ON THE ABSENCE OF THE AHLFORS AND SULLIVAN THEOREMS FOR KLEINIAN GROUPS IN HIGHER DIMENSIONS

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Introduction

One of the fundamental results of the theory of discontinuous groups of fractional linear transformations acting on the complex plane **C** is the following finiteness theorem of L. Ahlfors [1, 2]:

Let G be a finitely generated nonelementary discrete subgroup of $\mathrm{PSL}(2, \mathbf{C})$ *acting freely on a region of discontinuity* $\Omega(G)$. *Then the quotient surface* $\Omega(G)/G$ *consists of a finite number of Riemann surfaces* $S_1, \ldots, S_n$ *of finite hyperbolic area. In particular the groups* $\pi_1(S_i)$ *are finitely generated and the homotopy type of the surfaces* S_i *is finite* $(i = 1, \ldots, n)$.

Subsequently D. Sullivan [3] strengthened Ahlfors' finiteness theorem by showing that any finitely generated discrete group $G \subset \mathrm{PSL}(2, \mathbf{C})$ has at most a finite number of cusps (i.e., conjugacy classes of maximal parabolic subgroups).

Ahlfors [4] and Ohtake [5] attempted to develop analytic methods of studying the problem of finiteness of multidimensional Kleinian groups. However, the results obtained do not give any information about either the topology of the quotient spaces of Kleinian groups or the number of cusps.

In the present article we shall show that even a weakened version of Ahlfors' finiteness theorem fails in dimension 3 and also construct a counterexample to the analog of Sullivan's finiteness theorem in higher dimensions.

Theorem 1. *There exists a finitely generated torsion-free function group* $F \subset \mathrm{M\ddot{o}b}\,(S^3)$ *with invariant component* $\Omega \subset \Omega(F)$ *such that the fundamental group* $\pi_1(\Omega/F)$ *is infinitely generated. Moreover the group* F *itself is infinitely defined.*

Theorem 2. *There exists a finitely generated Kleinian group* $F' \subset \mathrm{M\ddot{o}b}\,(S^3)$ *such that*

a) F' *contains an infinite number of cusps (of rank* 1*),*

b) *if* F^n *is a conformal extension of the group* F' *to* S^n, *then* $\mathrm{rank}\,(H_{n-1}(\Omega(F^n)/F^n, \mathbf{Z})) = \infty$. *Thus the manifold* $\Omega(F^n)/F^n$ *has an infinite homotopy type.*

1. Preliminary information

Let $\mathrm{M\ddot{o}b}\,(\overline{\mathbf{R}}^n) \simeq \mathrm{Isom}\,(\mathbf{H}^{n+1})$ be the group of conformal automorphisms of the n-dimensional sphere $S^n = \overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$, where $\mathbf{H}^{n+1} = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbf{R}^{n+1} : x_{n+1} > 0\}$ is hyperbolic space.

A subgroup $G \subset \mathrm{M\ddot{o}b}\,(S^n)$ is called *Kleinian* if the action of G is discontinuous at some point $x \in S^n$, i.e., there exists a neighborhood $U(x)$ such that $g(U(x)) \cap U(x) \neq \varnothing$ for only a finite number of elements $g \in G$. The set of points where G acts discontinuously is called the *discontinuity set* $\Omega(G)$ and its complement $\Lambda(G) = S^n \setminus \Omega(G)$ the *limit set* of the group G.

A Kleinian group G is called a *function group* if there exists a connected component $\Omega \subset \Omega(G)$ that is invariant with respect to G. If G acts freely on Ω, then the quotient space $M(G) = \Omega/G$ is an n-dimensional manifold. We shall denote by $\mathcal{P}(G)$ the isometric fundamental region for G [6] and by $I(g)$ the isometric sphere $g \in \mathrm{M\ddot{o}b}\,(S^n)$.

In what follows we shall assume (if not otherwise specified) that all manifolds are three-dimensional and piecewise linear. Standard reductions by the theory of Kleinian groups and three-dimensional topology can be found in [2] and [6]–[8]. If $S \subset \mathbf{R}^3$ is a two-sphere, we shall denote by $\mathrm{ext}\,(S)$ and $\mathrm{int}\,(S)$ the components of $\overline{\mathbf{R}}^3 \setminus S$ such that $\infty \in \mathrm{ext}\,(S)$. The symbol cl ( ) denotes the closure of a set.

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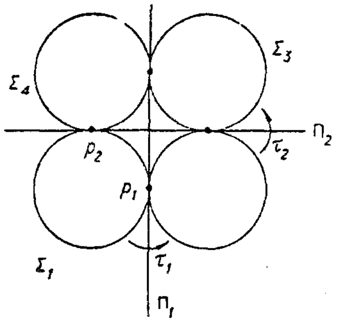

Fig. 1

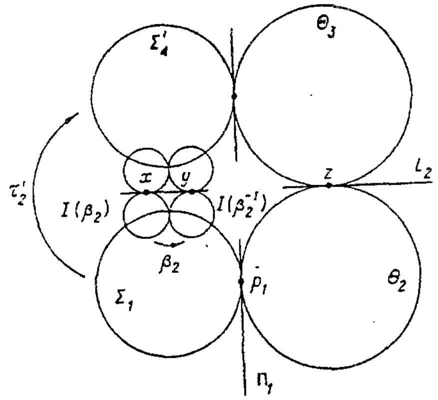

Fig. 2

2. Outline of the proof of Theorem 1

Consider the configuration consisting of four mutually tangent Euclidean spheres $\Sigma_i \subset \mathbf{R}^3$ (Fig. 1). Each sphere Σ_i is obtained from its neighbor by a reflection τ_j in the plane Π_j, $(i = 1, \ldots, 4)$, $j = 1, 2)$.

We shall construct discontinuous groups $\Gamma_i \subset \text{Möb}(\overline{\mathbf{R}}^3)$ such that the groups Γ_i are isomorphic to the fundamental group of a bundle over a circle with a "surface" as fiber. The group Γ_i leaves invariant the outside of the sphere Σ_i, $i = 1, \ldots, 4$. Using Maskit's combination method we shall show that both groups $G_1 = \langle \Gamma_1, \Gamma_2 \rangle$ and $G_2 = \langle \Gamma_3, \Gamma_4 \rangle$ are Kleinian and also isomorphic to the fundamental group of a bundle over a circle (cf. Lemma 3). Let F_i be normal subgroups in G_i corresponding to surface subgroups in G_i $(i = 1, 2)$. The proof of Theorem 1 concludes with Lemma 5, in which we establish that the group $\langle F_1, F_2 \rangle = F$ is the one sought. In particular F is a normal subgroup of the geometrically finite function group $G = \langle G_1, G_2 \rangle$.

The proof of Lemma 5 is based on the following reasoning. Using the involution τ_2 we represent the manifold $M(F) = \Omega/F$ in the form of a doubling of some manifold $M^-(F)$. There exists an infinite regular covering $M^-(F) \to M^-(G) \subset M(G)$ induced by the covering $M(F) \to M(G)$. The manifold $M^-(G)$ is not a bundle over a circle, since $\partial M^-(G)$ contains a surface of genus 2. It follows from this that the group $\pi_1(M^-(F))$ cannot be finitely generated [7]. It then follows immediately that the group $\pi_1(M(F))$ is also infinitely generated.

