M, \mathbb{R})^H = H^2(M, \mathbb{R})^H = \{0\}$. Consequently the third arrow above induces an isomorphism on H^0, H^1 and H^2 and the lemma follows by Theorem 2.4 of [GM1]. □

We will henceforth work only with $\mathrm{PGL}_2(\mathbb{C})$. We will change our notation and replace $\pi \circ \rho$ by ρ. Then we note that $\rho(\Gamma)$ is a finite subgroup of $\mathrm{PGL}_2(\mathbb{R})$, hence we may conjugate ρ by an element of $\mathrm{PGL}_2(\mathbb{R})$ and assume that ρ take values in PO(2). We first consider the deformations of ρ in $\mathrm{Hom}(\Gamma, \mathrm{SO}(3)) \subset \mathrm{Hom}(\Gamma, \mathrm{PGL}_2(\mathbb{C}))$.

Since *ab*, *bc*, and *ca* all have order 3, their conjugacy classes are fixed under deformation and the inclusion $\mathrm{Hom}(\Gamma, \mathrm{SO}(3)) \to \mathrm{Hom}(\Phi_3, \mathrm{SO}(3))$ induces a canonical isomorphism of germs (here $R = \{\langle ab \rangle, \langle bc \rangle, \langle ca \rangle\}$)

$$(\mathrm{Hom}(\Gamma, \mathrm{SO}(3)), \rho) \cong (\mathrm{Hom}(\Phi_3, R; \mathrm{SO}(3)).$$

Thus, by the results of Chapter 3, the deformation space of ρ is isomorphic to that of the spherical triangle obtained from the fixed-points of a, b and c on S^2. Since $\rho(\Gamma) \subset \mathrm{PO}(2)$ this triangle is contained in the equator of S^2 and it consists of the equator decomposed into three equal arcs. By [KM1], the germ of the real variety $\mathrm{Hom}(\Gamma, \mathrm{SO}(3))$ at ρ is isomorphic to the germ at $(I, 0)$ of $\mathrm{SO}(3) \times \mathrm{Spec}\,\mathbb{R}[t]/(t^2)$. Since $\mathrm{PGL}_2(\mathbb{C})$ is the complexification of its maximal compact subgroup SO(3), the deformation space of ρ in $\mathrm{PGL}_2(\mathbb{C})$ is the complexification of the above space. We obtain the result of Lubotzky–Magid, Goldman–Millson.

The non-trivial nilpotent deformation corresponds to the first-order deformation of the above degenerate triangular linkage obtained by assigning the zero vector to the first and second vertices u_1, u_2 and the normal vector to the equator (in the tangent space to S^2 at u_3) to the third vertex u_3. See the remark following Lemma 5.5 for a description of a first-order deformation of a linkage.

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