Consequently, $\dim[R(d_\Gamma)/R(d_V)]<\infty$. By Theorem 1, the operator d_V is compactly solvable. By Lemma 3, the operator d_Γ is compactly solvable. The theorem is proved.

Note that in [1] there have been constructed for every $k \neq 0$, $n-1$ and $p = q = 2$ examples of operators d_Γ which are not normally or compactly solvable.

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CONFORMALLY FLAT STRUCTURES ON 3-MANIFOLDS: EXISTENCE PROBLEM. I*

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INTRODUCTION

A conformally flat structure on a manifold M (of dimension $n\geqslant 3$) is a maximal atlas $K=\{(U_i,\varphi_i)\ \varphi_i\colon U_i\to V_i\subset\bar{\mathbf{R}}^n, i\in I\}$, in which the transition maps are conformal (i.e., $\varphi_i\circ\varphi_j^{-1}$ is a restriction of a Möbius automorphism of $\bar{\mathbf{R}}^n$). There is also another, classical definition of conformally flat structure (CFS), as the class of conformally equivalent conformally Euclidean metrics on M [i.e., metrics locally expressible as $\rho(x)|dx|^2$, where $\rho(x)$ is a smooth positive function]. That these definitions are equivalent was proved in [1, 2]. It is well known that metrics of constant sectional curvature are conformally Euclidean (see [3]). Yet another characterization of CFS makes use of Kleinian groups: if a Kleinian group Γ is free and acts discontinuously on a domain Ω (for the detailed definitions see below, Sec. 1), then the quotient manifold $M = \Omega/\Gamma$ admits a natural CFS K_Γ for which the cover $p\colon \Omega \to M$ is a conformal map. Such structures are said to be uniformizable, and Γ is a uniformizing group.

The particular interest in conformally flat structures on 3-manifolds is due largely to the fact that five of the eight homogeneous Riemann spaces in three dimensions are conformally Euclidean: S^3, $\mathbf{E}^3$, $\mathbf{H}^3$, $S^2\times\mathbf{R}$, $\mathbf{H}^2\times\mathbf{R}$ (see [4]). The following theorem of Thurston is well known [5, 6]:

THEOREM H. Let M be a closed atoroidal Haken manifold. Then there exists a hyperbolic structure (i.e., a metric of sectional curvature −1) on M.

Thus manifolds of this class admit CFSs. On the other hand, it follows from results of Goldman [7] that if M is a closed 3-manifold whose fundamental group is solvable but not a finite extension of an Abelian group (i.e., M is either a Sol- or a Nil-manifold; see [4]), then M does not admit a CFS.

Our aim is to prove the following theorem, according to which there exist CFSs on a broader (than atoroidal) class of Haken manifolds.

THEOREM C. Let M be a closed Haken 3-manifold with unsolvable fundamental group, such that M, when obtained by gluing hyperbolic and Seifert pieces together along tori, does not contain combinations of hyperbolic manifolds with hyperbolic or Euclidean manifolds (in the sense of [4]). Then there exists a finite-sheeted cover M_0 over M which admits a uniformizable CFS.

*Dedicated to Yurii Grigor'evich Reshetnyak on his sixtieth birthday.

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The proof will be divided into three steps. In this paper we carry out the first two; the third will be the subject of a forthcoming paper. In Sec. 1 we introduce the necessary definitions. In Sec. 2 we prove

THEOREM A. Let S(g, e) be a fiber space over a closed orientable surface S_g of genus g, with fiber S^1 and Euler number e > 0 such that $e \leqslant (g-1)/11$. Then the space of S(g, e) admits a uniformizable CFS.

COROLLARY. If M is a Seifer fiber space and $\pi_1(M)$ is unsolvable, then the conclusion of Theorem C is true for this manifold.

As an application we shall construct an example of a discrete uniformly quasiconformal group Γ which is not topologically conjugate to any subgroup of the Möbius group (Corollary 3).

In Sec. 3 we prove

THEOREM B. Let M be a closed manifold obtained by gluing Seifert fiber spaces $Z_1, \ldots, Z_s$ together along boundary tori (i.e., M is a "graph-manifold" in Waldhausen's sense), such that $\pi_1(M)$ is unsolvable. Then the conclusion of Theorem C is true for M.

In Sec. 3 we again construct an example: a manifold which itself does not admit a CFS, but has a conformally flat finite-sheeted cover. This manifold will be obtained from a certain Seifert fiber space by gluing together two components of the boundary (Theorem D).

In our forthcoming paper we shall prove Theorem C in the general case — when the manifold is obtained by gluing together both Seifert and hyperbolic components. The main idea of the proof of Theorem C is to deform the CFSs on finite-sheeted covers over the hyperbolic and Seifert components glued together to get M, in such a way that the gluing operation can be done conformally.

We recall that by a result of Kulkarni [2], if M_1 and M_2 are conformally flat manifolds, there exists a CFS on their connected sum. In view of Theorem C and Kulkarni's theorem, the following conjecture is plausible.

Conjecture. Let M be a closed 3-manifold satisfying Thurston's geometrization conjecture. Conjecture (see [4, 6]), i.e., obtained from manifolds admitting a geometric structure by gluing together along tori and connected sum operations. Assume further that the decomposition of M as a connected sum of primitive manifolds does not involve terms with Sol- or Nil-structure. Then M has a finite-sheeted cover that admits a CFS.

