Math. Ann. 299, 341–354 (1994)

Mathematische
Annalen

# Deformations of representations of discrete subgroups of SO(3, 1)

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Received April 27, 1993; in revised form September 23, 1993

*Mathematics Subject Classification (1991):* 57M50, 22E40, 30F40, 53C15

## 1 Introduction

Let $M$ be a closed hyperbolic 3-dimensional orbifold (see [T, Sc] for definitions), $\rho_0 : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ be its holonomy representation. Denote the conjugacy class of $\rho_0$ by $[\rho_0]$. In this paper we discuss whether for $n=4$ the point $[\rho_0]$ is isolated in the space

$$R(\pi_1(M), n) = \mathrm{Hom}(\pi_1(M), \mathrm{Isom}(\mathbb{H}^n))/\mathrm{Isom}_+(\mathbb{H}^n) .$$

If $[\rho_0]$ is isolated, then the corresponding representation is called ***locally rigid***.

A suborbifold $\Sigma$ in $M$ is said to be a virtual fiber in a fiber bundle over $\mathbb{S}^1$ if $M$ admits a finite-sheeted covering $p : M_0 \to M$ such that $M_0$ is fibered over a circle and a component of the preimage $p^{-1}\Sigma$ is a fiber of this fibration.

We start with the following

**Conjecture 1.** *The representation $\rho_0$ is not locally rigid if and only if $M$ contains an incompressible 2-suborbifold which is not a virtual fiber in a fiber bundle over $\mathbb{S}^1$* [Ka1, Ka2].

Certainly the case of manifolds is the most interesting and most complicated. The main aim of this paper is to show that Conjecture 1 is not absolutely groundless. First our results deal with reflection orbifolds. In this case we will prove Conjecture 1 and find that $R(\pi_1(M), 4)$ is a smooth manifold of dimension $(f-4)$, where $f$ is the number of "faces" of the reflection orbifold (Theorem 1). Then we shall consider orbifolds of finite volume and examine the "restriction" map

$$\partial_T : PR(\pi_1(M), 4) \to PR(\pi_1(T), 4)$$

where $T$ is an incompressible Euclidean suborbifold corresponding to a "cusp end" of $M$; $PR(\cdot)$ mean the space of representations whose restrictions on cyclic parabolic subgroups of $\pi_1(M) \subset \mathrm{Isom}(\mathbb{H}^3)$ are induced by conjugations in $\mathrm{Isom}(\mathbb{H}^4)$. Under some conditions the image of this map is 1-dimensional

(Example 1). The question about deformations of such kind arises naturally if we are trying to construct flat conformal structures on manifolds obtained by gluing of two hyperbolic ones along boundary tori [GLT, Ka1]. Unfortunately, in the general case, the map $\partial_T$ is zero to the first order (Theorem 3). We prove the "only if" part of Conjecture 1 for infinitely many non-Haken manifolds arising after Dehn surgery on 2-bridge knots (Theorem 2). These are the first examples of closed hyperbolic 3-manifolds $M$ whose fundamental group are locally rigid in $R(\pi_1(M), 4)$. Moreover, we prove that for each hyperbolic 2-bridge knot $K \subset \mathbb{S}^3$ there exists only one conjugacy class of discrete faithful representations of $\pi_1(\mathbb{S}^3 - K)$ into $\mathrm{Isom}(\mathbb{H}^4)$ (Sect. 5).

## 2 Preliminary geometric results

In this section we collect several elementary facts about geometry of Euclidean spheres in $\mathbb{S}^3$.

Denote by $\mathscr{C}$ the set of all Euclidean spheres in $\mathbb{S}^3$ of positive radius. Then $\mathscr{C}$ has a natural topology and is a smooth 4-manifold. By $\mathbf{Mob}(\mathbb{S}^n)$ we denote the group of Moebius transformations acting on $\mathbb{S}^n$. We shall suppose that the hyperbolic 3-space $\mathbb{H}^3$ is realized as a unit ball in $\mathbb{R}^3 \subset \overline{\mathbb{R}^3} = \mathbb{S}^3$; thus $\mathrm{Isom}(\mathbb{H}^3)$ is the group of Moebius transformations of $\mathbb{S}^3$ which leave $\mathbb{H}^3$ invariant. $\mathbf{Mob}_+(\mathbb{S}^n)$ is the subgroup of orientation-preserving Moebius transformations of $\mathbb{S}^n$.

**Lemma 2.1.** *Let $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \in \mathscr{C}^4$. Then the following trichotomy holds:*
*either* (i) *there is a sphere $\Sigma_0 \in \mathscr{C}$ orthogonal to all spheres $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$;*
*or* (ii) *$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \cap \Sigma_4 \ni p$, where $p$ is a point;*
*or* (iii) *spheres $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ are totally geodesic in some metric of constant positive curvature on $\mathbb{S}^3$.*

**Corollary 2.1.** *Let $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \in \mathscr{C}^4$. Denote by $\tau_j$ the inversion in the sphere $\Sigma_j$, let $\Gamma$ be the group generated by $\tau_j$. Then the following trichotomy holds:*
*either* (i) *$\Gamma$ is conjugate in* $\mathbf{Mob}(\mathbb{S}^3)$ *to a subgroup of* $\mathrm{Isom}(\mathbb{H}^3)$;
*or* (ii) *$\Gamma$ is conjugate in* $\mathbf{Mob}(\mathbb{S}^3)$ *to a subgroup of* $\mathrm{Isom}(\mathbb{E}^3)$;
*or* (iii) *$\Gamma$ is conjugate in* $\mathbf{Mob}(\mathbb{S}^3)$ *to a subgroup of* $\mathrm{Isom}(\mathbb{S}^3)$.

*Proof of Lemma 2.1.* We present the proof that was suggested to the author by N. Kuiper instead of the original one. Consider the sphere $\mathbb{S}^3$ as a round sphere in the affine space $\mathbb{A}^4 \subset \mathbb{RP}^4$; respectively $\mathbf{Mob}(\mathbb{S}^3) \subset PGL(5, \mathbb{R})$. Every sphere $\Sigma \in \mathscr{C}$ is the intersection of $\mathbb{S}^3$ with some affine hyperplane $P \subset \mathbb{A}^4$, $\Sigma = \Sigma(P)$. Denote by $P^* \in \mathbb{RP}^4$ the polar of $\Sigma$ with respect to $\mathbb{S}^3$ (i.e. such point that tangent cone from $P^*$ to $\mathbb{S}^3$ touches $\mathbb{S}^3$ at $\Sigma$). Denote by $\hat{P}$ the closure of $P$ in $\mathbb{RP}^4$.

Then it is easy to see that $\Sigma(P)$ is orthogonal to $\Sigma(Q)$ iff $P^* \in \hat{Q}$ (it is sufficient to consider first the case of $P^* \notin \mathbb{A}^4$ and then apply $\mathbf{Mob}(\mathbb{S}^3) \subset PGL(5, \mathbb{R})$). Now consider the polars $P_j^*$ corresponding to $\Sigma_j$ $(j = 1, \ldots, 4)$. Let $\hat{P}$ be the extended hypersubspace in $\mathbb{RP}^4$ which passes through these points. Then we have 3 possibilities:

(i) The intersection $P \cap \mathbb{S}^3$ is a sphere of positive radius. This is the desired sphere $\Sigma$.

