DEFORMATIONS OF THE REPRESENTATIONS OF THE FUNDAMENTAL GROUPS OF THREE-DIMENSIONAL MANIFOLDS

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The Weil theorem [1] on local rigidity is one of the fundamental results of the theory of discrete subgroups of Lie groups. The theorem asserts that for any connected semisimple Lie group G without compact components whose Lie algebra does not have any $sl(2, \mathbf{R})$ factors, the orbit of any uniform lattice $\Gamma \subset G$ (with respect to the adjoint action ad G) is open in $\mathrm{Hom}(\Gamma, G)$. The assertion follows from the fact that the cohomology group $H^1(\Gamma, \mathrm{Ad})$ is trivial (for the definitions, see e.g. [2]). The above result of Weil can often be generalized ([3-6], etc.). Garland and Raghunathan [7] proved the "disappearance theorem," according to which for any lattice Γ in a simple connected Lie group G of real rank 1 that is not locally isomorphic with SL_2, the cohomology group $H^1(\Gamma, \mathrm{Ad})$ is equal to zero. Thurston [8] proved that the corresponding "disappearance theorem" is no longer valid for nonuniform lattices in the case $G = SL_2(\mathbf{C})$. Namely, if Γ is a lattice in $SL_2(\mathbf{C})$ that has no finite-order elements and n is the number of conjugacy classes of the maximal parabolic subgroups of Γ, then the complex dimension of $\mathrm{Hom}(\Gamma, SL_2(\mathbf{C}))/\mathrm{ad}\, SL_2(\mathbf{C})$ at any point corresponding to an irreducible representation ρ is not less than n. Alternative proofs of this fact were presented in [9, 10]. However, each of the proofs rests upon some algebraic (or geometric) properties of the group $SL_2(\mathbf{C})$, the representations in which were considered in these articles. In particular, Thurston's proof was based on the fact that for any $a, b \in SL_2(\mathbf{C})$ and for any word $w(a, b) = 1$, the word $w(a^{-1}, b^{-1})$ is also equal to 1.

The goal of the present article is to explain the fact that the absence of rigidity in the above case is caused by the topology of $M = \mathbf{H}^3/\Gamma$, where $\mathbf{H}^3$ is a hyperbolic space, rather than by any algebraic or geometric properties of $SL_2(\mathbf{C})$. The fact that M is a three-dimensional manifold turns out to be essential (besides, Thurston's proof was also purely topological).

Let M be a three-dimensional compact nonspherical manifold, G be a Lie group with Lie $\mathfrak{G}$, $\partial M = T_1 \cup \ldots \cup T_n$ be a system of tori, and let ρ be a representation of $\pi_1(M)$ in G.

<u>THEOREM 1.</u> If the above conditions are satisfied, then the following inequality holds:

$$\dim H^1(\pi_1 M, \mathrm{Ad}\circ\rho) \geqslant \dim H^0(\pi_1 M, \mathrm{Ad}\circ\rho) - \dim H_0(\pi_1 M, \mathrm{Ad}\circ\rho) + \sum_{i=1}^{n} \dim H_0(\pi_1 T_i, \mathrm{Ad}\circ\rho|_{\pi_1(T_i)}). \tag{1}$$

<u>COROLLARY.</u> If G is a semisimple group, then the inequality

$$\dim H^1(\pi_1 M, \mathrm{Ad}\circ\rho) \geqslant d = \sum_{i=1}^{n} \dim Z_G(\rho(\pi_1 T_i)), \tag{2}$$

holds. $Z_G(A)$ denotes the centralizer of a subgroup A of G. If G is an infinite algebraic group and the groups $\rho(\pi_1 T_i)$ are infinite for all $i = 1, \ldots, n$, then $d \geqslant n$. In particular,

$$\dim H^1(\pi_1 M, \mathrm{Ad}\circ\rho) \geqslant n > 0. \tag{3}$$

<u>Remark.</u> If $G = SL_2(\mathbf{C})$, then (3) implies the above-mentioned result of Thurston.

The proof of the theorem is contained in Sec. 2. In Sec. 3 some consequences of the theorem are presented and questions connected with the problem of local rigidity for the natural embedding of the lattice $\Gamma \subset SO(3, 1)$ in $SO(4, 1)$ are discussed.

The results of the present article were announced by the author in [11].