1. DEFINITIONS AND NOTATION

1.1. Let $\mathscr{M}_n$ be the group of all orientation-preserving Möbius automorphisms of the n-dimensional sphere $S^n = \overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. If $\gamma \in \mathscr{M}_n$, we let Fix(γ) denote the set $\{x \in S^n\colon \gamma(x) = x\}$. The region of discontinuity of a group $\Gamma \subset \mathscr{M}_n$ is the set R(Γ) of all points $x \in S^n$, having a neighborhood U(x) such that the intersection $U(x) \cap \gamma U(x)$ is empty for all but a finite number of $\gamma \in \Gamma$. A connected component $R_0 \subset R(\Gamma)$, which is invariant under Γ is called an invariant component of Γ. The group Γ acts freely on R_0 is the stabilizer Γ_x of every point $x \in R_0$ is trivial. Thus, Γ acts freely on an invariant component $R_0 \subset R(\Gamma)$ if and only if the natural projection $q\colon R_0 \to R_0/\Gamma$ is a cover. A group $\Gamma < \mathscr{M}_n$, with a nonempty set R(Γ) is called a Kleinian group, and $L(\Gamma) = S^n \backslash R(\Gamma)$ is known as the limit set of Γ.

If Γ is a Kleinian group, R_0 an invariant component of Γ on which the group acts freely, then a set $\Phi_0 \subset R_0$ is called a fundamental region for the action of Γ on R_0 if (a) $\mathrm{cl}\,\Phi_0 = \mathrm{cl}\,\mathrm{int}\,\Phi_0$, $\mathrm{int}\,\Phi_0 = \mathrm{int}\,\mathrm{cl}\,\Phi_0$, (b) $\bigcup_{\gamma \in \Gamma} \gamma\Phi_0 = R_0$, (c) $\gamma\Phi_0 \cap \Phi_0 = \varnothing$ for all $\gamma \in \Gamma \backslash \{1\}$, (d) the family $\Gamma\,\mathrm{cl}\,\Phi_0$ is locally finite. The details may be found, e.g., in [8, 9]. Thus, a manifold $M(\Gamma) = R_0/\Gamma$ uniformizable by Γ is obtained from $\mathrm{cl}\,\Phi_0$ by identifying boundary points that are equivalent relative to Γ (i.e., x and γx, $\gamma \in \Gamma$).

1.2. A 3-manifold M is said to be irreducible if any polyhedral sphere embedded in M bounds a ball. An irreducible 3-manifold M is called a Haken manifold if it admits an embedding i: S → M of a closed surface, neither S^2 nor RP^2, such that the induced map $i_* : \pi_1 \times (S) \to \pi_1(M)$.

Remark. Throughout this paper we shall be concerned only with orientable 3-manifolds.

Thurston's hyperbolization theorem [5, 6] states that if M is Haken, ∂M is the union of finitely many tori $T_1 \cup \ldots \cup T_n$ and M is atoroidal [i.e., for any subgroup $\mathbf{Z} + \mathbf{Z} \subset \pi_1(M)$ there exists a conjugate subgroup $A\subset\pi_1(T_i)$ for some i], then there exists a complete metric of constant negative curvature on int M. A manifold satisfying this condition is said to be hyperbolic.

The main definitions and facts from the theory of orbifolds may be found, e.g., in [4], and the definition of compact three-dimensional Seifert fiber spaces in [4, 10, 11]. We mention only that (if $|\pi_1(M)|=\infty$) a manifold M is a Seifert fiber space if and only if it has a finite-sheeted cover which is an ordinary fiber space over an orientable surface (possibly with boundary) with fiber S^1. In addition, the fundamental group of a Seifert fiber space over an orbifold $\mathcal{O}$ can be embedded in a short exact sequence $1\to\mathbf{Z}\to\pi_1(M)\to\pi_1(\mathcal{O})\to 1$, where $\mathbf{Z}\subset\pi_1(M)$ is generated by a regular fiber of the Seifert fiber space (for short, we shall call this a fiber).

1.3. We shall also need the following geometric description of a fiber space S(g, e) with fiber S^1, base space S_g and Euler number $e\in\mathbf{Z}$.

Let $\Sigma_g=S_g\setminus\operatorname{int}B^2$, where B^2 is a closed disk, $x\in\partial B^2$, $\mathfrak{M}=\Sigma_g\times S^1$, $t=\{x\}\times S^1\subset\mathfrak{M}$, $\beta=\partial B^2\times\{\varphi\}$, where $\varphi\in S^1$, $T=\partial B^2\times S^1$ is the boundary of the manifold $\mathfrak{M}$. Let $\mathrm{T}=B^2\times S^1$ be a solid torus, $\tau=\{x\}\times S^1\subset\partial\mathrm{T}$, $\varkappa=\partial B^2\times\{\varphi\}\subset\partial\mathrm{T}$. The corresponding elements of $\pi_1(\mathrm{T})$ and $\pi_1(\partial\mathrm{T})$ are again denoted by t, β, τ, ϰ. Glue T to $\partial\mathfrak{M}$ so that the loop t is glued to τ and the loop β to the loop $\varkappa\tau^e$. The manifold thus obtained is precisely S(g, e) (clearly, only |e| is of topological significance).

1.4. Let M be a 3-manifold. We shall say that M admits a geometric structure (is geometrical) if it has the form X/Γ, where X is one of the eight three-dimensional homogeneous Riemannian spaces (see [4]): E^3, S^3, $\mathbf{H}^3$, $\mathbf{H}^2\times\mathbf{R}$, $S^2\times\mathbf{R}$, $\widetilde{SL}_2(\mathbf{R})$, Sol, Nil, and Γ is a discrete subgroup of subgroup of Isom(X) acting freely on X. In the case of a manifold admitting a Sol- (or Nil)-structure we shall speak of a Sol- (or Nil)-manifold.