(ii) The intersection $P \cap \mathbb{S}^3$ is a point $\infty \in \mathbb{S}^3$. Then $P_j \ni \infty$, so we have the case (ii) of lemma.

(iii) The intersection above is empty. Then $P^*\in \mathrm{int}(\mathbb{S}^3)$ and the point $P^*$ lies in the intersection $P_1\cap P_2\cap P_3\cap P_4$. Applying a projective transformation from $\mathbf{Mob}(\mathbb{S}^3)$ we can map the point $P^*$ in the center of $\mathbb{S}^3$. So, $\Sigma_j$ are "great spheres" in $\mathbb{S}^3$ which are totally geodesic in elliptic geometry. Lemma is proved. ☐

*Remark 1.* Independently a generalization of Lemma 2.2 to higher dimensions was proven in [Lu].

**Corollary 2.2.** *Let $l$ and $k$ be two unlinked Euclidean circles in $\mathbb{S}^3$, $l\cap k=\varnothing$. Then there exists a Euclidean sphere $\Sigma_0\subset\mathbb{S}^3$ which is orthogonal to $l$ and $k$.*

*Proof.* We can realize $l$ and $k$ as intersections $l=\Sigma_1\cap\Sigma_2$ and $k=\Sigma_3\cap\Sigma_4$, so that the case (i) of Lemma 2.1 holds for the collection

$$(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)\ .$$

Then the sphere $\Sigma_0$ from Lemma 2.1 is orthogonal to $l$ and $k$. ☐

*Remark 2.* If the sphere $\Sigma_0$ (case (i) of Lemma 2.1) is unique (i.e. when $\Sigma_j$ are not orthogonal to a common circle), then it smoothly depends on

$$(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)\in\mathscr{C}^4\ .$$

**Notation.** If $\Sigma_1$, $\Sigma_2$ are spheres in $\mathbb{E}^3$ then by

$$0\leqq\alpha(\Sigma_1, \Sigma_2)\leqq\pi$$

we shall denote the (external) angle between them. If $\Sigma_1\cap\Sigma_2=\varnothing$ then we put $\alpha(\Sigma_1, \Sigma_2)=0$.

If $X\subset\mathbb{S}^n$ then $Sp(X)$ will denote the round sphere in $\mathbb{S}^n$ which contains $X$ and has minimal dimension. Uniqueness of this sphere is evident.

## 3 Compact reflection orbifolds

Consider a compact convex finite-sided polyhedron $\Phi$ in $\mathbb{H}^3$, such that every vertex of $\Phi$ belongs to precisely 3 edges. Denote the numbers of vertices, edges and faces of $\Phi$ by $v, e, f$ respectively. Then we have: $3v=2e, 2=v-e+f$. So $e=3f-6$. Let $\Gamma_\Phi\subset\mathrm{Isom}(\mathbb{H}^3)$ be the isometry group generated by the reflections in the faces of $\Phi$. Suppose that $\Gamma_\Phi$ is discrete and $\Phi$ is a fundamental polyhedron for it. Then we shall identify the polyhedron $\Phi$ with the factor-orbifold $\mathbb{H}^3/\Gamma_\Phi$. This orbifold is sufficiently large (i.e. contains an incompressible suborbifold) iff $\Phi$ is not a tetrahedron [T]. Each (2-dimensional) face $\Pi_i\subset\partial\Phi$ is contained in the unique round sphere $\Sigma_i^0=Sp(\Pi_i)\in\mathscr{C}$.

According to the Poincare theorem about fundamental polyhedra for discrete groups, the group $\Gamma_\Phi$ has the presentation

$$\langle\tau_1, \ldots, \tau_f : (\tau_j\tau_i)^{n_{ij}}=1\rangle$$

where $n_{ii}=1$, $\mathbb{Z}\ni n_{ij}=\pi/\alpha(\Sigma_i^0, \Sigma_k^0)$ (see [Ma]); the elements $\tau_j$ are reflections in the faces $\Pi_j$ of $\Phi$.

Moreover, suppose that we have another configuration

$$(\Sigma_1, \ldots, \Sigma_f)$$

of spheres in $\mathbb{S}^3$, such that

$$\alpha(\Sigma_i^0, \Sigma_k^0) = \alpha(\Sigma_i, \Sigma_k)$$

for each $i, k$. Then the subgroup of SO(4, 1) generated by the reflections in the spheres $\Sigma_i$ is isomorphic to $\Gamma_\Phi$. Thus the problem of deforming the representation $\rho_0$ of $\Gamma_\Phi$ is equivalent to the problem of deforming the configuration of spheres preserving the angles between the neighboring spheres.

**Theorem 1.** *Near the class* $[I]$ *of the embedding* $\mathrm{id}: \Gamma_\Phi \to \mathrm{Isom}(\mathbb{H}^4)$ *the space* $R(\Gamma_\Phi, 4)$ *is smooth and has dimension* $f-4$.

3.1 *Proof.* Define $r = (\Sigma_1, \ldots, \Sigma_f) \in \mathscr{C}^f$. If faces $\Pi_i, \Pi_k$ of $\Phi$ have a common edge $\varepsilon_j$ then put $\alpha_j = \alpha(\Sigma_i, \Sigma_k)$. So, we have the map

$$\tilde{\xi}_3 = (\alpha_1, \ldots, \alpha_j, \ldots, \alpha_e): \mathscr{C}^f \to \mathbb{R}^e .$$

Denote $\alpha_j^0 = \alpha_j(r_0)$. Let $\mathscr{C}_2$ denote the space of spheres orthogonal to the sphere $\partial_\infty \mathbb{H}^3$. Then the groups $\mathbf{Mob}_+(\mathbb{S}^3)$ and $\mathbf{Mob}_+(S^2)$ act on $\mathscr{C}^f$ and $\mathscr{C}_2^f$ respectively. Drop the map $\tilde{\xi}_3$ to the maps $\xi_3: \mathscr{C}^f/\mathbf{Mob}_+(\mathbb{S}^3) \to \mathbb{R}^e$ and

$$\xi_2: \mathscr{C}_2^f/\mathbf{Mob}_+(S^2) \to \mathbb{R}^e .$$

Hence $[r_0] = (\xi_2)^{-1}(\alpha^0 = (\alpha_1^0, \ldots, \alpha_e^0))$. Moreover, the map $\xi_2$ is an immersion at the point $[r_0]$, since $\mathrm{H}^1(\Gamma_\Phi, so(3,1)) = 0$[W].