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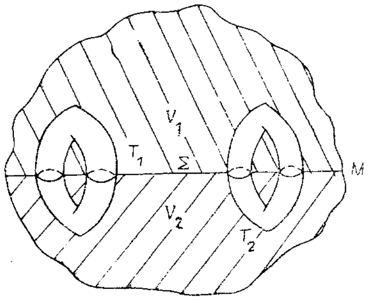

Fig. 1

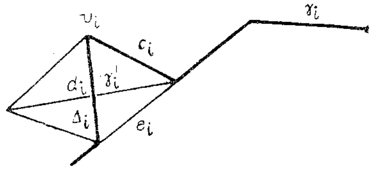

Fig. 2

1. HEEGAARD INTERLACINGS FOR MANIFOLDS WITH TOROIDAL BOUNDARIES

1.1. This section contains an auxiliary construction of a section of M, which is necessary to prove the theorem.

Definition. Let M be a three-dimensional manifold whose boundary consists of tori. A pair (V_1, V_2) of two homeomorphic bodies (generally speaking, nonorientable) with handles such that

(a) $V_1 \cup V_2 = M$, $V_1 \cap V_2 = \Sigma \equiv \mathrm{cl}(\partial V_1 \setminus (\partial V_1 \cap \partial M))$,

(b) the intersection of the surface Σ with any of the components T_i of the boundary is the union of two disjoint circles which yield a nontrivial element of $H_1(T_i)$ (Fig. 1), is called a Heegaard interlacing.

Remark. If $\partial M = \varnothing$, then (V_1, V_2) is an ordinary Heegaard interlacing (for example, see [12]).

1.2. Proposition 1. For every three-dimensional compact manifold M with toroidal boundary there exists a Heegaard interlacing.

Proof. Let M^* be a closed manifold obtained by attaching a solid torus $\mathcal{T}_i$ to each of the boundary tori T_i. We shall regard $\mathcal{T}_i$ as regular neighborhoods of simple loops $\gamma_i \subset M^*$ which are piecewise linear with respect to a sufficiently fine triangulation. The first barycentric subdivision of a triangulation K will be called K' and N(S, K) will denote a regular neighborhood of a complex $S \subset K$. We shall consider a triangulation K on M^* such that $\gamma = \gamma_1 \cup \ldots \cup \gamma_n$ is a part of its 1-skeleton Γ_1, and we denote by Γ_2 the dual skeleton to Γ_1 (i.e., the maximal 1-subcomplex K' that does not intersect Γ_1).

Then [12, Theorem 2.5] $V_i^* = N(\Gamma_i, K'')$ is a body with handles (i = 1, 2) and (V_1^*, V_2^*) is a Heegaard interlacing for M^*. Moreover, one can assume without loss of generality that V_1^* and V_2^* are simultaneously orientable (or nonorientable), and so V_1^* is homeomorphic with V_2^*.

Let Δ_i be any simplex from K' that intersects γ_i along the edge e_i (one can assume that $\Delta_i \cap \Delta_j = \varnothing$ if $i \neq j$). Let $v_i \in \Gamma_2$ be a vertex of Δ_i that does not lie on γ_i. Let us now replace γ_i by the piecewise linear loop $\gamma_i' = (\gamma_i \setminus e_i) \cup (c_i \cup d_i)$, where c_i and d_i are the edges of Δ_i that connect v_i with the end-points of e_i (Fig. 2). $\gamma' = \gamma_1' \cup \ldots \cup \gamma_n'$ is a union of disjoint simple loops.

We denote the manifold $N(\gamma', K''')$ by V. It is easily seen that $V_1^* \setminus \mathrm{int}\, V = V_1$ is homeomorphic with a body with handles, and so is $V_2 = V_2^* \setminus \mathrm{int}\, V$. Besides, these manifolds are simultaneously (orientable or nonorientable) and have the same genus (as bodies with handles). Moreover, each of the components of ∂V intersects ∂V_1 along two circles which divide ∂V into two rings. Now, it remains to note that since γ' and γ are isotopic in M^*, it follows that $M^* \setminus \mathrm{int}\, V$ is isomorphic with M. Therefore, (V_1, V_2) is a Heegaard interlacing for M. The proposition is proved.