It follows from results of [5, 6, 11, 12] that if M is a closed Haken manifold, then M can be cut into maximal geometrical components (in this case – open ones); up to isotopy this can be done in only one way.

2. CONFORMALLY FLAT STRUCTURES ON SEIFERT FIBER SPACES

2.1. We first observe that if M is a Seifert fiber space over a hyperbolic base with Euler number zero, there exists a Kleinian group F uniformizing M. Indeed, a Seifert fiber space satisfying this condition admits an $\mathbf{H}^2\times\mathbf{R}$-structure (see [4]), i.e., it has the form $\mathbf{H}^2\times\mathbf{R}/\Gamma$, where Γ is a subgroup of the group of isometries of $\mathbf{H}^2\times\mathbf{R}$. It is readily verified that a generator t of Γ generating a normal cyclic subgroup may be chosen as follows: $t(z,\varphi)=(z,\varphi+2\pi)$, where z is a coordinate on $\mathbf{H}^2$, and φ a coordinate on $\mathbf{R}$. Then $\mathbf{H}^2\times\mathbf{R}/\langle t\rangle$ is isometric to $X=R^3\setminus\{(x_1,x_2,x_3)\colon x_1=0\}$, where we have introduced the metric $ds^2=|dx|^2(x_2^2+x_3^2)^{-1}$, and the group F = Γ/⟨t⟩ acts freely on X as a discrete group of isometries. Clearly $F\subset\mathcal{M}_3$ is the required group uniformizing M.

At the same time, an invariant Riemannian metric on the group $\widetilde{SL}_2(\mathbf{R})$ is not conformally Euclidean, and so this kind if argument collapses entirely in the attempt to define a CFS on an $\widetilde{SL}_2(\mathbf{R})$-manifold.

2.2. Proof of Theorem A. Our main goal will be to construct a Kleinian group H = H(g, 1) such that R(H)/H = M(H) is homeomorphic to S(g, 1), where g = 12 [and in that case H ≃ $\pi_1(S_g)$]. The fundamental polyhedron Φ of H is homeomorphic to a solid torus and satisfies the following conditions.

(a) The faces of the polyhedron, $Q_1, R_1', Q_1', R_1, \ldots, Q_g, R_g', Q_g', R_g$, lie on Euclidean spheres in $\mathbf{R}^3$ and are homeomorphic to annuli. Two adjacent faces (i.e., appearing successively in the above chain, and also R_g and Q_1) intersect in a circle; faces which are not adjacent do not intersect (Fig. 1).

The faces of Φ are identified by Möbius transformations $A_1\colon Q_1\to Q_1'$, $B_1\colon R_1\to R_1'$, ..., $A_g\colon Q_g\to Q_g'$, $B_g\colon R_g\to R_g'$, which generate the group H.

Let $x_0\in Q_1\cap R_g$, $x_1=B_1^{-1}\circ A_1^{-1}\circ B_1\circ A_1\,(x_0)=[A_1,B_1](x_0)\in Q_2\cap R_1$ and so on, $x_g=[A_g,B_g]\circ\ldots\circ[A_1,B_1](x_0)\in R_g\cap Q_1$.

(b) We stipulate that $x_g = x_0$. If in addition the sum of the dihedral angles of Φ is 2π, then Φ is a fundamental region for the group $H=\langle A_1,B_1,\ldots,A_g,B_g\colon [A_g,B_g]\times\ldots\times[A_1,B_1]=1\rangle$. In order to see this, it suffices to extend Φ into the hyperbolic space $\mathbf{H}^4$ (every sphere can be extended to a geodesic hypersurface) and to apply the arguments of [13].

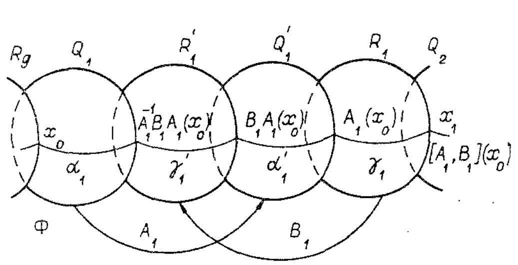

Fig. 1

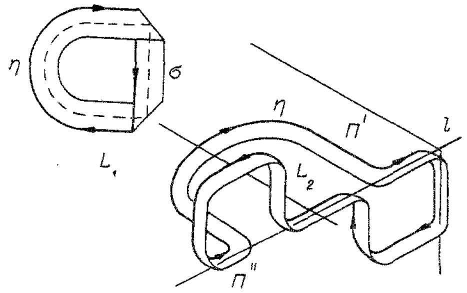

Fig. 2

Let α_1 be a simple curve on Q_1 connecting x_0 and $A_1^{-1}B_1A_1(x_0)$, let $\gamma_1 \subset R_1$ be a curve connecting $A_1(x_0)$ and x_1, $\alpha_1' = A_1(\alpha_1)$, $\gamma_1' = B_1(\gamma_1)$. Similar constructions yield curves $\alpha_2, \alpha_2', \ldots, \gamma_g, \gamma_g'$ (see Fig. 1). Thanks to condition (b), the union of these curves is a simple closed curve on $\partial\Phi$, which we denote by η. Assume that the following condition holds:

(c) The linking number of η and the axis of the solid torus $S^3\backslash\Phi$ is $|e| = 1$.