However, $\dim_{[r_0]} \mathscr{C}_2^f/\mathbf{Mob}_+(S^3) = 3f - 6 = \dim \mathbb{R}^e$, so the map $\xi_2$ is also a submersion near $[r_0]$. Hence the map $\xi_3$ is a submersion near $[r_0]$ too. Thus the variety $(\xi_3)^{-1}(\alpha^0 = (\alpha_1^0, \ldots, \alpha_e^0)) \cong R(\Gamma_\Phi, 4)$ is a manifold of dimension $4f - e - 10 = f - 4$ near $[r_0]$. □

**Corollary 3.1.** *The group* $\Gamma_\Phi$ *is rigid in* SO(4, 1) iff *the orbifold* $\Phi$ *is not sufficiently large.*

3.2. Now describe the basis of $\mathrm{H}^1(\Gamma_\Phi, so(4, 1))$. Realize the hyperbolic 3-space $\mathbb{H}^3$ as the upper half-space $\mathbb{R}^3_+ = \{(x_1, x_2, x_3): x_3 > 0\}$. Pick an arbitrary sphere $\Sigma_i^0$ as above, center of $\Sigma_i^0$ is a point $(x_1, x_2, 0)$. Then define the family $\Sigma_i^t$ of spheres with the same radius as $\Sigma_i^0$ and center at $(x_1, x_2, t), (t \in \mathbb{R})$. Let $r_t(i) = (\Sigma_1, \ldots, \Sigma_i^t, \ldots, \Sigma_f) \in \mathscr{C}^f$. For every sphere $\Sigma_j^0$ adjacent to $\Sigma_i^0$ the function $\alpha(\Sigma_j^0, \Sigma_i^t)$ has maximum at the point $t = 0$. So,

$$d/dt(\alpha(\Sigma_j^0, \Sigma_i^t))|_{t=0} = 0 .$$

Therefore, the vector $d/dt(r_t(i))|_{t=0}$ is tangent to $\mathrm{Hom}(\Gamma_\Phi, \mathrm{SO}(4, 1)) \subset \mathscr{C}^f$. By direct calculations it is possible to show that $\{d/dt(r_t(i))|_{t=0}: i = 1, \ldots, f-4\}$ forms a basis of $\mathrm{H}^1(\Gamma_\Phi, so(4, 1))$.

*Remark 3.* The same construction 3.2 works for hyperbolic reflection orbifolds of arbitrary dimension. Thus, if $\Gamma \subset \mathrm{SO}(n, 1)$ is any discrete reflection group then $\dim \mathrm{H}^1(\Gamma, so(n+1, 1)) = \max\{f - n - 1, 0\}$ where $f$ is the number of faces of the fundamental polyhedron of $\Gamma$.

## 4 Rigidity of 2-bridge knots

In this section we will need the following

**Lemma 4.1.** *Suppose that $G$ is a finitely generated group, $E$ a $G$-modulus, $R$ is an element of $G$ so that $R\cdot\xi=\xi$ for each $\xi\in E$. Denote by $\phi: G\to G/\langle\langle R\rangle\rangle=G'$ the natural projection epimorphism, where $\langle\langle R\rangle\rangle$ is the normal closure of $R$ in $G$. Let $[x]\in \mathrm{H}^1(G,E)$ be a class such that the restriction of $x$ to $\langle R\rangle$ is zero in $Z^1(\langle R\rangle, E)$. Then*

(i) *$E$ is $G'$-modulus: $g'\cdot\xi=g\cdot\xi$ for any $g\in\phi^{-1}(g')$;*

(ii) *there is a class $[x']=\phi_*(x)\in \mathrm{H}^1(G',E)$ such that $x'(g')=x(g)$ for each $g'\in G'$, $g\in\phi^{-1}(g')$.*

*Proof.* Define $x'$ as $x'(g')=x(g)$. Since the action of $R$ on $E$ is trivial, then the definitions of $g'\cdot\xi$ and $x'$ do not depend on the choice of $g\in\phi^{-1}(g')$. ☐

Let $K\subset\mathbb{S}^3$ be a 2-bridge knot (see [R]). Let $(p,q)$ be a pair of coprime integers. Remove from $\mathbb{S}^3$ an open regular neighborhood $\mathcal{N}(K)$ of the knot $K$ and denote the resulting manifold with boundary by $M_\infty=M(K;\infty)$. We shall consider only hyperbolic 2-bridge knots $K$, i.e. such that $\mathrm{int}(M(K;\infty)$ admits a complete hyperbolic structure. Denote by $\lambda$ a simple homotopically nontrivial loop on $\partial M(K;\infty)$ such that $\lambda$ bounds a disc in $\mathcal{N}(K)$; let $\mu\subset\partial M(K;\infty)$ be a simple homotopically nontrivial loop which is homologically trivial in $M(K;\infty)$. Denote by $M_{(p,q)}$ the manifold obtained from $M_\infty$ by attaching a solid torus $\tilde{T}$ along the boundary so that the loop $\lambda^p\mu^q$ bounds a disc in $\tilde{T}$. Suppose that $(p,q)$ are not coprime, and $k$ is their greatest common divisor. Denote by $\tilde{T}(k)$ the orbifold whose underlying set is $D^2\times\mathbb{S}^1$ and the singular set $\{0\}\times\mathbb{S}^1$ has order $k$.

Then $M_{(p,q)}$ is the orbifold obtained from $M_\infty$ by attaching $\tilde{T}(k)$ so that the loop $\lambda^{p/k}\mu^{q/k}$ bounds a disc in $\tilde{T}(k)$ with one singular point.

This procedure is called the ***generalized Dehn surgery on the knot*** $K$; $(p,q)$ are parameters of the surgery.

*Remark 4.* This definition is slightly different from the standard one.

Then for all but finite coprime parameters $(p,q)$ of Dehn surgery on $K$ the resulting manifolds are hyperbolic and are not sufficiently large [HT]. For a group $F$ and representation $\rho: F\to \mathrm{SO}(n,1)$ we denote by $Ad_n\circ\rho$ the corresponding adjoint representation on the Lie algebra $so(n,1)$.

**Theorem 2.** *For infinitely many coprime $(p,q)$ the groups $\pi_1 M_{(p,q)}$ are locally rigid in* $\mathrm{SO}(4,1)$ *and moreover* $\mathrm{H}^1(\pi_1 M_{(p,q)}, Ad_n)=0$.

*Proof of Theorem 2.* Consider the uniformization $M=\mathbb{S}^3\backslash K=\mathbb{H}^3/\Gamma$. Then $\Gamma=\langle x,y\,|\,xw=wy\rangle$, where $w=w(x,y)$, $x,y$ are parabolic elements of $\Gamma\subset\mathrm{PSL}(2,\mathbb{C})$. Denote by $A$ the maximal parabolic subgroup of $\Gamma$ which contains $\langle x\rangle$; $A=\langle x\rangle\oplus\langle z\rangle$ where the elements $x,z$ are represented by the loops $\lambda$ and $\mu$ respectively.

First consider the case of a "singular" Dehn surgery $(r,0)$ on the knot $K$ such that in the fundamental group of the hyperbolic orbifold $M_{(r,0)}$ the image of $x$ has the order $r$. Let $\rho_r:\Gamma\to\Gamma_r\cong\pi_1(M_{(r,0)})$ be the holonomy representation; $\mathbb{H}^3/\Gamma_r=M_{(r,0)}$.

Denote by

$$\Sigma=\{\rho_r:\Gamma\to\mathrm{PSL}(2,\mathbb{C})\}/\mathrm{PSL}(2,\mathbb{C})$$

the collection of conjugacy classes of such representations (where $r$ varies).