3. Outline of the proof of Theorem 2

Consider the configuration of four spheres Σ_1, Θ_2, Θ_3, Σ_4 shown in Fig. 2. We construct groups Γ'_1, Γ'_2, Γ'_3, Γ'_4 conjugate in $\text{Möb}(S^3)$ to the groups Γ_i of the preceding section such that their limit sets are respectively the spheres Σ_1, Θ_2, Θ_3, and Σ_4. The groups Γ'_3 and Γ'_4 are obtained from Γ'_2 and Γ'_1 by conjugation using a symmetry τ'_2 with respect to the plane L_2.

The group Γ'_1 contains a parabolic element β_2 such that the isometric spheres $I(\beta_2)$ and $I(\beta_2^{-1})$ are tangent to L_2. The point of tangency $x = I(\beta_2) \cap L_2$ is a fixed point for the parabolic transformation $u = \tau'_2 \beta_2^{-1} \tau'_2 \beta_2$. We shall show that the point x is cusped for the Kleinian group $G' = \langle \Gamma'_1, \Gamma'_2, \Gamma'_3, \Gamma'_4 \rangle$. The group G' contains a normal free subgroup $F' \ni u$ of finite rank, $G'/F' \simeq \mathbf{Z}$. The action of G'/F' on F' by conjugates is induced by a homeomorphism θ of a compact surface $\mathcal{F}$, $\pi_1(\mathcal{F}) \simeq F'$. The system of three loops $\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3$, which is invariant with respect to θ, gives a reduction of θ to irreducible homeomorphisms of infinite order [9]. Here $[\nu]$ is the image of u under the isomorphism $j^{-1} : F' \to \pi_1(\mathcal{F}')$; $[\nu]$ is not conjugate to any element of $\pi_1(\mathcal{F})$ corresponding to components of $\partial\mathcal{F} \cup \alpha$ (since $\langle u \rangle$ is a maximal parabolic subgroup of G'). It follows from this that $\theta_*^m([\nu])$ and $\theta_*^n([\nu])$ are nonconjugate elements of $\pi_1(\mathcal{F})$ for all $m \neq n \in \mathbf{Z}$. Thus the group F' contains an infinite number of conjugacy classes of maximal parabolic subgroups: $\{j(\theta_*^m([\nu])),\ m \in \mathbf{Z}\}$. Property (b) of the group F' follows from the fact that the point x is cusped in the group F^n.

4. Construction of the group F

Let M be an open manifold homeomorphic to the complement of a linkage of Borromean rings. It is known that M is a total bundle space over a circle with a "surface" for fiber [10]. In addition M admits a complete hyperbolic structure of finite volume, i.e., $M = \mathbf{H}^3/\Gamma$, $\Gamma \subset \mathrm{Isom}(\mathbf{H}^3)$ [8].

Definition. A group K with a subgroup S is called *S-finitely approximable* if for any element $k \in K \setminus S$ there exists a subgroup $K_1 \subset K$ of finite index that contains S but $k \notin K_1$.

4.1. Lemma 1. *The group Γ is S-finitely approximable for any geometrically finite subgroup $S \subset \Gamma$.*

Proof. Consider the regular ideal octahedron $P \subset \mathbf{H}^3$ whose dihedral angles all equal $\pi/2$ [8]. Let Q be the group of reflections in the faces of P, and let Q_1 be a finite extension of it using four automorphisms of order 3. Then Q_1 contains Γ as a subgroup of finite index [8]. The assertion of the lemma follows from [11] and the commensurability of the groups Γ and Q. Lemma 1 is now proved.

4.2. We denote by B the outside of the unit sphere $\Sigma \subset \mathbf{R}^3$ with center at the origin. We shall regard B as a model of the hyperbolic space $\mathbf{H}^3$. Further let H_i be certain nonconjugate maximal parabolic subgroups of Γ and $\Lambda(H_i) = \{p_i\}$, $i = 1, 2$. We shall assume that the points p_i have coordinates $(0, 1, 0)$ and $(0, 0, 1)$. Let Π_i be a Euclidean plane tangent to Σ_i at the point p_i (cf. Fig. 1) and Π_i^- the component of $\mathbf{R}^3 \setminus \Pi_i$ that does not intersect Σ $(i = 1, 2)$. We set $\overline{\Pi}_i = \Pi_i \cup \{\infty\}$.

In the next lemma we shall show that for some subgroup of finite index $\Gamma \subset \Gamma$ and planes Π_i the hypotheses of Maskit's combination theorem are fulfilled. Consider a certain neighborhood of Π_i, and let the sphere W_i be tangent to Σ at the point p_i, so that $W_i \subset \mathrm{cl}\, B$, $V_i = \mathrm{ext}\, W_i$, $\mathrm{cl}\, \Pi_i^- \setminus \{p_i\} \subset V_i$, $i = 1, 2$.

4.3. Lemma 2. *There exists a subgroup of finite index $\tilde{\Gamma}$ in the group Γ such that the following conditions hold:*

(a) *the group $\tilde{\Gamma}$ contains a normal subgroup $\tilde{F} \subset \tilde{\Gamma}$ for which $\tilde{\Gamma} = \langle \tilde{F}, t_i \rangle$, $t_i \in H_i \cap \tilde{\Gamma}$, $i = 1, 2$;*

(b) *the group $\tilde{\Gamma}$ has a fundamental set $\mathcal{P} \subset B$ such that $\mathcal{P} \cap V_i$ is a fundamental set for the action of the group $H_i \cap \tilde{\Gamma} = \tilde{H}_i$ on V_i, $(i = 1, 2)$.*

Proof. We denote by $I(g)$ the isometric sphere of the element $g \in \Gamma$. Then there exists at most a finite number of elements $h_k \in H_i$ such that $I(h_k) \cap (\Pi_j \cup (S^3 \setminus \mathcal{P}(H_j)) = \varnothing$, $i \neq j$, $i, j = \{1, 2\}$, $0 \leq k \leq N$. Using the finite approximability of the group Γ [12], we choose a subgroup of finite index $\Gamma^* \subset \Gamma$ for which $h_k \notin \Gamma^*$, $0 \leq k \leq N$. We set $H_i^* = \Gamma^* \cap H_i$, $i = 1, 2$.

(a) Let Φ be a normal subgroup of Γ corresponding to a fiber of M. Then $F^* = \Phi \cap \Gamma^*$ is a normal subgroup of Γ^* and $\Gamma^* = \langle F^*, l \rangle$. The action of the element l on F^* by conjugation is induced by the action of some homeomorphism λ of a compact surface S for which $\pi_1(S) \simeq F^*$. Let $\gamma_i \subset \partial S$ be oriented boundary curves whose homotopy classes $[\gamma_i]$ correspond to elements $\beta_i \in H_i^* \cap F^*$, $i = 1, 2$. Without loss of generality we may assume that $\lambda(\partial S) = \partial S$; therefore there exists a number $n \in \mathbf{Z} \setminus \{0\}$ such that $\lambda^n(\gamma_i) = \gamma_i$, $i = 1, 2$. We denote by M_0 the manifold obtained from $S \times [0, 1]$ by identifying the points $(x, 0)$ and $(\lambda^n(x), 1)$, $x \in S$. The manifold M_0 is a bundle over a circle and a typical fiber S^* of this bundle is the image of the surface $S \times \{0\}$ under the quotient mapping. Then the intersection of S^* with the component of ∂M_0 on which the image γ_i lies consists of only the curve γ_i, $i = 1, 2$. Hence it easily follows that there exist elements $t_i \in \tilde{H}_i = H_i^* \cap (\Gamma_0 = \langle F^*, l^n \rangle)$ such that $\langle F^*, t_i \rangle = \Gamma_0$. Obviously $|\Gamma : \Gamma_0| < \infty$.