It is easy to see that condition (c) is equivalent to the following: η is homotopic on $\partial\Phi$ to a loop $t+\varkappa$, where $t = Q_1 \cap R_g$, and the class $[\varkappa]$ generates the kernel of the homomorphism $\pi_1(\partial\Phi) \to \pi_1(\Phi)$ (the loop $\varkappa$ is homotopic in $S^3\backslash\Phi$ to the axis of the solid torus).

2.3. We claim that if conditions (a)-(c) are fulfilled, then H uniformizes S(g, 1) (the fiber space over S_g with fiber S^1 and Euler number 1). Let $T' \subset \Phi$ be a torus parallel to $\partial\Phi$ and $\mathscr{F}$ a component of $\Phi\backslash T'$, lying between $\partial\Phi$ and T'. The manifold M(H) = R(H)/H is homeomorphic to Φ, provided that points of the boundary equivalent relative to H are identified. Let q: $\Phi \to$ M(H) be the natural projection, $\mathfrak{M} = q(\mathscr{F})$, $\beta = q(\beta')$, where $\beta' \subset T'$ is a loop parallel in $\Phi\backslash\mathscr{F}$ to η. Then the manifold M(H) is obtained by gluing together $\mathfrak{M}$ (which is homeomorphic to $\Sigma_g \times S^1$) and $T = q(\Phi\backslash\mathscr{F})$ – but this is precisely the construction of Sec. 1.3 for the case $|e| = 1$.

2.4. We now proceed to the construction of Φ. Note that on the twice twisted tape L_1 (Fig. 2) the linking number of the central line σ and the curve η is 1. In the same figure we also see an equivalent tape L_2 in which the folded-over sections have been "separated." Our problem will be to "pave" L_2 with spheres in such a way that conditions (a)-(c) of 2.2 will be satisfied.

Dividing L_2 into two parts: L_2', lying in the horizontal plane Π', and L_2'' in which the central line σ lies in the vertical plane Π''. Let $l = \Pi' \cap \Pi''$ and let $\Lambda' \subset \Pi'$ be the axis of symmetry of L_2, $O = l \cap \Lambda'$. We shall treat l and Λ' as coordinate axes in Π' (Fig. 3).

Let O_1 and O_2 be the points with coordinates (0, 1) and (2, 1), respectively, and $l_1 \subset \Pi'$ the straight line through O_1 and O_2. Let $\alpha = \pi/8$, $\varepsilon = \pi/24$, and let C_1 be the point with coordinates $(1, 1 - \tan(\alpha/2))$. Define Q_1 (the same letter will denote the sphere and the face of the polyhedron Φ on it) to be the sphere with center C_1 and radius $r = \tan(\alpha/2)/\cos(\varepsilon/2)$. The spheres R_1', Q_1', R_1 and Q_2 are obtained from Q_1 by rotations about O_2 through angles α, 2α, 3α, 4α. Similarly, the spheres R_{12}, Q_{12}', E_{12} and Q_{12}' are obtained by rotating the same sphere about O_1 through the same angles (see Fig. 3). It is readily seen that the angles between adjacent spheres are ε, and the centers of R_1 and Q_2 lie on the axis ℓ. We have thus constructed the required "paving" of L_2'. Let J_1 be inversion with respect to Q_1 and σ_1 symmetry with respect to the plane orthogonal to Π' and passing through O_1 and the center of the sphere R_1'; define $A_1 = \sigma_1 \circ J_1$. Similarly, we let I_1 be inversion with respect to R_1 and θ_1 symmetry with respect to the plane orthogonal to Π_1 and passing through O_1 and the center of Q_1', $B_1 = \theta_1 \circ I_1$. It is easy to see that $A_1(Q_1) = Q_1'$, $B_1(R_1) = R_1'$, $A_1(Q_1 \cap R_1') = R_1' \cap Q_1'$ and so on.

We now turn to the plane Π''. Let $\Lambda'' \subset \Pi''$ be the straight line orthogonal to l and passing through O. Introduce a coordinate system (l, O, Λ'') on Π'' (see Fig. 3). Let $O_3 = (2, 1)$, $O_4 = (1, 0)$ be points on Π''. The spheres $R_2', Q_2', R_2, \ldots, R_4, Q_5$ are obtained from Q_2 by rotation about O_3 through angles $\alpha, 2\alpha, 3\alpha, \ldots, 11\alpha, 12\alpha$. All these spheres are orthogonal to

Π'' and the angles between them are ε. Finally, the spheres R_5', Q_5' and R_5 are obtained from Q_5 by rotation about O_4 through angles α, 2α, 3α. The center of R_5 is on the line ℓ.

The system of spheres $Q_6, R_6', \ldots, Q_{11}', R_{11}$ is obtained by symmetry about the axis Λ' from the already constructed family of spheres. The angle between any two adjacent spheres is ε. The exterior of the spheres $Q_1,\ldots,R_{12}$ is the required polyhedron Φ. Indeed, the sum of its dihedral angles is $48\varepsilon = 2\pi$. The generators $A_2, B_2,\ldots,A_{12}, B_{12}$ are constructed by analogy with A_1 and B_1: $A_i = \sigma_i \circ J_i$, $B_i = \theta_i \circ I_i$, where J_i and I_i are inversions with respect to Q_i and R_i, and σ_i and θ_i symmetry with respect to planes equidistant from the centers of Q_i, Q_i' and R_i, R_i', respectively.