2. PROOF OF THEOREM 1

2.1. We denote by (V_1, V_2) an arbitrary Heegaard interlacing for a manifold M (which satisfies the assumptions of the theorem) and we consider the actions of $\Gamma_i = \mathrm{Ad} \circ \rho(\pi_1(V_i))$ on $\mathfrak{G}$ and $\Gamma_i^* = * \circ \mathrm{Ad} \circ \rho(\pi_1(V_i))$ on $\mathfrak{G}^*$, where $\mathfrak{G}^*$ is the dual space to $\mathfrak{G}$. If X is a vector space and $H \subset GL(X)$, then we denote by fix(H) the set of points $x \in X$ such that h(x) = x for all $h \in H$. We recall that $\mathrm{fix}(\Gamma_i) \simeq H^0(\pi_1 V_i, \mathrm{Ad} \circ \rho)$, $\mathrm{fix}(\Gamma_i^*) \simeq H_0(\pi_1 V_i, \mathrm{Ad} \circ \rho)$; $\mathrm{fix}(\Gamma) \simeq H^0(\pi_1 M, \mathrm{Ad} \circ \rho)$, and $\mathrm{fix}(\Gamma^*) \simeq H_0(\pi_1 M, \mathrm{Ad} \circ \rho)$, where $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ and $\Gamma^* = \langle \Gamma_1^*, \Gamma_2^* \rangle$ are the images of $\pi_1(M)$. Our goal is to find a Heegaard interlacing such that $\mathrm{fix}(\Gamma) = \mathrm{fix}(\Gamma_i)$ and $\mathrm{fix}(\Gamma^*) = \mathrm{fix}(\Gamma_i^*)$ for i = 1, 2.

Let (V_1, V_2) be an arbitrary Heegaard interlacing for M. We denote by n_i the codimension of fix(Γ) in fix(Γ_i), and we denote by n_i^* the codimension of fix(Γ^*) in fix(Γ_i^*). We assume that $n = n_1 + n_2 > 0$ and, consequently, one of these numbers (for example n_1) is greater than zero. We assume that the desired modification of the Heegaard interlacing exists for all $m < n$. We denote by $v_1, \ldots, v_g$ the standard system of generators for the free group $\pi_1(V_2, x)$, where $x \in \Sigma$. Then, since $n_1 > 0$, there is an element of the system (for example v_1) such that $\mathrm{fix}(\mathrm{Ad}\circ\rho\langle v_1\rangle)$ does not contain fix(Γ_1). As a representative of the class $v_1 \in \pi_1(V_2, x)$ we choose a loop w that is unknotted in V_2 (i.e., $\mathrm{int}\, V_2 \setminus w$ is homeomorphic with an open body with handles). Such a choice is possible due to the fact that v_1 is a standard generator of $\pi_1(V_2, x)$. Let N(w) be a regular neighborhood of w in V_2, $V_1' = V_2 \cup N(w)$ and $V_2' = V_2 \setminus \mathrm{int}\, N(w)$. It is easily seen that (V_1', V_2') defines a new Heegaard interlacing for M, $\mathrm{codim}(\mathrm{fix}(\Gamma)$, $\mathrm{fix}(\mathrm{Ad}\circ\rho(\pi_1 V_1'))) < n_1$, and the numbers n_2 and n_i^* for the new Heegaard interlacing do not exceed the corresponding codimensions for the original interlacing (V_1, V_2).

It follows that one can use an inductive argument. With the aid of analogous considerations one can ensure that $n_1^* + n_2^*$ is equal to zero. We denote the resulting Heegaard interlacing (such that $n_1 + n_2 = n_1^* + n_2^* = 0$) anew by (V_1, V_2). The genus of V_i is equal to g. Now, we can immediately set about proving Theorem 1.

2.2. In the discussion below we shall find it expedient to pass from the cohomology of the group $\pi_1 M$ to the cohomology of M itself (with coefficients in some bundle). Let $L_\rho = M \times_{\mathrm{Ad}\circ\rho} \mathfrak{G}$ be a fiber bundle over M constructed from the representation $\mathrm{Ad}\circ\rho\colon \pi_1 M \to \mathrm{GL}(\mathfrak{G})$, where $\mathfrak{G}$ is equipped with the discrete topology, and let $\mathscr{L}_\rho$ be the bundle of continuous sections of L_ρ.

Then (since M is nonspherical) there is a natural isomorphism between the groups $H^p(\pi_1 M, \mathrm{Ad}\circ\rho)$ and $H^p(M, \mathscr{L}_\rho)$ (for example, see [2, Chap. 7]). In what follows we shall suppress the given bundles in the notation for the cohomology groups (assuming that either $\mathscr{L}_\rho$ or the restriction of $\mathscr{L}_\rho$ to the appropriate submanifold of M is the bundle in question).