(b) We have already shown that $\mathrm{cl}\,(S^3 \setminus \mathcal{P}(\tilde{H}_i)) \subset \mathcal{P}(\tilde{H}_j)$, $i \neq j$, $i, j \in \{1, 2\}$. Therefore by Klein's combination theorem the set $\mathcal{P}(\tilde{H}_1) \cap \mathcal{P}(\tilde{H}_2)$ is a fundamental region for a group of Schottky type $\tilde{H} = \langle \tilde{H}_1, \tilde{H}_2 \rangle \simeq \tilde{H}_1 * \tilde{H}_2$ [6]. Thus the set $R = \mathcal{P}(\tilde{H}_1) \cap \mathcal{P}(\tilde{H}_2) \cap \mathrm{cl}\,(V_1 \cup V_2)$ cannot have equivalent points with respect to the action of the group $\tilde{H}$. The closure of the set $T = R \cap (W_1 \cup W_2)$ is compact in B and therefore there exists at most a finite number of elements $g_m \in \Gamma_0$ such that $g_m(T) \cap T \neq \varnothing$, $m = 1, \ldots, K$. The group $\tilde{H}$ is geometrically finite [6] and by Lemma 1 Γ_0 is an $\tilde{H}$-finitely approximable group. Consequently there exists a subgroup of finite index $\tilde{\Gamma} \subset \Gamma_0$, in which $\tilde{H}$ is contained, but which contains none of the elements g_m. Obviously for all $g \in \tilde{\Gamma}$ we have $g(R) \cap R = \varnothing$ and R is a fundamental set for the action of the group $\tilde{\Gamma}$ in the orbit $\tilde{\Gamma}(\mathrm{cl}\,(V_1 \cup V_2))$. It is also clear that $\tilde{\Gamma}$ satisfies condition (a) in the statement of this lemma.

We choose an arbitrary fundamental set A for the action of the group $\tilde{\Gamma}$ in $B \setminus \tilde{\Gamma}(\mathrm{cl}(V_1 \cup V_2))$. The set $\mathcal{P} = R \cup A$ will be fundamental in B, and condition (b) will hold for it. Lemma 2 is now proved.

4.4. Let τ_i be reflection in the plane Π_i ($i = 1, 2$). We introduce the following notation: $\Gamma_1 = \tilde{\Gamma}$, $\Gamma_2 = \tau_1 \Gamma_1 \tau_1$, $G_1 = \langle \Gamma_1, \Gamma_2 \rangle$, $G_2 = \tau_2 G_1 \tau_2$, $G = \langle G_1, G_2 \rangle$. In what follows we shall write H_1 and H_2 instead of $\tilde{H}_1$ and $\tilde{H}_2$.

Lemma 3. *The group G_1 is discontinuous and contains a finitely generated normal subgroup F_1 such that $G_1/F_1 \simeq \mathbf{Z}$, $G_1 = \langle F_1, t \rangle$, $t \in H_2$.*

Proof. Consider the fundamental set $\mathcal{P}_1 = \mathcal{P}$ of the group Γ_1, which was constructed in Lemma 2. The group $H_1 = \Gamma_1 \cap \Gamma_2$ stabilizes the plane $\overline{\Pi}_1$ in the groups Γ_1 and Γ_2. By assertion (b) of Lemma 2 and the maximality of the parabolic subgroup $H_1 \subset \Gamma_1$ the region $\mathrm{cl}\,\Pi_1^-$ is precisely invariant with respect to H_1 in the group Γ_1. Similarly the region $\tau_1(\mathrm{cl}\,\Pi_1^-)$ is precisely invariant with respect to H_1 in the group Γ_2. Thus all the hypotheses of the first combination theorem of Maskit [13] are satisfied (the multidimensional variant of the combination theorem can be found, for example, in [14]). Consequently the group G_1 is discontinuous and isomorphic to $\Gamma_1 *_{H_1} \Gamma_2$, and the set $R_1 = \mathcal{P}_1 \cap \tau_1(\mathcal{P}_1)$ is fundamental for the action of G_1 on the invariant component $\Omega_1 \subset \Omega(G_1)$ containing the point ∞.

Suppose further that $F_1 = \langle \tilde{F}, \tau_1 \tilde{F} \tau_1 \rangle$, where the group $\tilde{F}$ is the normal subgroup of $\Gamma_1 = \Gamma$ constructed in Lemma 2. Then $G_1 = \langle F_1, t_1 \rangle$ and F_1 is normal in G_1 and finitely generated. It remains only to remark that $t_1 \in \Gamma_1 = \langle \tilde{F}, t_2 \rangle$, and so $G_1 = \langle F_1, t \rangle$, $t = t_2 \in H_2$. Lemma 3 is now proved.

Proposition 1 (cf. also [15]). *The manifold $M(G_1) = \Omega_1/G_1$ is homeomorphic to the interior of a bundle over a circle whose fiber is a compact surface with $\pi_1(\Omega_1) = \{1\}$.*

Proof. It follows from the geometrical decomposition of the group $G_1 = \Gamma_1 *_{H_1} \Gamma_2$ that $M(G_1)$ is obtained by gluing together two manifolds M_1 and M_2, where $M_1 = M(\Gamma_1) \setminus (\Pi_1^-/H_1)$ and $M_2 = M(\Gamma_2) \setminus (\tau_2 \Pi_1^-/H_1)$. We further have $\Pi_1^-/H_1 \cong \tau_1 \Pi_1^-/H_1 \cong S^1 \times S^1 \times (0,1)$. Consequently each of the manifolds M_i is homeomorphic to a bundle over a circle and the interior of M_i is a finite-sheeted covering of the original manifold M.

The gluing homeomorphism $f : \partial M_1 \to \partial M_2$ preserves the bundle structure, since it is covered by the identity homeomorphism $\tilde{f} : \overline{\Pi}_1 \to \overline{\Pi}_1$. It follows from the Seifert-van Kampen theorem that $\pi_1(M(G_1)) \simeq \Gamma_1 *_{H_1} \Gamma_2 \simeq G_1$. The group $G_1 \subset \mathrm{M\ddot{o}b}\,(S^3)$ is a Hopf group [12], and therefore $\pi_1(\Omega_1) = \{1\}$. Each of the manifolds M_i admits a compactification (by adjoining tori); therefore $M(G_1)$ is also compactifiable. Proposition 1 is now proved.