**Lemma 4.2.** *For every* $(r, 0)$*-surgery we have*

$$\mathrm{H}^1(\Gamma_r, \mathrm{Ad}_4)=0\,.$$

*Proof.* The group $\Gamma_r$ is generated by two elliptic elements $x_r=\rho_r(x)$, $y_r=\rho_r(y)$, which are conjugate in $\Gamma_r$. Consider the group

$$\Gamma_r^*=\langle x \mid x^r=1\rangle * \langle y \mid y^r=1\rangle$$

and the natural projection $\varphi_r:\Gamma_r^*\to\Gamma_r$. Then we have $\mathrm{H}^2(\Gamma_r^*, \mathrm{Ad}_4\circ\varphi_r)=0$ since $\Gamma_r^*$ is almost free. Let $[\xi]\in\mathrm{H}^1(\Gamma_r, \mathrm{Ad}_4]$, then

$$[\xi^*]=\varphi_r^*[\xi]\in\mathrm{H}^1(\Gamma_r^*, \mathrm{Ad}_4)$$

is an integrable infinitesimal deformation. Let $\theta_t:\Gamma_r^*\to\mathrm{SO}(4, 1)$ be a curve tangent to $[\xi^*]$.

**Proposition.** *For every $t$ the group $\theta_t(\Gamma_r^*)$ is conjugate in* SO(4, 1) *to a subgroup of* SO(3, 1).

*Proof.* The group $\theta_t(\Gamma_r^*)$ is generated by two elliptic transformations with the fixed-point sets $l_t$, $k_t$ which are unlinked Euclidean circles in $\mathbb{S}^3$. Then the proposition follows from Corollary 2.2. ☐

Thus we can find a coboundary

$$\delta_\eta\in B^1(\Gamma_r^*, \mathrm{Ad}_4\circ\varphi_r)=B^1(\Gamma_r, \mathrm{Ad}_4)$$

such that $\delta_\eta-\xi\in Z^1(\Gamma_r, \mathrm{Ad}_3)$. However,

$$\mathrm{H}^1(\Gamma_r, \mathrm{Ad}_3)=0$$

by Weil's rigidity theorem [W]. So $[\xi]=0$. This proves Lemma 4.2. ☐

**Corollary 4.1.** (i) *The restriction map*

$$\mathrm{res}:\mathrm{H}^1(\Gamma, \mathrm{Ad}_4\circ\rho_r)\to\mathrm{H}^1(\langle x\rangle, \mathrm{Ad}_4\circ\rho_r)$$

*is injective.*

(ii) *The space* $\mathrm{H}^1(\Gamma, \mathrm{Ad}_4\circ\rho_r)$ *has dimension* 2.

*Proof.* Consider (i). Let $[\psi]\in\mathrm{Ker}(\mathrm{res})$. Then $\psi(x^m)=\beta-\mathrm{Ad}_4\circ\rho_r(x^m)\beta$, where $\beta\in so(4, 1)$. Define a cocycles $\sigma, \xi$ in $Z^1(\Gamma, \mathrm{Ad}_4\circ\rho_r)$ as $\sigma(g)=\beta-\mathrm{Ad}_4\circ\rho_r(g)\beta$, $\xi(g)=\psi(g)-\sigma(g)$. The cocycle $\psi$ is cohomologous to $\xi$ and the restriction of $\xi$ to $\langle x\rangle$ is identically zero.

Now we can apply Lemma 4.1 to the projection $\rho_r:\Gamma\to\Gamma_r$. Then $\xi$ induces the cocycle $\rho_{r*}(\xi)=\xi'\in Z^1(\Gamma_r, \mathrm{Ad}_4)=B^1(\Gamma_r, \mathrm{Ad}_4)$ (according to Lemma 4.2). Hence for some $\alpha\in so(4, 1)$ we have $\xi'(g')=\alpha-\mathrm{Ad}_4(g')(\alpha)$. However, $\xi'(g')=\xi(g)=\alpha-\mathrm{Ad}_4\circ\rho_r(g)(\alpha)$. Therefore,

$$\xi\in B^1(\Gamma, \mathrm{Ad}_4\circ\rho_r)$$

and (i) is proved.

Consider (ii). We have the following diagram

$$\begin{tikzcd}
\mathrm{H}^1(\Gamma, \mathrm{Ad}_3\circ\rho_r) \arrow[r, "\mathrm{res}_0"] \arrow[d, dash, "\eta"] & \mathrm{H}^1(A, \mathrm{Ad}_3\circ\rho_r) \arrow[r, "\mathrm{res}_3"] \arrow[d, dash, "\psi"] & \mathrm{H}^1(\langle x\rangle, \mathrm{Ad}_3\circ\rho_r) \arrow[d, dash, "\vartheta"] \\
\mathrm{H}^1(\Gamma, \mathrm{Ad}_4\circ\rho_r) \arrow[r, "\mathrm{res}_1"] & \mathrm{H}^1(A, \mathrm{Ad}_4\circ\rho_r) \arrow[r, "\mathrm{res}_2"] & \mathrm{H}^1(\langle x\rangle, \mathrm{Ad}_4\circ\rho_r)\,.
\end{tikzcd}$$

The Abelian group $\rho_r A$ is generated by one hyperbolic and one elliptic element; thus its action on $\mathbb{R}^{3,1}$ has no nonzero fixed vectors and

$$0 = 2\dim \mathrm{H}^0(A, \mathbb{R}^{3,1}_{\rho_r}) = \mathrm{H}^1(A, \mathbb{R}^{3,1}_{\rho_r}) .$$

Therefore, in the exact sequence

$$\begin{aligned} 0 = \mathrm{H}^0(A, \mathbb{R}^{3,1}_{\rho_r}) &\to \mathrm{H}^0(A, \mathrm{A}d_3 \circ \rho_r) \\ &\to \mathrm{H}^0(A, \mathrm{A}d_4 \circ \rho_r) \to \mathrm{H}^1(A, \mathbb{R}^{3,1}_{\rho_r}) = 0 \end{aligned}$$

the homomorphism $\psi : \mathrm{H}^0(A, \mathrm{A}d_3 \circ \rho_r) \to \mathrm{H}^0(A, \mathrm{A}d_4 \circ \rho_r)$ is an isomorphism. On the other hand:

$$\begin{aligned} \vartheta : \mathrm{H}^1(\langle x \rangle, \mathrm{A}d_3 \circ \rho_r) \cong \mathbb{R}^2 &\to \mathrm{H}^1(\langle x \rangle, \mathrm{A}d_4 \circ \rho_r) \cong \mathbb{R}^4 \\ &\to \mathrm{H}^1(\langle x \rangle, \mathbb{R}^{3,1}_{\rho_r}) \cong \mathbb{R}^2 \to 0 . \end{aligned}$$

Hence $\vartheta$ is a monomorphism with 2-dimensional image.