It follows from the discussion in Sec. 2.1 that $H^0(M) \simeq H^0(V_i) \simeq H^0(\partial V_i)$ [the latter equality follows from the fact that the homomorphism $\pi_1(\partial V_i) \to \pi_1(V_i)$) is an epimorphism] and $H_0(M) \simeq H_0(V_i) \simeq H_0(\partial V_i)$. The dimensions of these linear spaces will be denoted by h' and h, respectively.

Let N(Q) be a regular neighborhood of the complex $Q = \partial V_1 \cup \partial M$, $N(Q) = N(\partial V_1) \cup N(\partial M)$, and let $N(C) = N(\partial V_1) \cap N(\partial M) = N(\partial V_1 \cap (T_1 \cup \ldots \cup T_n))$ be a regular neighborhood of the system of cylinders $C_j = \partial V_1 \cap T_j$ in M. Since the Euler characteristic $\chi(\partial V_1)$ is equal to $2 - 2g$, $\chi(\partial M) = 0$, $\chi(N(C)) = 0$, it follows from Poincaré's duality that $H^0(N(C)) \simeq H^1(N(C))$, $H^1(\partial M) \simeq H^0(\partial M) \oplus H_0(\partial M)$, and $\dim H^1(\partial V_1) = (2g-2)\dim\mathfrak{G} + \dim H^0(\partial V_1) + \dim H_0(\partial V_1) = (2g-2)\dim\mathfrak{G} + h' + h$.

2.3. Let us write down the Mayer–Vietoris sequence [13] for the covering of N(Q) by the pair $(N(\partial V_1), N(\partial M))$ of closed sets:

$$0 \to H^0(N(Q)) \to H^0(\partial V_1) \oplus H^0(\partial M) \to H^0(N(C)) \to H^1(N(Q)) \to H^1(\partial V_1) \oplus H^1(\partial M) \to H^1(N(C)) \to \ldots .$$

Since the sequence is exact, we have the inequality

$$\dim H^1(Q) \geqslant \dim H^0(Q) + (2g-2)\dim\mathfrak{G} + h + \dim H_0(\partial M).$$

2.4. Since $\pi_1(V_i)$ is a free group of rank g, it follows that $\dim H^1(V_1') = h' + (g-1) \times \dim \mathfrak{G}$, where $V_i' \subset V_i$ is a component of the manifold $M \setminus \mathrm{int}\, N(Q)$, which is a deformation retract for V_i. We shall now consider the covering of M by the pair $(V_1' \cup V_2', N(Q))$ of closed sets. The surfaces $S_1 = N(Q) \cap V_1'$ and $S_2 = N(Q) \cap V_2'$ are homotopic with ∂V_1 and ∂V_2 in M, and so $H^1(S_i) \simeq H^1(\partial V_i)$ the space being of dimension $(2g-2)\dim\mathfrak{G} + h' + h$, $i = 1, 2$. Analogously, $\dim \times H^1(V_i') = (g-1)\dim\mathfrak{G} + h'$, $i = 1, 2$.

2.5. Taking the above observations into account, let us write down the Mayer–Vietoris sequence for the covering $(V_1' \cup V_2', N(Q))$ of M:

$$0 \to H^0(M) \to H^0(V_1') \oplus H^0(V_2') \oplus H^0(N(Q)) \to H^0(S_1 \cup S_2) \to$$
$$\to H^1(M) \to H^1(V_1') \oplus H^1(V_2^1) \oplus H^1(N(Q)) \to H^1(S_1 \cup S_2) \to \ldots .$$

From (4) and the fact that the sequence is exact there follows the estimate

$$\dim H^1(\pi_1 M, \mathrm{Ad}\circ\rho) = \dim H^1(M) \geqslant h' + 2h' - 2h' - \dim H^0(N(Q)) +$$
$$+ (2h' + 2(g-1)\dim\mathfrak{G} + \dim H_0(\partial M) + \dim H^0(Q) + h) -$$

$$-2(g-1)\dim\mathfrak{G}-2h'-2h=h'-h+\dim H_0(\partial M)=$$

$$=\dim H^0(\pi_1 M,\ \mathrm{Ad}\circ\rho)+\dim H_0(\pi_1 M,\ \mathrm{Ad}\circ\rho)+\sum_{i=1}^{n}\dim H_0(\pi_1 T_i,\ \mathrm{Ad}\circ\rho|_{\pi_1(T_i)}).$$

The theorem is proved.