4.5. We set $F = \langle F_1, F_2 \rangle$, where $F_2 = \tau_2 F_1 \tau_2$ and $G = \langle G_1, G_2 \rangle$.

Lemma 4. *The following assertions hold:*

(A) *The group G is the result of Maskit combination of the groups G_1 and G_2.*

(B) *The group G is discontinuous and possesses an invariant component $\Omega \subset \Omega(G)$ containing the point ∞.*

(C) *The finitely generated group F is normal in G.*

(D) *The manifold $M(G) = \Omega/G$ is homeomorphic to thc interior of a compact manifold.*

Proof. (A) Let $H_3 = \tau_1 H_2 \tau_1$ and $H = \langle H_2, H_3 \rangle$. By Lemma 3 the group G_1 acts discontinuously on $\Omega_1 \ni \infty$ and has a fundamental set $R_1 = \mathcal{P}_1 \cap \tau_1 \mathcal{P}_1$. It follows from Lemma 2 that $R_1 \cap \mathrm{cl}\,\Pi_2^-$ is a fundamental set for the action of the group H on $\mathrm{cl}\,\Pi_2^-$. Moreover in the neighborhood $V = V_2 \cap \tau_1 V_2$ of the set $\overline{\Pi}_2 \setminus \Lambda(H)$ we have $R_1 \cap V = \mathcal{P}(H_2) \cap \mathcal{P}(H_3) \cap V$ (Lemma 2) and the open surface $\overline{\Pi}_2 \setminus \Lambda(H)$ is precisely invariant with respect to H in the group G_1. Consequently there exists a neighborhood $\mathcal{N}$ of the surface $\overline{\Pi}_2 \setminus \Lambda(H) \subset \Omega(G_1)$ such that $\mathcal{N} \subset \Omega(G_1)$ and $\mathcal{N}$ is precisely invariant with respect to H in G_1. To verify assertion (A) it now remains to prove the following result.

Proposition 2. *The sphere $\overline{\Pi}_2$ is precisely invariant under the action of the group $H \subset G_1$.*

Proof. Assume that there exists an element $g \in G_1 \setminus H$ such that $g(\Pi_2) \cap \Pi_2 = \{x\} \subset \Lambda(H)$. The group H of Schottky type is geometrically finite [6, 16], and so the following alternative holds [17]: either 1) x is an approximation point for the group H, or 2) x is a fixed point of a parabolic element $\gamma \in H$.

In the first case there exists a sequence $h_n \in H$ such that $\lim h_n(x) = x_0 \in \Pi_2$ and $x_0 \neq y_0 = \lim_{n\to\infty} h_n(z)$ for any point $z \in \operatorname{cl}\Pi_2^- \setminus \{x\}$, and $y_0 \in \Pi_2$. It follows from this that the sequence of spheres $h_n g(\overline{\Pi}_2)$ converges to $\overline{\Pi}_2 \subset S^3$. Consequently $h_n g(\mathcal{N}) \cap \mathcal{N} \neq \varnothing$ for large values of n. The latter is impossible by the precise invariance of $\mathcal{N}$ with respect to H in the group G_1.

In the second case there exist elements h and h' such that $hgh'(\{p_2, p_3\}) = \{p_2, p_3\}$, where $p_3 = \tau_1(p_2)$. From the maximality and nonconjugacy of the parabolic subgroups $H_2, H_3 \subset G_1$ it follows that $g \in H$ and this is impossible. Proposition 2 is now proved.

(B) Assertion (B) follows immediately from (A) and Maskit's combination theorem.

(C) We shall verify that the inclusion $g_1 F_2 g_1^{-1} \subset F = \langle F_1, F_2\rangle$ holds for any $g_1 \in G_1$. The element g_1 has the form $f_1 t^n$, where $f_1 \in F_1$, $t \in H_2 \subset G_2 \cap G_1$, $G_2 = \langle F_2, t\rangle$ (cf. Lemma 3). Thus $g_1 F_2 g_1^{-1} = f_1 t^n F_2 t^{-n} f_1^{-1} = f_1 F_2 f_1^{-1} \subset F$. Analogously $g_2 F_1 g_2^{-1} \subset F$ for any $g_2 \in G_2$. It follows from this that F is normal in G and assertion (C) is thus verified.

(D) As we have already seen, both manifolds $M(G_1)$ and $M(G_2)$ admit a natural compactification by the adjunction of cusped tori. Consequently the manifolds $M^-(G_1) = M(G_1) \setminus (\Pi_2^-/H)$ and $M^-(G_2) = M(G_2) \setminus (\tau_2(\Pi_2^-))/H$ also admit a compactification. Therefore the manifold $M(G)$ obtained by gluing together $M^-(G_1)$ and $M^-(G_2)$ along the compact boundary surface $S_2 = (\overline{\Pi}_2 \setminus \Lambda(H))/H$ is also compactifiable. Lemma 4 is now proved.

By assertions (B) and (C) of Lemma 4 the groups G and F have a common invariant component $\Omega \ni \infty$. We set $M(F) = \Omega/F$.

4.6. Lemma 5. *The group $\pi_1(M(F))$ is not finitely generated.*

Proof. Step 1. We begin by verifying that the orbits of $G_1(\Pi_2^-)$ and $F_1(\Pi_2^-)$ coincide. Indeed $G_1 = \langle F_1, t\rangle$, $t \in H_2$, $t(\Pi_2^-) = \Pi_2^-$. Hence $G_1\Pi_2^- = F_1\Pi_2^-$. We shall further show that $G = \langle F, t\rangle$. For any element $g \in G$ the decomposition $g = g_1 g_2 \cdots g_n$ $(g_i \in G_1 \cup G_2)$ holds and from the equality $g_i = f_i t^{m_i}$ $(f_i \in F_1 \cup F_2,\ t \in H \setminus F)$ we obtain $g = ft^m$, $f \in F$, $m \in \mathbf{Z}$. In analogy with Lemma 3 the subgroup F is normal in G.

Remark. We are not asserting here that $G/F \simeq \mathbf{Z}$. This will follow from the reasoning below.

Step 2. By the construction we have $\tau_2 G \tau_2 = G$. Therefore using the covering $p : \Omega \to \Omega/G = M(G)$ the involution τ_2 projects to an involution $\bar\tau_2 : M(G) \to M(G)$. Obviously the surface $S_2 = p(\overline{\Pi}_2 \setminus \Lambda(H))$ is the fixed set for this involution. Similarly the involution τ_2 projects to an involution $\hat\tau_2 : \Omega/F \to \Omega/F = M(F)$. Thus we have the commutative diagram

$$\begin{array}{ccccc} \Omega & \xrightarrow{q} & M(F) & \xrightarrow{r} & M(G) \\ \downarrow \tau_2 & & \downarrow \hat\tau_2 & & \downarrow \bar\tau_2 \\ \Omega & \xrightarrow{q} & M(F) & \xrightarrow{r} & M(G) \end{array},$$

where $p = r \circ q$ and r is a regular covering with the group of covering transformations G/F. The surface $\hat{S} = r^{-1}(S) = q(\overline{\Pi}_2 \setminus \Lambda(H))$ is connected (Step 1) and coincides with the fixed set of the involution $\hat\tau_2$.