Recall that for $H$-modulus $F$ the space $F^H$ is the set of elements of $F$ fixed under the action of $H$. Now, consider the exact sequence:

$$\begin{aligned} 0 \to \mathbb{R}^2 &= \mathrm{H}^1(\langle z \rangle, so(3,1)^{\rho_r(x)}) \\ &\to \mathbb{R}^4 = \mathrm{H}^1(A, \mathrm{A}d_3 \circ \rho_r) \to \mathrm{H}^1(\langle x \rangle, \mathrm{A}d_3 \circ \rho_r)^{\langle z \rangle} \to 0 . \end{aligned}$$

However, $y$ acts trivially on $\mathrm{H}^1(\langle x \rangle, \mathrm{A}d_3 \circ \rho_r)$ because $A$ is Abelian and

$$\mathrm{H}^1(\langle x \rangle, \mathrm{A}d_3 \circ \rho_r) = \mathrm{H}^1(\langle x \rangle, co(2))$$

where $co(2)$ is the Lie algebra of the centralizer of $\rho_r(x)$ and $\rho_r(z)$ in $\mathrm{SO}(3,1)$. This follows that the restriction map $\mathrm{res}_3$ is surjective.

Therefore, $\mathrm{Im}(\mathrm{res}_2) = \mathrm{Im}(\mathrm{res}_2 \circ \psi) = \mathrm{Im}(\vartheta \circ \mathrm{res}_3) = \mathrm{Im}(\vartheta) \cong \mathbb{R}^2$. Notice also that $\mathrm{Ker}(\mathrm{res}_1) = \mathrm{Ker}(\mathrm{res}_2 \circ \mathrm{res}_1) = 0$ (according to (i)) and $\mathrm{Ker}(\eta) = 0$ (for instance because $\psi \circ \mathrm{res}_0$ is injective). Thus $\mathrm{res}_2 \circ \mathrm{res}_1$ injects $\mathrm{H}^1(\Gamma, \mathrm{A}d_4 \circ \rho_r)$ into the image of $\vartheta$ which is 2-dimensional. Thus,

$$\dim \mathrm{H}^1(\Gamma, \mathrm{A}d_4 \circ \rho_r) \leqq 2 .$$

On the other hand,

$$\mathrm{H}^1(\Gamma, \mathrm{A}d_4 \circ \rho_r) \supset \eta(\mathrm{H}^1(\Gamma, \mathrm{A}d_3 \circ \rho_r))$$

and thus

$$\dim \mathrm{H}^1(\Gamma, \mathrm{A}d_4 \circ \rho_r) \geqq 2 .$$

This implies the second assertion of the corollary. □

We continue proof of Theorem 2. The space $R(\Gamma, 3)$ has the natural complex structure since we can identify $\mathrm{SO}(3,1)_+$ with $\mathrm{PSL}(2, \mathbb{C})$. Denote by $E$ the projection to $R(\Gamma, 3)$ of the set of representations $\rho_{p,q} : \Gamma \to \Gamma_{p,q} \subset \mathrm{SO}(3,1)$ which are the holonomy representations of hyperbolic manifolds $M_{(p,q)}$.

*Remark 4.* Here and below $p$ and $q$ are coprime integers.

Denote by $R_0(\Gamma, 3)$ the connected component of $R(\Gamma, 3)$ containing $\rho_0$.

**Lemma 4.3.** *The set $E$ is Zariski dense (over $\mathbb{R}$) in $R_0(\Gamma, 3)$.*

*Proof. Step 1.* The element $\rho_0(x)$ is a parabolic element in $\mathrm{PSL}(2, \mathbb{C})$. Take a simply connected neighborhood $V$ of $\rho_0(x)$ in $\mathrm{PSL}(2, \mathbb{C})$. For each $g \in \mathrm{PSL}(2, \mathbb{C})$

choose a lift $\tilde{g}$ of $g$ to $\mathrm{SL}(2, \mathbb{C})$ and define $\lambda(g)$ to be the ratio of eigenvalues of $\tilde{g}$ (this is well defined up to inversion).

We can choose $V$ to be so small that the image of $\lambda$ does not intersect the set of nonpositive real numbers. Then extend the function $\lambda$ to the orbit of $V$ under the conjugation by $\mathrm{PSL}(2, \mathbb{C})$ as:

$$\lambda(hgh^{-1}) = \lambda(g) .$$

Finally we put

$$u: [\rho] \mapsto \log(\lambda(\rho(x))/2)$$

where we choose that branch of logarithm so that $\log(1) = 0$. Then, $u[\rho_0] = 0$. The function $u([\rho])$ is well defined up to the multiple $\pm 1$ and there is a way to choose this multiple so that function $u$ is a holomorphic embedding of $W$ into $\mathbb{C}$ (see [NZ]). Put $E^* = u(E)$ and

$$U = \{|z|/z, z \in E^*\} .$$

The set $U$ is dense on the unit circle (see [NZ]) and the points of $E^*$ accumulate to zero. Therefore, the set $E^*$ cannot lie on any real-analytic subset of $u(W)$.

*Step 2.* Suppose that $E$ is not Zariski-dense and $E \subset f^{-1}(0)$ for some nontrivial polynomial $f$. Then $E^* \cap W$ is contained in the real-analytic set $(f \circ u)^{-1}(0)$ which is impossible. ☐

**Corollary 4.2.** *If $E_0$ is any finite subset of $E$, then $E \backslash E_0$ is Zariski dense in $R_0(\Gamma, 3)$ over $\mathbb{R}$.*

**Lemma 4.4.** *For infinitely many elements $\rho \in E$*

$$\dim \mathrm{H}^1(\Gamma, \mathrm{Ad}_4 \circ \rho) = 2 .$$

*Proof.* Denote by $(L, c_\rho)$ the $so(4, 1)$-bundle over $M_\infty$ with the flat connection $c_\rho$ associated with the representation $\mathrm{Ad}_n \circ \rho$ (see [JM]). The group cohomology $\mathrm{H}^1(\Gamma, \mathrm{Ad}_4 \circ \rho)$ can be calculated via simplicial cochains of $M_\infty$ with coefficients in the parallel sections of $(L, c_\rho)$ (see [JM]). Thus the spaces of $i$-chains $C^i(\mathscr{K}_\rho)$ of the corresponding complex $\mathscr{K}_\rho$ is finite-dimensional. We shall identify $C^i = C^i(\mathscr{K}_\rho)$ for different $\rho$ so that the coboundary operators $\delta^i_\rho$ are linear operators between finite-dimensional spaces

$$\delta^i_\rho : C^i \to C^{i+1}$$

which depend algebraically on the parameter $\rho$.

Suppose now that there exists a finite set $E_0 \subset E$ such that

$$\dim \mathrm{H}^1(\Gamma, \mathrm{Ad}_4 \circ \rho) \geqq 3$$

for every $\rho \in E \backslash E_0$.

Denote the dimension of $C^i$ by $N_i$. The space $\mathrm{Im}(\delta^0_\rho)$ has constant dimension $\Delta_0$ since $\mathrm{H}^0(\Gamma, \mathrm{Ad}_4 \circ \rho) = 0$ for each $[\rho] \in E$. If $d_\rho = \dim \mathrm{Ker}(\delta^1_\rho) \geqq 3 + \Delta_0$ for $\rho \in E \backslash E_0$ then

$$\mathrm{Im}(\delta^1_\rho) = N_1 - d_\rho \leqq N_1 - 3 - \Delta_0 .$$

Denote by $\{\mu_s, s = 1, 2, \ldots\}$ the complete set of minors of order $(N_1 - 2 - \Delta_0)$ in the matrix $\delta^1_\rho$. Then

$$\mu(\rho) = \sum_{s \geqq 1} \mu_s^2(\rho) = 0$$

for every $\rho \in E \backslash E_0$.