2.6. It is obvious that the proof of Theorem 1 is valid in the case of a $\pi_1(M)$-module of a more general form than $\mathfrak{G}_{\mathrm{Ad}\circ\rho}$. Namely, there holds the following result.

THEOREM 2. Under the assumptions of Theorem 1, let E be an arbitrary finite-dimensional $\pi_1(M)$-module (over a field of characteristic 0). Then $\dim H^1(\pi_1 M,\ E)\geqslant\dim H^0(\pi_1 M,\ E)-\dim H_0(\pi_1 M,\ E)+\sum_{i=1}^{n}\dim H_0(\pi_1(T_i),E)$.

3. SOME CONSEQUENCES OF THEOREM 1 AND REMARKS

3.1. Proof of the Corollary (for the Formulation, see the Introduction). We assume that G is a semisimple Lie group. Then the Killing metric on $\mathfrak{G}$ defines a nondegenerate Ad-invariant bilinear coupling on $\mathfrak{G}$, and so $H_0(\Gamma,\ \mathrm{Ad}\circ\rho)\simeq H^0(\Gamma,\ *\circ\mathrm{Ad}\circ\rho)\simeq H^0(\Gamma,\ \mathrm{Ad}\circ\rho)$ for any group Γ. Therefore $\dim H^0(\pi_1 M,\ \mathrm{Ad}\circ\rho)=\dim H_0(\pi_1 M,\ \mathrm{Ad}\circ\rho)$. It is easily seen that $\dim H^0(\pi_1 T_i,\ \mathrm{Ad}\circ\rho|_{\pi_1(T_i)})=\dim Z_G(\rho(\pi_1 T_i))$, from which there follows inequality (2).

We shall now demonstrate that for any connected semisimple Lie group $G\neq 1$, the dimension of $Z_G(A)$ is greater than zero, A being an arbitrary infinite Abelian subgroup of G [in particular, $\rho(\pi_1(T_i))$]. G is an algebraic group. The algebraic envelope $\mathfrak{Z}(A)$ of A is also an Abelian group and consists of a finite number of connected components. Therefore [since $\mathfrak{Z}(A)$) is infinite] the dimension of $\mathfrak{Z}(A)$ is greater than zero, and so $\dim Z_G(A)>0$. The corollary is proved.

3.2. Let $G=\mathrm{SL}_2(\mathbf{C})$, let M be a three-dimensional compact manifold such that $\mathbf{H}^3/\Gamma=\mathrm{int}\,M$, where Γ is a torsion-free nonuniform lattice in G, and let n be the number of components of the boundary of M; $\Gamma\simeq\pi_1 M$.

Proposition 2. If $\rho=\mathrm{id}\colon\Gamma\to G$, then $\dim_{\mathbf{C}}H^1(\Gamma,\ \mathrm{Ad})=n=(1/2)\dim_{\mathbf{C}}H^1(\partial M,\mathscr{L}_\rho)=(1/2)\sum_{i=1}^{n}\dim\times H^1(\pi_1 T_i,\ \mathrm{Ad})$.

Proof. We denote by $Z^1_{\mathrm{par}}(\pi_1 M,\ \mathrm{Ad})$ the space of cocycles c such that $c|_{\langle\gamma\rangle}$ is the coboundary in $Z^1(\langle\gamma\rangle,\ \mathrm{Ad})$ for any $\gamma\in\pi_1(T_i)$, where $i=1,\ldots,n$. $Z^1_{\mathrm{par}}(\Gamma,\ \mathrm{Ad})/B^1(\Gamma,\mathrm{Ad})$ is the space of parabolic cohomologies of $H^1_{\mathrm{par}}(\pi_1 M,\mathrm{Ad})$ (see, for example, [14]). It is easily seen that $\dim_{\mathbf{C}}H^1_{\mathrm{par}}(\pi_1 T_i,\ \mathrm{Ad})=1=(1/2)\dim_{\mathbf{C}}H^1(\pi_1 T_i,\mathrm{Ad})$ in the case under consideration. We set $H^1_{\mathrm{par}}\times(\partial M,\ \mathscr{L}_\rho)=\bigoplus_i H^1_{\mathrm{par}}(\pi_1 T_i,\ \mathrm{Ad})$, $i_*\colon H^1(M,\mathscr{L}_\rho)\to H^1(\partial M,\mathscr{L}_\rho)$ is the natural "restriction" homomorphism. By virtue of the results of [7], $H^1_{\mathrm{par}}(M,\mathscr{L}_\rho)=0$, and so $i_*(H^1(M,\mathscr{L}_\rho))$ intersects $H^1_{\mathrm{par}}(\partial M,\mathscr{L}_\rho)$ at the point 0 only and i_* is a monomorphism. Hence it follows immediately that $\dim_{\mathbf{C}}H^1(M,\ \mathscr{L}_\rho)\leqslant(1/2)\dim_{\mathbf{C}}H^1(\partial M,\mathscr{L}_\rho)$. On the other hand, by virtue of (3), $\dim_{\mathbf{C}}H^1(\pi_1 M,\ \mathrm{Ad})\geqslant n$, and so $\dim_{\mathbf{C}}H^1(\pi_1 M,\ \mathrm{Ad})=n$. Proposition 2 is proved.