Let $x_0 \in Q_1 \cap l_1$ be the point nearest O_2 It is readily seen that $[A_{12}, B_{12}]\circ\ldots\circ[A_1, B_1](x_0) = x_0$, and the curve η and $\partial\Phi$ constructed as in Sec. 2.2 has linking number 1 with the axis of the solid torus $\mathbf{R}^3\backslash\Phi$. We have thus constructed the required group H = H(12, 1) uniformizing S(12, 1).

2.5. We now show that for any g and e [such that $1 \leqslant |e| \leqslant (g-1)/11$) there exists a Kleinian group H(g, e) uniformizing S(g, e). Let H be a subgroup of H(12, 1) of index j. It follows at once from Lemma 3.5 of [4] and the Riemann–Hurwitz formula that H = H(11j + 1, j). If H(12, 1) = H + h_1H + ... + h_jH is the coset decomposition of this group, then the fundamental polyhedron Ψ of H is the union $\Phi \cup h_1(\Phi) \cup \ldots \cup h_j(\Phi)$. The elements $h_1,\ldots,h_j$ may be so chosen that Ψ is homeomorphic to a solid torus. We may assume that the boundary of Ψ contains the piece $h_1(\Phi \cap (Q_{11} \cup \ldots \cup R_{12}))$. The transformations $A_{11}' = h_1 A_{11} h_1^{-1}$, $B_{11}' = h_1 B_{11} h_1^{-1}$, $A_{12}' = h_1 A_{12} h_1^{-1}$ and $B_{12}' = h_1 B_{12} h_1^{-1}$ of H, which identify the faces of this piece, leave invariant a certain circle C [the image under h_1 of the circle about O_1 of radius $1 - r^2 \sin^2(\varepsilon/2)$, in the plane Π']. Let Γ_m be a Kleinian group leaving C invariant (as well as the Euclidean disc D spanned by the circle), such that $(D\backslash L(\Gamma_m))/\Gamma_m$ is homeomorphic to a surface of genus m + 2 with one boundary component $\Gamma_m = \langle E_{11}, D_{11}, \ldots, E_{12+m}, D_{12+m}\rangle$, $[A_{12}', B_{12}'][A_{11}', B_{11}'] = [E_{12+m}, D_{12+m}] \times \ldots \times [E_{11}, D_{11}]$. Then Γ_m can be combined in Maskit's sense (see [14], also [15, Chap. IV. Sec. 1, p. 169]) with the group H' generated by the elements of H that identify the faces of the polyhedron $\Psi\backslash h_1(Q_{11} \cup \ldots \cup R_{12})$ (the amalgamated subgroup is $\langle h = [A_{12}', B_{12}'][A_{11}', B_{11}']\rangle$). It is not hard to see that the combined group thus formed, $H^{(m)} = H' \underset{(h)}{*} \Gamma_m$, uniformizes the manifold S(11j + 1 + m, j); hence, setting m = g − (11j + 1), j = |e|, we obtain the required group H(g, e), completing the proof of the theorem.

2.6. Let $\tilde{H}$(g, e) be an extension of H(g, e) to $\overline{\mathbf{R}}_+^4 = \{(x_1, x_2, x_3, x_4)\colon x_4 \geqslant 0\} \cup \{\infty\} = \mathbf{H}^4 \cup S^3$, $M(g, e) = \overline{\mathbf{R}}_+^4 \setminus L(H(g, e))/H(g, e)$. Note that the manifold M(g, e) is a fiber space over S_g whose fiber is a "closed disk," and the absolute value of its Euler number is e. In order to see that M(g, e) is the total space of the fibration, it will suffice to extend the fundamental region Φ of H(g, e) to a polyhedron $\tilde{\Phi}$ in $\mathbf{H}^4$, whose faces are hyperplanes based on corresponding spheres in S^3. The natural foliation of $\partial\Phi$ into circles extends to a foliation of $\partial\tilde{\Phi}$ into two-dimensional planes in $\mathbf{H}^4$, which in turn extends to a foliation of $\tilde{\Phi}$ having the local structure of a product. The structure of the foliation is now dropped to M(g, e), which becomes a fiber space over S_h with fiber D^2. The Euler class of the resulting fibration is equal in absolute value to e; this follows from the fact that ∂M(g, e) = S(g, e) is a fiber space with Euler number e.

COROLLARY 1. Let E → S_g be a fibration with fiber $\mathbf{R}^2$ and Euler number $e \in \mathbf{Z}$, such that $|e| \leqslant |\chi(S_g)|/22$, $g \geqslant 12$ [where $\chi(S_g)$ is the Euler characteristic of S_g]. Then there exists a complete metric of constant negative curvature on E.

Remark. Analogues of Theorem A and Corollary 1 – though without explicit estimates of |e| – have been proved independently in a preprint of Gromov, Lawson, and Thurston [16].

COROLLARY 2. Any Seifert fiber space with hyperbolic base (see [4]) is almost conformally flat (i.e., it has a finite-sheeted cover by which is a manifold admitting a CFS).