Obviously, $\mu(\rho)$ is an algebraic function; however $\mu(\rho_r) \neq 0$ for every $\rho_r \in \Sigma$ since $\dim \mathrm{H}^1(\Gamma, \mathrm{Ad}_4 \circ \rho_r) = 2$. So, $E \backslash E_0$ is contained in a proper real-algebraic subset of $R(\Gamma, 3)$ which contradicts to assertion of Corollary 4.2. □

Now we can finish the proof of Theorem 2. We have an infinite subset $F \subset E$ such that $\dim \mathrm{H}^1(\Gamma, \mathrm{Ad}_4 \circ \rho) = 2$ for every $\rho \in F$. However $\dim \mathrm{H}^1(\Gamma, \mathrm{Ad}_3 \circ \rho) = 2$, so $\dim \mathrm{H}^1(\Gamma_{(p,q)}, \mathrm{Ad}_4 \circ \rho_{(p,q)}) = 0$ for $\rho_{(p,q)} \in F$. Theorem 2 is proved. □

## 5 Deformations of nonuniform lattices

5.1. Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is an arbitrary nonuniform lattice (i.e. $\mathrm{vol}(\mathbb{H}^3/\Gamma) < \infty$ but $\mathbb{H}^3/\Gamma$ is not compact).

**Definition.** *Let* $\mathrm{PZ}^1(\Gamma, so(4,1))$ *be the subspace of cocycles* $\xi \in \mathrm{Z}^1(\Gamma, so(4,1))$ *such that the restriction of* $\xi$ *to each cyclic parabolic subgroup* $\langle \gamma \rangle \subset \Gamma$ *is a coboundary in* $\mathrm{Z}^1(\langle \gamma \rangle, so(4,1))$. *Then put*

$$\mathrm{PH}^1(\Gamma, so(4,1)) = \mathrm{PZ}^1(\Gamma, so(4,1)) / \mathrm{B}^1(\Gamma, so(4,1)) .$$

*The space* $\mathrm{PH}^1(\Gamma, so(4,1))$ *is called the space of* ***parabolic cohomology classes*** *and* $\mathrm{PZ}^1(\Gamma, so(4,1))$ *is the space of* ***parabolic cocycles.***

**Theorem 3.** *For every maximal parabolic subgroup* $A \subset \Gamma$ *we have*

$$\mathrm{res}_A : \mathrm{PH}^1(\Gamma, so(4,1)) \to \mathrm{PH}^1(A, so(4,1))$$

*is identically zero.*

*Proof.* The space $so(4,1)$ admits the $\mathrm{Ad}_\Gamma$-invariant decomposition $so(4,1) = so(3,1) \oplus V$, where $V \cong \mathbb{R}^{3,1}$ is the Lorentz vector space [JM]. This splitting induces the natural decomposition

$$\mathrm{PH}^1(\Gamma, so(4,1)) = \mathrm{PH}^1(\Gamma, so(3,1)) \oplus \mathrm{PH}^1(\Gamma, V) .$$

However, $\mathrm{PH}^1(\Gamma, so(3,1)) = 0$ by Weil-Garland-Raghunathan Rigidity theorem [GR, R]. Therefore, projections of every $[\xi] \in \mathrm{PH}^1(\Gamma, so(4,1))$ to

$$\mathrm{PH}^1(\Gamma, so(3,1)) \quad \text{and} \quad \mathrm{PH}^1(A, so(3,1))$$

are zero. However, $\mathrm{PH}^1(A, V) = 0$ since

$$\mathrm{PH}^1(A, so(4,1)) \cong \mathrm{PH}^1(A, so(3,1)) .$$

So, $\mathrm{res}_A([\xi]) = 0$. □

5.2 *Example.* Let $\Gamma \subset \mathrm{SO}(3,1)$ be the fundamental group of the complement to any hyperbolic 2-bridge knot (as in the Sect. 4).

**Theorem 4.** $\mathrm{PH}^1(\Gamma, so(4,1)) = 0$.

*Proof.* Suppose that $\xi \in \mathrm{PZ}^1(\Gamma, so(4,1))$ be a nonzero cocycle. Denote by $\Gamma^*$ the free group generated by $x, y$. Then, applying the arguments of Lemma 4.2, we construct a smooth family of representations $\theta_t$ of $\Gamma^*$ into $\mathrm{SO}(4,1)$ such that:

(i) $\theta_0 = \mathrm{id}$;
(ii) $\theta_t(x)$ and $\theta_t(y)$ are all conjugate to $x$ and $y$ for all $t$;
(iii) the curve $\theta_t$ is tangent to the lift $\psi$ of $\xi$ to $\Gamma^*$.

Each transformation $\theta_t(x)$ and $\theta_t(y)$ is conjugate in SO(4, 1) to a Euclidean translation. This implies that every 3-dimensional hyperbolic subspace of $\mathbb{H}^4$ which contains 1-dimensional horocycle of $\theta_t(x)$ is invariant under $\theta_t(x)$; the same is true for $\theta_t(y)$. We can assume that $\theta_t(x)$ is a Euclidean translation along a line $\ell$ in the upper-halfspace model of $\mathbb{H}^4$. Let $P$ be a Euclidean 3-dimensional subspace in $\mathbb{R}^4$ which contains $\ell$, horocycle of $\theta_t(y)$ and orthogonal to the absolute of $\mathbb{H}^4$. It follows that $\theta_t(\Gamma)$ has an invariant 3-dimensional hyperbolic hyperplane $P \cap \mathbb{H}^4$. The same arguments as in Lemma 4.2 imply that $\xi \in B^1(\Gamma, so(4, 1))$. ☐

*Remark 5.* The arguments of the proof show that if $[\rho] \in PR(\Gamma, 4)$, then the class $[\rho]$ contains a representation $\rho_1$ with image in SO(3, 1). However Riley in [Ri] described completely all representations of $\Gamma$ in PSL(2, $\mathbb{C}$). Any representation which preserves the conjugacy classes of $x$ and $z$ is conjugate to id. Therefore $PR(\Gamma, 4)$ consists of a single point. This implies that if $[\rho] \in R(\Gamma, 4)$ is the conjugacy class of a discrete faithful representation $\rho$ then $[\rho] = [\mathrm{id}]$. Indeed, such representation $\rho$ must preserve the conjugacy class of $x$ because $\rho(\langle x \rangle \oplus \langle z \rangle)$ is conjugate to a lattice in $\mathbb{R}^2$.

## 6 Three examples

**Notation.** For any orbifold $\mathcal{O}$ we shall denote by $|\mathcal{O}|$ its underlying set. For a face $P$ of a polyhedra $\Phi$ we shall denote by $St_P$ the set off all those faces of $\Phi$ which have nonempty intersection with $P$; $St_P^* = St_P - \{P\}$. We shall suppose that $\mathbb{H}^3$ is realized as a unit ball in $\mathbb{R}^3$.

*Bending deformations.* The following is not the most general description of the "bending", but it is enough for our aims. Suppose that $G \subset SO(n+1, 1)$ is any group which splits as the amalgamated free product $G = G_1 *_J G_2$ so that:
(1) $G_1$ and $G_2$ have finite centralizers in $SO(n+1, 1)$;
(2) the centralizer $Z_J$ of the group $J$ in $SO(n+1, 1)$ is 1-dimensional.