We remark that $\dim_{\mathbf{C}}H^1(\Gamma,\ \mathrm{Ad})=n$ is the dimension of the tangent space (in the sense of Zariski) to $R(\Gamma,\ G)=\mathrm{Hom}(\Gamma,\ G)/\mathrm{ad}(G)$ at the point $[\rho]$ (see [15, Sec. 2]). Therefore, from inequality (3), Proposition 2, and the fact that simple points are dense in the complex-algebraic manifold $R(\Gamma,\ G)$, it follows that $[\rho=\mathrm{id}]$ is a simple point in $R(\Gamma,\ G)$ and there is a smooth manifold of complex dimension n in a neighborhood of this point. However, this fact can also be proved directly [without referring to the complex-algebraic nature of $R(\Gamma,\ G)$].

3.3. Let G be a Lie group, M be a three-dimensional manifold that satisfies the assumptions of Theorem 1, and let $\rho\colon\Gamma\to G$ be a homomorphism, where $\Gamma=\pi_1(M)$. We now assume that

(a) there holds the equality in (1),

(b) $H_0(\Gamma,\ \mathrm{Ad}\circ\rho)=0$,

(c) for all $i=1,\ldots,n$, $(\rho|_{\pi_1(T_i)})$ is a nonsingular point of the algebraic set $\mathrm{Hom}(\pi_1(T_i),\ G)$.

THEOREM 3. Under the above assumptions (a)-(c), $R(\Gamma,\ G)$ is a smooth manifold of dimension $\dim H^1(\Gamma,\ \mathrm{Ad}\circ\rho)$ in a neighborhood of $[\rho]$.

Proof. Since there holds the equality in (1), there also holds the equality in (4), and so the homomorphisms $\alpha\colon H^1(\partial V_1)\oplus H^1(\partial M)\to H^1(N(C))$ and $\beta\colon H^1(V_1'\cup V_2')\oplus H^1(N(Q))\to H^1(S_1)\oplus H^1(S_2)$

from the corresponding Mayer–Vietoris sequences (see Secs. 2.3 and 2.5) are endomorphisms. We remark that assumption (b) implies that $H^2(S_1)=H^2(S_2)=0$ and the point $[\rho|_{\pi_1(S_i)}]$ is nonsingular in $R(\pi_1(S_i), G)$ [16].

Remark. In what follows we find it convenient to pass from considering the representation spaces $R(\pi_1 Y, G)$ to the corrsponding spaces of flat connections (because we shall deal with disconnected manifolds). If Y is a manifold and $\rho: \pi_1(Y) \to G$ is a representation of its fundamental group, then we shall denote by $E = E(Y)$ the fiber bundle over Y with fiber $\mathfrak{G}$ (equipped with the standard vector space topology) constructed from the representation $\mathrm{Ad}\circ\rho: \pi_1(Y) \to GL(\mathfrak{G})$.

We consider the space of flat connections on E(Y) and its quotient space R(E(Y)) with respect to the group of gauge transformations. Then R(E(Y)) is diffeomorphic with the connected component of $R(\pi_1 Y, G)$ that contains $[\rho]$ (see [17]). We denote the corrsponding diffeomorphism by hol: $R(\pi_1 Y, G) \to R(E)$. There is a natural isomorphism between the "tangent space" $H^1(\pi_1 Y, \mathrm{Ad}\circ\rho)$ to $R(\pi_1 Y, G)$ (at the point $[\rho]$) and the "tangent space" $\mathcal{T}_A R(E)$ to R(E) (at $A = \mathrm{hol}[\rho]$). $\mathcal{T}_A R(E)$ is nothing but the quotient space $\mathrm{Ker}(d_A: \Lambda^1(\hat{Y}, E) \to \Lambda^2(Y, E))/d_A(\lambda^0(Y, E))$.