Proof. It will suffice to consider the case of a closed Seifert fiber space with Euler number zero. The group π_1(M) can be embedded in a short exact sequence $1 \to \mathbf{Z} \to \pi_1(M) \overset{\varphi}{\to} F \to 1$, where F is isomorphic to a discrete subgroup of $PSL(2, \mathbf{R})$. Then F contains a subgroup of finite index F_0 which is isomorphic to $\pi_1(S_g)$, where the genus of S_g is at least 12. Let $G_0 = \varphi^{-1}(F_0)$. Then G_0 has a corepresentation $\langle a_1, b_1, \ldots, a_g, b_g, t\colon [a_i, t] = [b_j, t] = [a_1, b_1] \times \ldots \times [a_g, b_g] t^{-e} = 1\rangle$, where e ≠ 0. If $\tau = t^e$, then the index of the subgroup $G_0' = \langle a_1, b_1, \ldots, a_g, b_g, \tau\colon [a_1, b_1] \times \ldots \times [a_g, b_g]\tau^{-1} = 1\rangle$ in π_1(M) is finite.

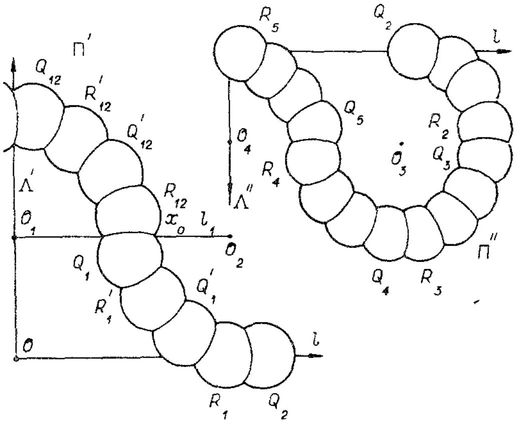

Fig. 3

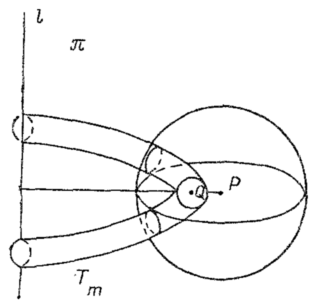

Fig. 4

The cover constructed on the basis of this subgroup is homeomorphic to S(g, 1) and is the required conformally flat manifold (since $g \geqslant 12$ and Theorem A is applicable).

Remark. The analogous assertion for the case g = 1, i.e., when the base space is Euclidean, is no longer true [7].

2.7. Later we shall need a certain modification of the groups H(g, e) constructed in Theorem A. Consider a circle in a plane π, say $O(P, \rho)$ with center P and radius ρ; let l be a straight line in the same plane, whose distance from P is $\rho + R$, where $R > 0$. Rotating $O(P, \rho)$ in $\mathbf{R}^3$ about l, we obtain a torus, denoted by $T(R, \rho)$; call ρ the inner and R the outer radius of the torus.

Note that the exterior of the fundamental polyhedron Φ of any group H(g, e) as constructed in Sec. 2.3 is contained in a ball of radius 4 (centered at O), and the radius of any sphere (containing a face of Φ) is at most $r = \tan\alpha/\cos\varepsilon < 0.2$. For every natural number $m \geqslant 0$, let us consider the torus $T = T(10(m + 1), 8)$ with rotation axis l. Within this torus, consider the solid torus T_m obtained by rotating the disk D(Q, 0.5) about l (Fig. 4), where the center Q of the disk is situated on the perpendicular dropped from P to l, at a distance 2 from P. Then for given m and Euler number e there exists a number $g_0 = g_0(m, e)$ such that for all $g \geqslant g_0$ there is a Kleinian group $H_m(g, e)$ [the above-mentioned modification of H(g, e)] with the following properties:

(a) $H_m(g, e)$ uniformizes S(g, e);

(b) $H_m(g, e)$ has a fundamental polyhedron $\Phi_m(g, e)$ homeomorphic to a solid torus, whose complement in S^3 (1) lies in the union of the solid torus T_m and the ball B(P, 8) of radius 8 about P, (2) forms a link of index 1 (as the construction of this group is entirely analogous to the construction of Sec. 2.5, we shall not go into details).

2.8. Recall that a group Γ of homeomorphisms of S^n is said to be (uniformly) quasiconformal if $\sup\{K(\gamma), \gamma \in \Gamma\} < \infty$, where $K(\gamma)$ is the quasiconformality coefficient (see, e.g., [17]). Various examples have been constructed [18-20] to refute the conjecture, advanced in [21], that any such group is quasiconformally conjugate to a conformal group. We are going to show how Theorem A can be used to construct an example of a quasiconformal topologically nonstandard (i.e., not conjugate to a topologically conformal) action of the group $\pi_1(S_g) \times \mathbf{Z}_n$ on the 3-sphere.

Let H = H(12, 1) be the group constructed in Theorem A, φ: $M(H) \to M(H)$ a diffeomorphism of order $n \geqslant 2$, isotopic to the identity (which exists because Seifert fiber spaces admit an S^1-action [10]). Let $\tilde{\varphi}$ denote a lifting of order n of φ to the region of discontinuity R(H). Then $K(\tilde{\varphi}) < \infty$, $\varphi \circ h = h \circ \tilde{\varphi}$ for all $h \in H$, so $\tilde{\varphi}$ extends to a quasiconformal homeomorphism on the whole of S^3 (see [22-24]).

Remark. We have thus proved that L(H) is an unknotted circle in S^3 for any group H that uniformizes a Seifert fiber space over a hyperbolic orbifold [24]. Denote the extension of $\tilde{\varphi}$ to S^3 by f. Then $\Gamma = H \times \langle f \rangle \simeq \pi_1(S_g) \times \mathbf{Z}_n$ is a discrete quasiconformal group. In addition, every element of Γ is quasiconformally conjugate to some Möbius transformation, and Γ itself is isomorphic to a subgroup of $\mathscr{M}_3$.