Step 3. Since the group G is the result of the Maskit combination of the groups G_1 and G_2, the region $\Omega_1 \setminus G_1(\Pi_2^-)$ is contained in Ω and $p(\Omega_1 \setminus G_1(\Pi_2^-))$ is the closure of one of the components of $M(G) \setminus S_2$. We denote this closure by $M^-(G)$ and use $M^-(F)$ to denote the preimage $r^{-1}(M^-(G))$. On the other hand, $M^-(F)$ and $M^-(G)$ are homeomorphic to $M(F_1) \setminus (\Pi_2^-/H \cap F)$ and $M(G_1) \setminus (\Pi_2^-/H)$ respectively. Thus the covering $r : M^-(F) \to M^-(G)$ is the restriction of the infinite cyclic covering $M(F_1) \to M(G_1)$.

Step 4. As we have already seen in Lemma 4, the manifold $M^-(G)$ can be compactified to a manifold $N^-(G)$. The boundary component $S_2 \subset \partial \operatorname{cl} M^-(G)$ is a compact surface of genus 2 (since $H \simeq (\mathbf{Z} \oplus \mathbf{Z}) * (\mathbf{Z} \oplus \mathbf{Z})$ acts as a group of Schottky type on the sphere $\overline{\Pi}_2$). Consequently the manifold $N^-(G)$ cannot be a total bundle space over the circle. Moreover neither of the manifolds $M^-(G)$ and $N^-(G)$ contains any fake cells, since they can be covered by a region in $\mathbf{R}^3$.

Step 5. *The group $\pi_1(M^-(F))$ is not finitely generated.*

Proof. By Step 3 we have an exact sequence

$$1 \to \pi_1(M^-(F)) \to \pi_1(N^-(G)) \to \mathbf{Z} \to 1.$$

Assume that the group $\pi_1(M^-(F))$ is finitely generated. The manifold $M^-(G)$ contains no projective planes in view of the $\mathbf{P}^2$-irreducibility of the manifold $M(G_1)$. Further $\pi_1(M^-(F))$ is not an abelian group, and so it follows from [7, Theorem 11.1] that $N^-(G)$ is homeomorphic to a bundle over S^1. The last result contradicts Step 4.

Step 6. It remains for us to verify that $\pi_1(M(F))$ is also not a finitely generated group. Let $\omega : \tilde{M} \to M(F)$ be the universal covering with group of covering transformations $\pi = \pi_1(M(F))$. We remark that the manifold $M^-(F)$ is homeomorphic $M(F)/\hat{\tau}_2$. Consider the lifting $\tilde{\tau}_2 : \tilde{M} \to \tilde{M}$ of the involution $\hat{\tau}_2$. We have $\tilde{\tau}_2 \pi \tilde{\tau}_2 = \pi$ and the group $\mathcal{G} = \langle \pi_1(M(F)), \tilde{\tau}_2 \rangle$ acts discontinuously on $\tilde{M}$. We denote the normal subgroup of $\mathcal{G}$ generated by the elements of finite order by TORS.

By Armstrong's theorem [18] the group $\pi_1(M^-(F))$ is isomorphic to $\mathcal{G}/\mathrm{TORS}$ and so $\mathcal{G}$ is infinitely generated. One can easily see that the group $\pi_1(M(F))$, being a subgroup of index 2 in the infinitely generated group $\mathcal{G}$, also cannot be finitely generated. Lemma 5 is now proved.

By construction the group $F = \langle F_1, F_2 \rangle \subset \mathrm{M\ddot{o}b}\,(S^3)$ is finitely generated and its quotient manifold $M(F) = \Omega/F$ has an infinitely generated fundamental group. We shall show finally that the group F is infinitely defined.

The group $I = H \cap F$ is the stabilizer in F of the sphere $\overline{\Pi}_2$. It follows immediately from the fact that G is the result of the Maskit combination of the groups G_1 and G_2 that F is also obtained from F_1 and F_2 by a Maskit combination. Therefore $F \simeq F_1 *_I F_2$ is the free product with the combined subgroup I. We remark that the subgroup I is normal in H (by the normality of F in G) and has infinite index, since $G/F \simeq \mathbf{Z}$ and $t^n \notin F$ for $n \in \mathbf{Z} \setminus \{0\}$. It follows easily from this that the group I is infinitely generated. The fact that the group F is infinitely defined now follows immediately from the results of [19].

Theorem 1 is now proved.

5. Proof of Theorem 2

Consider the group $\Gamma_1' = \Gamma_1$ constructed in Lemma 2; $\Gamma_2 = \tau_1 \Gamma_1 \tau_1$, and in the group Γ_1 there is a parabolic element $\beta \in \tilde{F} \subset \Gamma_1$ (cf. the proof of Lemma 2, part (a)). We denote by L_2 the plane parallel to Π_2 and tangent to the isometric spheres of the elements β_2 and β_2^{-1}, $L_2 \subset \Pi_2^-$ (cf. Fig. 2). We set $\bar{L}_2 = L_2 \cup \{\infty\}$ and let L_2^- be the component of $\mathbf{R}^3 \setminus L_2$ contained in Π_2^-. Let Θ_2 be a sphere tangent to Σ_1 at the point p_1 so that $\operatorname{int}\Theta_2 \supset \operatorname{int}\Sigma_2$; $x = L_2 \cap I(\beta_2)$, $y = L_2 \cap I(\beta_2^{-1})$, $z = \Theta_2 \cap L_2$. Then there exists a unique transformation $T \in \mathrm{M\ddot{o}b}\,(S^3)$, that commutes with each element of the group H_1 and maps the point p_3 to the point z.

Remark. Passing to a subgroup of finite index $\langle \tilde{F}, t_2^n \rangle \subset \Gamma_1$ if necessary, we may assume that for any $h \in H_2 \setminus \{\beta_2, \beta_2^{-1}\}$ the intersection $I(h) \cap L_2$ is empty.

5.1. It is easy to see that $\tau_1 \circ T^{-1}(L_2^-) \subset \Pi_2^-$ and the sphere $\tau_1 \circ T^{-1}(\bar{L}_2)$ is tangent to Σ_1 at the point p_1. It therefore follows from Lemma 2 that L_2^- is precisely invariant with respect to the subgroup $H_3' = TH_3T^{-1}$ in the group $\Gamma_2' = T\Gamma_2T^{-1}$. Also by Lemma 2 the hypotheses of Maskit's first combination theorem hold for the groups Γ_1' and Γ_2'.

We denote by G_1' the group $\langle \Gamma_1', \Gamma_2' \rangle$. It follows from Maskit's combination theorem that the group G_1' is Kleinian and has an invariant component $\Omega_1' \ni \infty$.