Let us go back to the proof of Theorem 3. We consider the natural "restriction" mappings (r_1, r_2): $R(E(\partial V_1)) \times R(E(\partial M)) \to R(E(C))^2$ and (r_3, r_4): $R(E(V_1' \cup V_2')) \times R(E(N(Q))) \to R(E(S_1 \cup S_2))^2$ [for $Y \subset M$, we denote by E(Y) the restriction of the corresponding fiber bundle $E(M, \rho)$ constructed from the homomorphisms $\rho: \pi_1(M) \to G$].

Let us now remark that the space R(E(N(Q))) of connections is diffeomorphic with the inverse image of the diagonal of the Cartesian product $R(E(N(C)))^2$ in $R(E(\partial V_1)) \times R(E(\partial M))$ and, analogously, $R(E(V_1' \cup V_2' \cup N(Q) = M))$ is diffeomorphic with the inverse iamge of the diagonal of $R(E(S_1 \cup S_2))^2$. The analytic sets $R(E(\partial M))$ and $R(E(\partial V_1))$ are smooth (in neighborhoods of the points hol$[\rho]$) by virtue of (b) and (c). From the fact that α is an epimorphism and the above remark it follows that the mapping (r_1, r_2) is transversal with respect to the diagonal of R(E(N(Q))). Hence it follows immediately that hol$[\rho]$ is a nonsingular point of the analytic set R(E(N(Q))). By analogy (owing to the fact that β is an epimorphism), the smooth mapping (r_3, r_4) is transversal with respect to the diagonal $R(E(S_1 \cup S_2))^2$, and so $[\rho]$ is a nonsingular point in $R(\pi_1 M, G)$ and the dimension of $R(\Gamma, G)$ is equal to $\dim H^1[\Gamma, \mathrm{Ad}\circ\rho)$ in a neighborhood of $[\rho]$. The theorem is proved.

3.4. We go back to the case where int M is the hyperbolic manifold $\mathbb{H}^3/\Gamma$, $\Gamma \subset PSL_2(\mathbb{C}) = G$, and $\rho: \pi_1 M \to \Gamma$ is the natural isomorphism. Then $\rho(\pi_1 T_i)$ is a group generated by two parabolic elements and $R(\pi_1 T_i, G)$ is a smooth manifold of complex dimension $2 = 2\dim H^0(\pi_1 T_i, \mathrm{Ad}\circ\rho)$, in a neighborhood of $[\rho|_{\pi_1(T_i)}]$. Therefore (by virtue of Theorem 3), $R(\Gamma, G)$ is a smooth manifold of complex dimension n in a neighborhood of $[\rho]$.

3.5. We consider the natural embeddings $i_1: \Gamma \subset PSL_2(\mathbb{C}) = SO_+(3,1)$ and $i_2: \Gamma \subset SO(4,1)$. Then, by virtue of the corollary, $\dim H^1(\Gamma, \mathrm{Ad}\circ i_2) \geqslant 3n > \dim H^1(\Gamma, \mathrm{Ad}\circ i_1) = 2n$. Therefore, there exist infinitesimal deformations of Γ in SO(4, 1) that move the group out of $PSL_2(\mathbb{C}) = SO_+(3, 1)$.

Let N be a closed hyperbolic manifold, and let $\rho: \pi_1 N \to \Gamma \subset PSL_2(\mathbb{C})$ be the natural representation of its fundamental group.

Conjecture. The embedding $\Gamma \subset SO(4, 1)$ is not locally rigid if and only if there exists an incompressible surface W in N that is not a virtual fiber of the fibration over S^1 (i.e., no connected component $p^{-1}(W)$ of any finite-sheeted covering $p: M \to N$ is a fiber for the fibration of M over a circle).

LITERATURE CITED

1. A. Weil, "Discrete subgroups of Lie groups. I, II," Ann. Math., 72, 369-384 (1960); 75, 578-602 (1962).
2. M. S. Raghunathan, Discrete Subgroups of Lie Groups, Springer-Verlag, New York–Heidelberg (1972).
3. G. Mostow, Strong Rigidity of Locally Symmetric Spaces, Princeton Univ. Press, Princeton (1973).
4. G. A. Margulis, "Arithmetic properties of discrete groups," Usp. Mat. Nauk, 29, No. 1, 50-96 (1974).
5. G. Prasad, "Strong rigidity of Q-rank 1 lattices," Invent. Math., 21, No. 4, 225-286 (1973).