COROLLARY 3. The group Γ is not topologically conjugate to any subgroup of $\mathscr{M}_3$.

Proof. Suppose that there is such a conjugation g, then the group $G = g\Gamma g^{-1} \subset \mathcal{M}_3$ leaves the Euclidean circle Fix(g/g^{-1}) invariant. But the manifold $M(gHg^{-1})$ is homeomorphic to M(H) and has a nontrivial Euler class, which is impossible since there is an $\mathbf{H}^2 \times \mathbf{R}$-structure on $M(gHg^{-1})$ (cf. Sec. 2.1 in this paper, and also [4, Sec. 4]).

3. CONFORMAL GLUING OF SEIFERT FIBER SPACES

3.1. Let $Z_1, \ldots, Z_s$ be a collection of Seifert fiber spaces and M an orientable manifold obtained by gluing them together at boundary tori (i.e., M is a "graph-manifold"). Assume that $\pi_1(M)$ is not solvable. In this section we shall prove that there exists a finite-sheeted cover M_0 of M which admits a uniformizable conformally flat structure.

Before proceeding to the proof, we outline the main idea. Let $Z_1 = S'_{g_1} \times S^1$, $Z_2 = S'_{g_2} \times S^1$, where S'_{g_i} is a surface of genus $g_i > 0$ with one boundary component. Splitting Z_i into a direct product determines a "natural" basis in $\pi_1(\partial Z_i)$ (for more details, see Sec. 3.3). Suppose that M is obtained by gluing Z_1 and Z_2 together by means of a homeomorphism $f\colon \partial Z_1 \to \partial Z_2$, defined relative to the natural bases by a matrix $A \in GL_2^-(\mathbf{Z})$, where $a_{21} = 1$. Take the groups $H(g_1, a_{22})$ and $H(g_2, a_{11})$. constructed in Theorem A (they exist if g_1 and g_2 are sufficiently large), and place them in S^3 in such a way that the complements of the fundamental polyhedra form a link of index 1. It is not hard to see that the Klein combination $G = H(g_1, a_{22}) * H(g_2, a_{11})$, of these groups uniformizes M (note that with this method of constructing the condition $a_{21} = 1$ is absolutely unavoidable). Our goal will be to construct a finite-sheeted cover of M (in Theorem B) obtained by gluing products of surfaces of large genus to a circle, with coefficients a_{21} equal to unity for all the gluing homeomorphisms.

3.2. Proof of Theorem B. By Theorem A, we may assume without loss of generality that M is not a Seifert fiber space. Our first task is to construct a cover over M which, when cut along incompressible tori, will contain as components only trivial Seifert fiber spaces (i.e., products of a surface and a circle). Let Z_i be a fiber space over an orbifold $\mathcal{O}_i$, other than $S^1 \times [0, 1]$ (we may assume without loss of generality that there are no components $T^2 \times [0, 1]$ among the Z_i). To each component $\beta_{ij} \subset \partial\mathcal{O}_i$ we glue a disk $\mathcal{D}_{ij}$ with a singular conical point ζ_{ij} (with angle $2\pi/p$, $7 \leqslant p$ a prime). Denote the resulting orbifold by $\mathcal{O}'_i$. It is readily seen that $\mathcal{O}'_i$ is a "good" orbifold (see [4, Sec. 2]), and therefore there exists an even-sheeted regular cover $\varphi_i\colon \mathcal{P}'_i \to \mathcal{O}'_i$ of the orbifold which is orientable by a surface. Remove the disks $\varphi_i^{-1}(\mathcal{D}_{ij})$, from $\mathcal{P}'_i$. The resulting surface $\mathcal{P}_i$ covers our original orbifold $\mathcal{O}_i$. It is not hard to see that there exist a Seifert fiber space W_i over $\mathcal{P}_i$ and a cover $\psi_i\colon W_i \to Z_i$, corresponding to a cover $\varphi_i\colon \mathcal{P}_i \to \mathcal{O}_i$ of the bases and a p-fold cover of the fiber of Z_i by the fiber of Z_i by the fiber of W_i (cf. [25]). Since $\partial W_i \neq \varnothing$, the surface $\mathcal{P}_i$ is orientable and the Seifert fibration $W_i \to \mathcal{P}_i$ has no singular fibers, it follows that W_i is homeomorphic to $\mathcal{P}_i \times S^1$ [4]. The cover ψ_i has the property that if T_{ij} is a component of ∂Z_i and $\psi_{ij}\colon \tilde{T}_{ij} \to T_{ij}$ is the restriction of ψ_i to a component of $\psi_i^{-1}(T_{ij})$, the the defining subgroup of ψ_{ij} is the subgroup $p(\mathbf{Z}+\mathbf{Z}) \subset \mathbf{Z}+\mathbf{Z} \simeq \pi_1(T_{ij})$. Thanks to this property we can glue the manifolds W_i together to get a cover M_1 over M (cf. [25, Proposition 1.1]).