Lemma 6. *The region Ω_1' is simply connected. The quotient manifold Ω_1'/G_1' is a bundle over the circle formed by gluing together two hyperbolic manifolds that are bundles over S^1 and homeomorphic to $(B \setminus \Gamma_1'(\Pi_1^-))/\Gamma_1'$. The group $F_1' = \langle \tilde{F}, T\tau_1\tilde{F}\tau_1T^{-1} \rangle$ is normal in G_1' and corresponds to the fundamental group of a fiber of the manifold $M(G_1')$. There exists a fundamental set D_1 for the action of G_1 on Ω_1' such that*

1) *$(D_1 \cap \operatorname{cl} L^-) \cup \{y\}$ is a fundamental region for the action of the group H_3' in $\operatorname{cl} L_2^-$;*

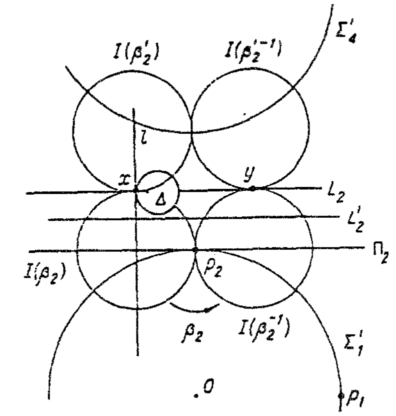

Fig. 3

2) *for some plane L_2' parallel to L_2 and lying between L_2 and Π_2 the intersection $L_2' \cap D_1 \cap \tau_1(\Pi_1^-)$ coincides with $L_2' \cap \mathcal{P}(\langle\beta_2\rangle) \cap \tau_1(\Pi_1^-)$.*

Proof. All the assertions except the last are proved as in Lemma 2 and Proposition 1. To prove property 2) of the set D_1 it suffices to use the remark preceding Lemma 6. As the set D_1 we choose $(\mathcal{P} \setminus \Pi_1^-) \cup (T\tau_1(\mathcal{P}) \setminus \tau_1(\Pi_1^-))$. For the definition of the set $\mathcal{P}$ see the proof of Lemma 2. Lemma 6 is now proved.

5.2. We introduce the following notation: $J = H_3'$, $X_2 = L_2^- \cap (\bar{L}_2 \setminus J(\{x,y,z\})$, τ_2' is a symmetry in the plane L_2, $X_1 = \tau_2'(X_2)$, $G_2' = \tau_2' G_1' \tau_2'$, $F_2' = \tau_2' F_1' \tau_2'$, $D_2 = \tau_2 D_1$.

Direct verification shows that the sets X_1 and X_2 are interactive (in the sense of Maskit [20, Ch. VII]) for the pair of groups (G_1', G_2'), i.e., $J(X_i) = X_i$ and any element $G_1' \setminus J$ maps X_1 into X_2. In addition $D_i \cap X_i \cap \bar{L}_2$ is a fundamental set for the action of the group J in $X_i \cap \bar{L}_2$, $i = 1, 2$. It follows from this that $g(D_1 \cap X_2) \subset X_2$ for any element $g \in G_1'$ and conversely, for any $g \in G_2'$ the inclusion $g(D_2 \cap X_1) \subset X_1$ holds.

Lemma 7. 1. *The groups G_1' and G_2' satisfy the hypotheses of the weak combination theorem of Maskit* [20, Ch. VII, Theorem A.15]).

2. *The group $G' = \langle G_1', G_2' \rangle$ is isomorphic to $G_1' *_J G_2'$.*

3. *The set $D = (D_1 \cap X_1) \cup (D_2 \cap X_2)$ contains no points that are equivalent with respect to G', and* $\operatorname{int} D$ *is contained in $\Omega(G')$.*

Proof. The first assertion follows from the reasoning preceding the lemma. The second and third follow from Maskit's weak combination theorem. Lemma 7 is now proved.

We denote by Ω' the component of $\Omega(G')$ containing ∞. It is easy to see that Ω' is invariant with respect to G'. Let $\beta_2' = \tau_2' \beta_2 \tau_2'$. Obviously the element $u = (\beta_2')^{-1} \circ \beta_2$ is parabolic, is conjugate to a shift in $\bar{\mathbf{R}}^3$, and leaves the point x fixed.

5.3. Lemma 8. *Let G^n be a conformal extension of the group G' to $\mathbf{R}^n$, $n \geq 3$. Then x is a cusped parabolic point in the group G^n.* (For the definition of a cusped point cf., for example [21].)

Proof. We begin by constructing a cusped neighborhood for the point $x \in \bar{\mathbf{R}}^3$. We denote by Π the plane passing through the points 0, p_1, and p_2. Let l be the line containing the point x and perpendicular to the plane Π_2 (Fig. 3). Let Δ be any closed disk lying in the plane Π and tangent to l at the point x such that the diameter of Δ is less than the radius of $I(\beta_2)$ and $\Delta \cap L_2' = \varnothing$ (cf. Lemma 6).

As a cusped neighborhood $\mathcal{O}$ of the point x we choose the set obtained from Δ by rotation about the axis l. It is obvious that $u(\mathcal{O}) = \mathcal{O}$. We shall show that $\mathcal{O}$ is precisely invariant in G' with respect to $\langle u \rangle$. By Lemmas 6 and 7 the intersection $\mathcal{O}_- = (\mathcal{O} \setminus \{x\}) \cap \operatorname{cl}\operatorname{ext}(I(\beta_2')) \cap \operatorname{cl}\operatorname{ext}(I(\beta_2))$ lies in the region D and contains no points that are equivalent with respect to G'. Let w be the shift in $\mathbf{R}^3$ along the line $L_2 \cap \Pi$ that takes the point x to the point y. Then the set $w(\mathcal{O}_-) \setminus (I(\beta_2^{-1}) \cup I((\beta_2')^{-1}) = \mathcal{O}_-$ is also contained in D. It follows from this that $\mathcal{O}_- \cup \beta_2^{-1}(\mathcal{O}_-)$ has no points that are equivalent (with respect to G') and is a fundamental set for the action of $\langle u \rangle$ in $\mathcal{O}$. Thus $\mathcal{O}$ is a cusped neighborhood of the parabolic point x in the group G'.

Consider the fractional linear transformation γ that makes the element u conjugate to the shift $U : \mathbf{x} \mapsto \mathbf{x} + \mathbf{e}_1$, where $\mathbf{e}_1$ is the vector with coordinates $(1, 0, 0)$. Then $Q = \mathbf{R}^3 \setminus \gamma(\mathcal{O})$ is a solid Euclidean cylinder with axis $\lambda_1 = \mathbf{R} \cdot \mathbf{e}_1$ and a certain radius r. In the group $G_* = \gamma G' \gamma^{-1}$ the region $\mathcal{O}_* = \mathbf{R}^3 \setminus Q$ is precisely invariant with respect to $\langle U \rangle$. Let G_*^n be a conformal extension of the group G_* in $\bar{\mathbf{R}}^n$ and Q^n the solid Euclidean cylinder in $\mathbf{R}^n$ of radius $3r$ with axis λ_1. We shall show that the region $\mathbf{R}^n \setminus Q^n$ is precisely invariant with respect to $\langle U \rangle$ in the group G_*^n.