6. D. Sullivan, "On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions," Ann. Math. Stud., No. 97, 465-496 (1981).
7. H. Garland and M. Raghunathan, "Fundamental domains for lattices in R-rank 1 semisimple Lie groups," Ann. Math., 92, 272-326 (1970).
8. W. Thurston, Geometry and Topology of 3-Manifolds, Princeton Univ. Press. Lecture Notes, Princeton (1978).
9. M. Culler and P. Shalen, "Varieties of group representations and splittings of 3-manifolds," Ann. Math., 117, No. 1, 109-146 (1983).
10. M. Neumann and D. Zagier, "Volumes of hyperbolic manifolds," Topology, 24, No. 3, 303-332 (1985).
11. M. É. Kapovich, "Deformations of the representations of the fundamental groups of three-dimensional manifolds," in: Proceedings of the All-Union Conference on Geometry, Kishinev, September 1988, Kishinev (1988).
12. J. Hempel, 3-Manifolds, Princeton Univ. Press, Princeton (1976).
13. G. E. Bredon, Sheaf Theory, McGraw-Hill, New York (1967).
14. A. Weil, "Remarks on the cohomology of groups," Ann. Math., 80, No. 1, 149-157 (1964).
15. A. Lubotzky and A. Magid, "Varieties of representations of finitely generated groups," Mem. Am. Math. Soc., 58, No. 336 (1985).
16. W. Goldman and J. Millson, "Deformations of flat bundles over Kahler manifolds," Lect. Notes Pure Appl. Math., 105, 129-145 (1987).
17. W. Goldman, "The symplectic nature of fundamental groups of surfaces," Adv. Math., 54, No. 2, 200-225 (1984).

TRANSITION PHENOMENA FOR THE TOTAL NUMBER OF OFFSPRINGS IN A GALTON–WATSON BRANCHING PROCESS

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INTRODUCTION

We consider a Galton–Watson branching process, starting with one particle at generation zero. By Z_n, n = 0, 1,..., we denote the number of particles in the n-th generation. In our case $Z_0 = 1$. We set $p_k = P(Z_1 = k)$; $f_n(x) = E(x^{Z_n})$, $|x| \leqslant 1$, $f(x) = f_1(x)$. Let λ be the smallest root of the equation $s = f(s)$, $0 \leqslant s \leqslant 1$. We shall make use of the following notations: $A = f'(1)$, $B = f''(1)$, $L = f'''(1)$, $A_0 = f'(\lambda)$, $B_0 = f''(\lambda)$. If $A \leqslant 1$, then λ = 1 and, therefore, $A_0 = A$, $B_0 = B$. If A > 1, then λ < 1 and $A_0 < 1$.

In this paper we prove limit theorems for the distribution $S_n = \sum_0^n Z_i$. As in [1], we investigate the conditional distribution $P(S_n < x | Z_n > 0)$, but, unlike the cases A = const, we consider the case when simultaneously $n \to \infty$, $A \to 1$. Limit theorems of this type are proved for $P(Z_n | Z_n > 0)$ in [2, 3], while for $P(S_n | Z_n = 0, Z_{n-1} > 0)$ in [4]. We mention that the limit law for $P(S_n/m_n < x | Z_n > 0)$ depends on the rate and the direction of the convergence of A to 1 with the increase of n. As normalizing constant we take $m_n = E(S_n | Z_n > 0)$. In connection with this, the asymptotic behavior of $E(S_n | Z_n > 0)$ is investigated.

We shall assume that the convergence for $n \to \infty$, $A \to 1$ is carried out with respect to the class K of distributions, satisfying the following conditions:

A) $\sum_2^\infty l(l-1) p_l(F) > \beta_0 > 0$ for some β_0 and for any $F \in K$;

B) $\lim_{n\to\infty} \sup_{F\in K} \sum_n^\infty l^2 p_l(F) = 0$;

C) $p_0(F) > \alpha_0 > 0$ for all $F \in K$.

Here $p_\ell(F)$ is the atom of the distribution F at the point ℓ. We note that by virtue of B) there exists β_1 such that for each $F \in K$ we have

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