Take a nondegenerate curve $\theta_t$ in $Z_J$ which contains 1. Then put $G_t$ to be the group generated by $G_1$ and $\theta_t G_2 \theta_t^{-1}$. It is easy to see that $G_t = \rho_t(G)$, where $\{\rho_t : t \in [0, 1]\}$ is a continuous curve of homomorphisms of $G$ in $SO(n+1, 1)$. This curve defines a nontrivial deformation of the identity representation of $G$ in $SO(n+1, 1)$. Such deformation is called the **bending in *J*.**

*Remark 6.* Bending deformations of representations of fundamental groups of hyperbolic manifolds (and orbifolds) of dimension $n$ were constructed by several authors: by Thurston [T] for $n = 2$; in the case of certain reflection groups in $\mathbb{H}^3$ – by Apanasov and Tetenov [AT]; then in infinitesimal form – by Lafontane [L]; and later by Kuorouniotis [K]. In the most general form (for graphs of groups) this conception is explained by Johnson and Millson [JM] (see also [G]). There are examples of Apanasov [A] of "pea-pod" groups which admit "stamping" deformations; this construction was generalized by Tan [Ta]. For further generalizations see also [KM].

6.1 *Example 1.* Suppose that we are given a finite-sided convex polyhedron $\Omega \subset \mathbb{H}^3$ with the following properties:

(a) for some compact face $P$ of $\Omega$ all but one faces of $St_P^*$ are orthogonal to a common geodesic plane $\Pi\subset\mathbb{H}^3$;
(b) among the faces in $St_P^*$ there is a face $Q_2$ which enters a cusp made by the faces $Q_1, Q_2, Q_3, Q_4$ where $Q_3, Q_4\in St_P^*$.

Then $Q_2$ is orthogonal to $Q_3, Q_4$. Suppose that $\Omega$ is the fundamental polyhedron for the discrete group $\Gamma_\Omega$ generated by reflection in its faces.

Denote by $A$ the group generated by reflections in $Q_1, Q_2, Q_3, Q_4$.

Consider a nontrivial continuous family $Sp(Q)_2^\varepsilon$ of Euclidean spheres ($\varepsilon\in[0,1]$) such that:

(i) $Sp(Q)_2^0 = Sp(Q)_2$;
(ii) $Sp(Q)_2^\varepsilon$ is orthogonal to $Sp(Q_4)$, $Sp(Q_3)$ and tangent to $Sp(Q_1)$;
(iii) the closed ball in $\mathbb{R}^3$ bounded by $(Sp(Q)_2^\varepsilon)$ contains $Sp(Q_2)$; ($\varepsilon\in[0,1]$).

Then $\alpha(Sp(P), Sp(Q)_2^\varepsilon)\geqq\alpha(Sp(P), Sp(Q_2))$. For all sufficiently small values of $\varepsilon$ there is an elliptic rotation $\varphi^\varepsilon\in\mathbf{Mob}(\mathbb{S}^3)$ around the circle $\partial_\infty\Pi$ such that:

$$\alpha(Sp(P)^\varepsilon, Sp(Q)_2^\varepsilon)=\alpha(Sp(P), Sp(Q_2)), \quad \text{where } \varphi^\varepsilon Sp(P)=Sp(P)^\varepsilon .$$

Define the new configuration of spheres $Sp(\Omega)^\varepsilon$, that consists of the same spheres as $Sp(\Omega)$, except of $Sp(P)$ and $Sp(Q_2)$ which are deformed to $Sp(P)^\varepsilon$ and $Sp(Q)_2^\varepsilon$ respectively. Then $Sp(\Omega)^\varepsilon$ has the same combinatorial type as $Sp(\Omega)$ and the same angles between spheres. The group generated by the reflections in the spheres of $Sp(\Omega)^\varepsilon$ defines the deformation $\rho_\varepsilon:\Gamma_\Omega\to G_\varepsilon$ with the following properties:
(a) for sufficiently small values of $\varepsilon$ the representation $\rho_\varepsilon$ is discrete and faithful;
(b) projection $\partial_A$ of $\rho_\varepsilon$ to $\mathrm{Hom}(A, \mathrm{SO}(4,1))/\mathrm{SO}(4,1)$ is a nontrivial path;
(c) $[\rho_\varepsilon]\in PR(\Gamma_\Omega, 3)$ (see Introduction).

One can generalize this example, however in general case it is rather difficult to determine: whether or not we obtain nontrivial deformations of cusps.

6.2 *Example 2.* Consider the convex polyhedron $\Phi$ in $\mathbb{H}^3$ which is drawn on Fig. 1. As usual, if an edge $e$ of $\Phi$ is labelled by the integer $n$ then the dihedral angle of $\Phi$ at $e$ is $\pi/n$. Let $G$ be the group generated by reflections in the faces of $\Phi$. First, find all totally geodesic suborbifolds in $\Phi$. There are only 3 incompressible suborbifolds $\mathscr{D}_i$ in $\Phi$: $\partial|\mathscr{D}_i|=\alpha_j$, $i=1,2,3$ (see Fig. 1).

(i) Suppose $\mathscr{D}_1$ is totally geodesic; then we split $\Phi$ along $\mathscr{D}_1$ and consider the "upper half" $\Phi_1^+$ (Fig. 2):

Then $\Phi_1^+$ contains an incompressible Euclidean rectangle suborbifold, that is impossible. So, $\alpha_1$ cannot be the boundary of the underlying set of any totally geodesic suborbifold in $\Phi$.

(ii) Consider $\mathscr{D}_2$. According to Andreev's theorem (see [T]) there exists a convex polyhedron $\Phi_2^+\subset\mathbb{H}^3$ as on Fig. 3. The face $\mathscr{D}_2\subset\partial|\Phi_2^+|$ is a rectangle symmetric under rotation $\theta$ of order 2 around the axis $\ell$. Let $\Phi_2^-=\theta(\Phi_2^+)$; then $\Phi_2^-\cup\Phi_2^+$ is a convex polyhedron isometric to the initial one $\Phi$. So $\mathscr{D}_2$ is a totally geodesic suborbifold.

(iii) The same arguments imply that the orbifold $\mathscr{D}_3$ is totally geodesic.

Thus the orbifold $\Phi$ has exactly two totally geodesic 2-dimensional suborbifolds $\mathscr{D}_2, \mathscr{D}_3$. Fundamental groups of $\mathscr{D}_2, \mathscr{D}_3$ have 1-dimensional centralizers in $\mathrm{Isom}(\mathbb{H}^4)$. The corresponding bending deformations $\beta_2, \beta_3$ of the group $G$ in $\mathrm{Isom}(\mathbb{H}^4)$ define classes $\beta_2', \beta_3'$ which span the 2-dimensional space $\mathrm{H}^1(G, so(4,1))$.