We remark that for an arbitrary element $g \in G_*$ the center of the isometric sphere $I(g)$ lies inside Q (otherwise the limit point $g^{-1}(\infty)$ would lie in the precisely invariant region $\mathcal{O}_*$). We shall verify that the radius of $I(g)$ is less than $2r$. Indeed elementary computations show that for radius of the sphere $I(g)$ larger than $2r$ the area of $I(g) \cap Q$ is less than half the area of $I(g)$. However the radii of $I(g)$ and $I(g^{-1})$ are equal and $g : I(g) \to I(g^{-1})$ is a Euclidean isometry. Therefore $g(I(g) \cap \mathcal{O}_*) \cap \mathcal{O}_* \neq \varnothing$, contradicting the precise invariance of the region $\mathcal{O}_*$. Thus each sphere $I(g)$ lies inside a cylinder Q^n, from which it follows that the region $\mathbf{R}^n \setminus Q^n$ is precisely invariant. Lemma 8 is now proved.

Remark. Unfortunately we were unable to use either Lemma 4.15 of [22] or its proof in our reasoning, since there are errors in the proof [22, p. 94].

Corollary. *The group $\langle u \rangle$ is a maximal parabolic subgroup of G'.*

We shall use the notation $\mathcal{O}_0^n$ below to denote the cusped neighborhood $\gamma^{-1}(\mathbf{R}^n \setminus Q^n)$.

5.4. As already noted in Lemma 6, the quotient manifolds $M(G_1')$ and $M(G_2')$ are bundles over a circle whose fibers $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond to normal subgroups $F_1' \subset G_1'$ and $F_2' \subset G_2'$. Let T_i be the peripheral tori in $M(G_i')$ corresponding to the parabolic subgroup $J = G_1' \cap G_2'$ and $\tilde{T}_i$ a component of $M(G_i') \setminus T_i$ homeomorphic to $(-\infty, \infty) \times T^2$. Consider the manifold N obtained from $M(G_i') \setminus \tilde{T}_i$ by compactification and gluing using a homeomorphism $h : T_1 \to T_2$ that induces the identity mapping $J \to J$.

We set $\beta_3 = \tau_1 T \beta_2 T^{-1} \tau_1$, $I = \langle \beta_3 \rangle = F_i' \cap J$, $i = 1, 2$. Without loss of generality we may assume that $h(\mathcal{F}_1 \cap T_1) = \mathcal{F}_2 \cap T_2$, so that the manifold N is also homeomorphic to a bundle over a circle whose typical fiber $\mathcal{F}$ is formed by gluing together the compactifications of the surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$ along $\mathcal{F}_i \cap T_i$. It follows from the Seifert-van Kampen theorem and Lemmas 6 and 7 that there exists an isomorphism $j : \pi_1(N) \to G'$, $j : \pi_1(\mathcal{F}_i) \to F_i'$, $i = 1, 2$. Thus the group $F' = \langle F_1', F_2' \rangle$ is isomorphic to $F_1' *_I F_2'$ and normal in G', and the action of the cyclic quotient group G'/F' is induced by some homeomorphism $\theta : \mathcal{F} \to \mathcal{F}$. The manifold N is formed by gluing together four copies $\bar{M}_i$ of the compactification of the manifold $M(\Gamma_i) = \mathbf{H}^3/\Gamma_1$. We denote the boundary tori of the $\bar{M}_i$ along which the gluing is done by $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$; they correspond to parabolic subgroups of rank 2 in G'. Here $\mathcal{F} \cap \mathcal{T}_i$ consists of a single loop α_i (Lemma 2). We shall denote the element of $\pi_1(N)$ corresponding to it by $[\alpha_i]$.

We remark that for all i there exists a parabolic subgroup of rank 2 in G' containing $j([\alpha_i])$ while for the element u there is no such subgroup (cf. the corollary to Lemma 8). Therefore $j([\alpha_i])$ and u are not conjugate in the group G' and *a fortiori* they are not conjugate in $F' \ni u = \beta_2(\beta_2')^{-1}$. For the same reasons for any loop $\delta \subset \partial\mathcal{F}$ the elements $j([\delta])$ and u are not conjugate in G'. We note also that $\theta(\alpha_i) = \alpha_i$, $i = 1, 2, 3$.

5.5. The manifold N can be obtained from $\mathcal{F} \times [0, 1]$ by identifying the points $(x, 0)$ and $(\theta(x), 1)$ for $x \in \mathcal{F}$. We denote by $\omega : S^1 \to \mathcal{F}$ the loop corresponding to the element u under the isomorphism $j : \pi_1(\mathcal{F}) \to F'$.

5.6. Lemma 9. *For any $m, k \in \mathbf{Z}$, $m \neq k$, the loops $\theta^k(\omega)$ and $\theta^m(\omega)$ are not freely homotopic on the surface $\mathcal{F}$.*

Proof. Denote by ν the loop $\theta^k(\omega)$. Then $\theta^m(\omega) = \theta^n(\omega)$, $n = m - k \neq 0$. Assume that the loops ν and $\theta^n(\nu)$ are freely homotopic on $\mathcal{F}$ and that $\mu : S^1 \times [-1, 0] \to \mathcal{F}$ is the corresponding homotopy. The manifold $\tilde{N} = \mathcal{F} \times [0, 1]/\theta^n$ is an n-sheeted regular covering of N. Consider the continuous mapping $\tilde{\eta} : [-1, 1] \times S^1 \to \mathcal{F} \times [0, 1]$ such that the restriction of $\tilde{\eta}$ to $[-1, 0] \times S^1$ coincides with μ and the restriction of $\tilde{\eta}$ to $[0, 1]$ is given by the formula $\tilde{\eta}(t, x) = (t, \theta^n \circ \nu(x))$. It is obvious that $\tilde{\eta}$ projects to a continuous mapping $\eta : S^1 \times S^1 \to \mathcal{F} \times [0,1]/\theta^n = \tilde{N}$. Passing to a covering of $\tilde{N}$ with defining subgroup $\pi_1(\mathcal{F})$, we see that the nontriviality of the loop ω implies that the mapping $\eta_* : \pi_1(S^1 \times S^1) \to \tilde{N}$ is injective. Thus $\eta(T^2)$ is an incompressible singular torus in the manifold $\tilde{N}$.

We now lift the tori $\mathcal{T}_i$ to tori $\tilde{\mathcal{T}}_i \subset \tilde{N}$, $i = 1,2,3$. Each component of $\tilde{N} \setminus (\tilde{\mathcal{T}}_1 \cup \tilde{\mathcal{T}}_2 \cup \tilde{\mathcal{T}}_3)$, is an n-sheeted covering of one of the hyperbolic manifolds $N \setminus (\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$, and is therefore itself hyperbolic and atoroidal. Consequently $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_1 \cup \tilde{\mathcal{T}}_2 \cup \tilde{\mathcal{T}}_3$ defines a "canonical system of tori" of the manifold $\tilde{N}$ (cf. [23]). Thus the components of a regular neighborhood $\mathfrak{X}$ of the submanifold $\partial\tilde{N} \cup \tilde{\mathcal{T}}$ are a complete set of characteristic submanifolds of $\tilde{N}$ [23]. By the results of [23] the continuous mapping $\eta : T^2 \to \tilde{N}$ is homotopic to some mapping $\eta : T^2 \to \mathfrak{X}$. It follows from this that the loop $\bar{\nu} = \eta\big|_{\{0\}\times S^1}$ is homotopic to a loop ϑ of $\mathfrak{X}$. Considering the elements $[\bar{\nu}]$ and $[\vartheta] \in \pi_1(\tilde{N})$ corresponding to $\bar{\nu}$ and ϑ, we verify that they are conjugate in the group $\pi_1(\tilde{N})$. However $[\bar{\nu}] = [\nu] \in \pi_1(\mathcal{T} \times \{0\})$ is the fundamental group of the fiber of the bundle $\tilde{N}$ and is normal in $\pi_1(\tilde{N}) \subset \pi(N)$.