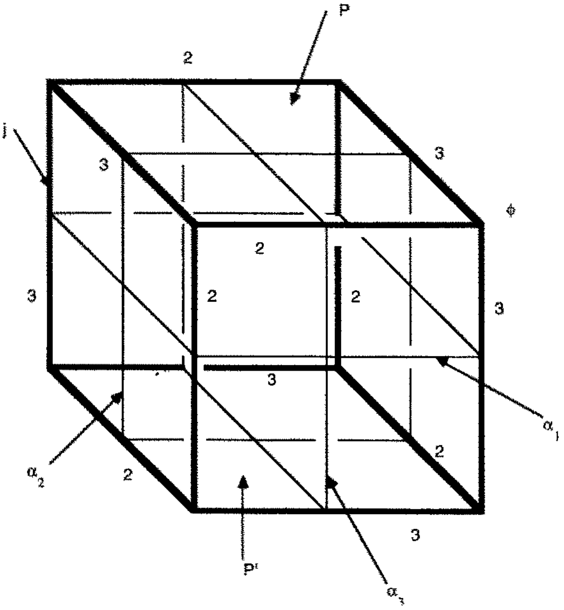

Fig. 1

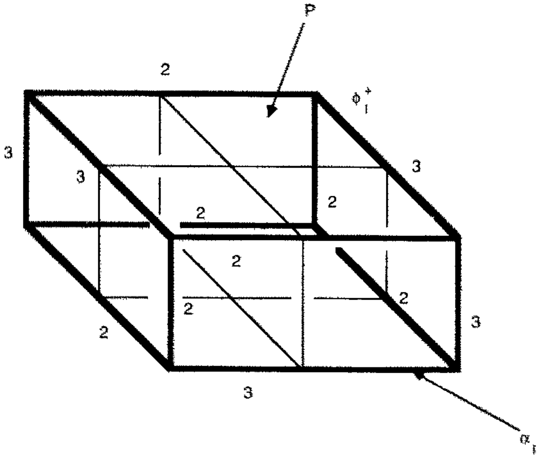

Fig. 2

On the other hand, deformation space $R(G, 4)$ is a smooth 2-dimensional surface. This shows that bending deformations in intersecting surfaces can span a plane tangent to a smooth surface in the representation variety. This result shows the striking difference between lattices in $\mathrm{Isom}(\mathbb{H}^3)$ and $\mathrm{Isom}(\mathbb{H}^n)$ $(n > 3)$ since in higher dimensions there are examples [JM] when a linear combination of two bending cocycles is not tangent to any smooth curve in the representation variety.

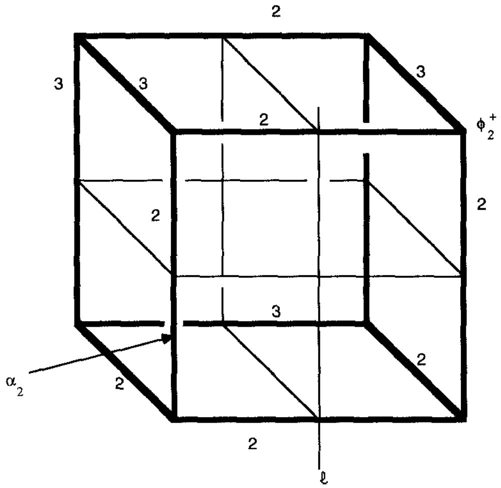

**Fig. 3**

6.3 *Example 3.* Our next aim is to construct an example of group $H$ which does not admit bending deformations at all, while it is possible to deform $H$ since its fundamental polyhedron has six faces.

Change only one dihedral angle of the polyhedron $\Phi$: instead of the angle $\pi/3$ at the edge $j$ we consider the angle $\pi/5$. Denote the new polyhedron by $\Psi$.

(i) The same arguments as in Example 2 imply that $\alpha_1$ cannot be boundary of a totally geodesic suborbifold $\mathcal{D}_1$ of $\Psi$.

Next notice that both $\alpha_2, \alpha_3$ do not intersect the edge $j$.

(ii) Consider the curve $\alpha_2 \subset \partial|\Psi|$ and suppose that $\alpha_2 = \partial|\mathcal{E}_2|$, $\mathcal{E}_2 \subset \Psi$ is a totally geodesic suborbifold. Then split $\Psi$ along $\mathcal{E}_2$ and obtain two parts: $\Psi_2^+ \supset j$ and $\Psi_2^-$ which does not contain $j$. The hyperbolic polyhedron $\Psi_2^-$ is isometric to $\Phi_2^-$. Then the rectangles $\mathcal{D}_2$ and $\mathcal{E}_2$ are also isometric. Fix the polyhedron $\Psi_2^+$, the face $\mathcal{E}_2 \subset \partial|\Psi_2^+|$ and denote the faces in $St^*_{\mathcal{E}_2}$ by $Q_i$ $(i = 1, \ldots, 4)$ so that $Q_1 \supset j$. The remaining face in $\partial\Psi_2^+ \backslash St_{\mathcal{E}_2}$ will be denoted by $S$. Then:

$$S \text{ meets } Q_i \ (i = 1, 2, 3) \text{ by the angles } \pi/2, \pi/3, \pi/2\,, \tag{a}$$

$$Sp(S) \text{ is orthogonal to } \partial\mathbb{H}^3\,. \tag{b}$$

The sphere $Sp(S)$ with the properties (a), (b) is unique up to the reflection $\tau_2$ in $\mathcal{E}_2$ ($\tau_2$ preserves $Q_1$). Thus, (a) and (b) $\Leftrightarrow$ the angle between $S$ and $Q_1$ is equal to $\pi/5$. However, we can consider $Sp(Q_i)$ as spheres which contain faces of the polyhedron $\Phi_2^+$ (since $\mathcal{D}_2$ is isometric to $\mathcal{E}_2$). Let $R$ be the remaining face of $\partial|\Phi_2^+|\backslash St_{\mathcal{D}_2}$. Then $R$ has the same properties (a), (b), however the angle between $R$ and $Q_1$ is equal to $\pi/3$. This contradiction implies that $\mathcal{E}_2$ cannot be a totally geodesic suborbifold in $\Psi$.

(iii) The same arguments as above are valid for the curve $\alpha_3$.

So the curves $\alpha_k$ $(k = 1, 2, 3)$ cannot be boundaries of underlying sets of totally geodesic suborbifolds of $\Psi$. Hence $\Psi$ does not contain totally geodesic suborbifolds at all.

Let $H$ be the discrete group generated by reflections in faces of $\Psi$. Then $H$ does not admit bending deformations, however Theorem 1 implies that the deformation space $R(H, 4)$ is 2-dimensional.

6.4 **Conjecture 2.** For every cocompact discrete subgroup $G \subset \mathrm{Isom}(\mathbb{H}^3)$ the variety $R(G, 4)$ is smooth at the point $[\mathrm{id}]$.

*Remark 7.* As John Millson explained to me, the first obstruction for deformations in this case is always zero because it belongs to the subspace $\mathrm{H}^2(G, so(3,1))$ of $\mathrm{H}^2(G, so(4,1))$, however $\mathrm{H}^2(G, so(3,1)) \cong \mathrm{H}^1(G, so(3,1)) = 0$.

*Acknowledgements.* Author is very grateful to Nicolaas Kuiper and John Millson for interesting discussions and to the referee of this paper for helpful remarks. A considerable part of this text was written during the author's stay in IHES [Ka3]. This work was also supported by NSF grant numbers 8505550 and 8902619 administered through MSRI and University of Maryland at College Park.

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