Consequently $[\vartheta]$ also lies in $\pi_1(\mathcal{F}) \subset \pi_1(\tilde{N}) \subset \pi(N)$. Thus the loop $\nu \subset \mathcal{F} \subset \tilde{N}$ is freely homotopic to a loop of a regular neighborhood of $\partial\mathcal{F} \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$, which is impossible by Sec. 6.4. This contradiction proves Lemma 9.

5.7. Proof of assertion (a) of Theorem 2. We choose some element representing a generator of the group G'/F', for example $t_2 \in H_2$. By Lemma 9 for any $k \neq m \in \mathbf{Z}$ the elements $t_2^k u t_2^{-k}$ and $t_2^m u t_2^{-m}$ are not conjugate in the group F'. However $t_2^m \langle u \rangle t_2^{-m}$ is a maximal parabolic subgroup of G' (and hence also of F') for $m \in \mathbf{Z}$. Thus the group F', being a free group of finite rank, contains an infinite number of conjugacy classes of the maximal parabolic subgroups $\langle u_m \rangle = \langle t_2^m u t_2^{-m} \rangle$. This proves assertion (a).

5.8. Proof of assertion (b) of Theorem 2. The point x is a cusped parabolic point of the group $G^n \subset \text{Möb}\,(S^n)$ for $n \geq 3$. A cusped neighborhood of this point $\mathcal{O}_0^n$ was constructed in the proof of Lemma 8. Since the elements u_m and $u = u_1$ are conjugate in G', the point $t_2^m(x)$ is also cusped and $\mathcal{O}_m^n = t_2^m(\mathcal{O}_0^n)$ is a cusped neighborhood of it (for the group F^n). We denote by $E(n,m)$ the projection of $\mathcal{O}_m^n$ in the manifold $M(F^n) = \Omega(F^n)/F^n$. The manifold $E(n,m)$ is homeomorphic to $S^{n-2} \times S^1 \times [0,\infty)$, and the closed orientable submanifold $\partial E(n,m)$ in $M(F^n)$ is the boundary of a "parabolic end." If $m \neq k$, the parabolic ends corresponding to $\partial E(n,m)$ and $\partial E(n,k)$ are distinct and the manifold $M(F^n)$ possesses an infinite number of ends. It is easy to see that the system of cycles $\{[\partial E(n,m)],\ m \in \mathbf{Z}\}$ is linearly independent in $H_{n-1}(M(F^n), \mathbf{Z})$. Thus $\operatorname{rank} H_{n-1}(M(F^n), \mathbf{Z}) = \infty$, and Theorem 2 is proved completely.

6. Concluding remarks

6.1. In the theory of discrete subgroups of Lie groups the following theorem of Selberg is well-known [24].

Theorem C. *For any finitely generated subgroup Γ in the Lie group G the number of G-conjugacy classes of elements of Γ of finite order is finite.*

The following result also holds.

Theorem 3. *There exists a sequence of representations $\rho_n : F' \to \text{Möb}\,(S^3)$ that converges to $\rho_\infty = \text{id}$ and is such that for all $n, m \in \mathbf{Z}$ the order of the element $\rho_n(u_m)$ is finite.*

In a subsequent publication we shall show that the elements $\rho_n(u_m)$ and $\rho_n(u_i)$ are not conjugate in the group $\rho_n(F')$ for any $n \in \mathbf{N}$, $m, i \in \mathbf{Z}$, $m \neq i$.

Proof. We denote by $\Sigma_1(s)$ the sheaf of spheres tangent to one another at the point p_1, where $\Sigma_1(0) = \Sigma_1$ and $\Sigma_1(1)$ is the sphere whose radius is equal to the distance from the center of Σ_1 to the plane L_2. Let $p_2(s)$ be the point of $\Sigma_1(s)$ closest to the plane L_2.

We choose a parabolic transformation ζ_s that commutes with the group H_1 and maps the point p_2 to the point $p_2(s)$; $\zeta_s(\overline{\Pi}) = \overline{\Pi}$. Consider the parabolic element $\beta_2(s) = \zeta_s \beta_2 \zeta_s^{-1}$. It is easy to see that the isometric spheres $I(\beta_2(s))$ and $I(\beta_2^{-1}(s))$ meet L_2 at equal angles $\varphi(s)$, $\varphi(0) = 0$, $\varphi(1) = \pi/2$, $\varphi(s)$ being a continuous function. Let $s(n)$ be a sequence of numbers, $0 \leq s(n) \leq 1$, such that $\varphi(s(n)) = \pi/2n$. Let $\rho_n : \Gamma_1 \to \text{Möb}\,(S^3)$ be the homomorphism defined by the conjugation $\rho_n(\gamma) = \zeta_{s(n)} \gamma \zeta_{s(n)}^{-1}$; the restriction of ρ_n to Γ_2' is the identity. By the equality $G_1' = \Gamma_1' *_{H_1} \Gamma_2'$ and the fact that $\zeta_{s(n)}$ commutes with the group H_1, the mapping $\rho_n : G_1' \to \text{Möb}\,(S^3)$ is a homomorphism. We define a mapping $\rho_n : G_2' \to \text{Möb}\,(S^3)$ by the formula $\tau_2'\rho_n(\tau_2' g \tau_2')\tau_2' = \rho_n(g)$, $g \in G_2'$. It is obvious that the extension of ρ_n to the group G' is a homomorphism and that $\lim_{n\to\infty} \rho_n = \mathrm{id}$.

At the same time $\rho_n(u)$ is an elliptic element of order $n \in \mathbf{Z}$. Since $u = u_0$ and $u_m \in G'$ are conjugate in G', it follows that $\rho_n(u_m)$ is an element of finite order for all $n, m \in \mathbf{Z}$. Thus the sequence of homomorphisms ρ_n is the one sought. Theorem 3 is now proved.

6.2. Numerous variant proofs of the Ahlfors finiteness theorem based on topological and other ideas appeared in the mid-80's [25–27].

The idea that a normal subgroup in a geometrically finite group could be a counterexample to Ahlfors' Theorem in dimension 3 occurred to the second author of this paper while working on [15] (taking account of the Jaco-Hempel theorem [7, Theorem 11.1]). The first example was constructed by the authors in a joint paper [28]. As B. I. Apanasov has pointed out to us, a configuration of spheres similar to [28] covering a "trefoil" knot was used in his Theorem 7.21 of [22]. The group constructed in Theorem 7.21 of [22] was a free, geometrically finite group having a wild knot as limit set (cf. also [20, VIII.F]). In the present article the example of [28] has been significantly simplified and the original configuration of 52 spheres has been replaced by the four spheres Σ_i, $i = 1, \ldots, 4$ (Theorem 1). Theorems 2 and 3 are due to the second author.

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