3.3. As $\pi_1(M)$ is not solvable, we may assume that the toric decomposition of M_1 does not contain components $T^2 \times [0, 1]$ (since a fiber space over S^1 with toric fiber can finitely cover only manifolds that admit E^3, Sol- or Nil-structure [4]). All components of the decomposition of M_1 are products $S^1 \times \mathcal{P}_i$, where $\mathcal{P}_i$ has an even number of boundary components. Fix the orientation on all the W_i's so that the homeomorphisms gluing them together to get M_1 reverse the induced orientation of the boundary (recall that M is orientable). Let σ_{ij} be a component of $\partial\mathcal{P}_i$, – we shall use the same symbol to denote its natural embedding in $S^1 \times \mathcal{P}_i$, – and let $t_{ij} = S^1 \times \{x_0\}$ $(x_0 \in \sigma_{ij})$ denote a representative of the fiber of $S^1 \times \mathcal{P}_i$ on the boundary component $S^1 \times \sigma_{ij} = \mathcal{T}_{ij}$. Orient all $t_{i1}, t_{i2}, \ldots$ in the same way and $\sigma_{i1}, \sigma_{i2}, \ldots$ in such a way that the sum of the corresponding elements of $H_1(W_i, \mathbf{Z})$ vanishes and the orientation of the pairs (t_{i1}, σ_{i1}), (t_{i2}, σ_{i2}), ... coincides with the chosen orientation of ∂W_i. The same letters t_{ij}, σ_{ij} will denote basis elements of the groups $\pi_1(\mathcal{T}'_{ij}) = \langle t_{ij}\rangle \oplus \langle\sigma_{ij}\rangle$. From now on we shall call these bases "natural." Let W_i and W_k be components of the toric decomposition of M_1, $\mathcal{T}'_{ij} \subset \partial W_i$, $\mathcal{T}'_{kn} \subset \partial W_{kn}$ components of the boundary glued together by the homeomorphism $f = f_{ij}^{kn}\colon \mathcal{T}'_{ij} \to \mathcal{T}'_{kn}$, assuming that the manifold thus obtained is not a Seifert fiber space. Then $f_*(t_{ij}) = a_{11}t_{kn} + a_{21}\sigma_{kn}$, $f_*(\sigma_{ij}) = a_{12}t_{kn} + a_{22}\sigma_{kn}$ (where $a_{21} \neq 0$, otherwise the gluing operation produces a Seifert fiber space). We shall call $A = (a_{\alpha\beta}) \in \mathrm{GL}_2^-(\mathbf{Z})$ the gluing matrix (relative to the natural bases). Let $\tilde{\sigma}_{ij} = a_{21}\sigma_{ij}$ and $\tilde{\sigma}_{kn} = a_{21}\sigma_{kn}$. Then $f_*(t_{ij}) = a_{11}t_{kn} + \tilde{\sigma}_{kn}$, $f_*(\tilde{\sigma}_{ij}) = a_{21}a_{12}t_{kn} + a_{22}\tilde{\sigma}_{kn}$, therefore $f_*(\langle t_{ij}\rangle \oplus \langle\tilde{\sigma}_{ij}\rangle) = \langle t_{kn}\rangle \oplus \langle\tilde{\sigma}_{kn}\rangle$. We thus select loops $\tilde{\sigma}_{ij}$ on all the tori $\mathcal{T}'_{ij}$, along

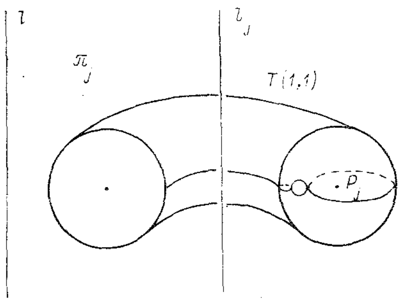

Fig. 5

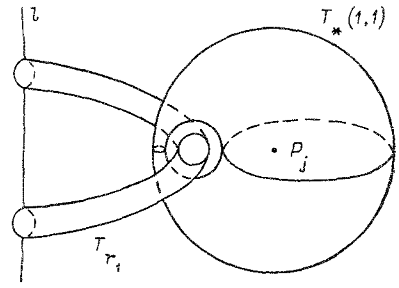

Fig. 6

which the manifold M_1 will be cut. For all surfaces $\mathscr{P}_i$, construct covers p_i: $\tilde{\mathscr{P}_i} \to \mathscr{P}_i$ such that for each component $\sigma_{ij} \subset \partial\mathscr{P}_{ij}$ the defining subgroup of the corresponding restriction of p_i is the subgroup $\langle\sigma_{ij}\rangle$ (cf. Sec. 3.2). Let Π_i: $\tilde{W}_i \to W_i$ be the cover induced by the cover p_i of the base space and the trivial cover of the fiber S^1. Lifting the loops $\tilde{\sigma}_{ij}$ and t_{ij} to $\tilde{W}_i$ clearly yields natural bases for the components $\Pi_i^{-1}(\mathscr{T}'_{ij})$, relative to which the gluing matrix $\tilde{A}=(\tilde{a}_{\alpha\beta})$ has its entry $\tilde{a}_{21}$, equal to 1 (the gluing is carried out by lifting the map f_{ij}^{hn} to the covering spaces).

3.4. Let $M_2 \to M_1$ be a finite-sheeted cover, glued together from Seifert fiber spaces Y_i (each of which is homeomorphic to some one of the $\tilde{W}_i$'s). Associated with each Y_i, which has r_i boundary components, we have a collection of numbers $\tilde{a}_{22}(i, j)$, $j=1, \ldots, r_i$, – the elements of the gluing matrix $\tilde{A}(i, j)$ (see Sec. 3.2). Let $e_i=|\tilde{a}_{22}(i, 1)+\ldots+\tilde{a}_{22}(i, r_i)|$, and let g_i be the genus of the surface $\tilde{\mathscr{P}_i}$ (the base space of Y_i).

Recall that by construction (see Sec. 3.2) the numbers r_i are even for all i. Hence each surface $\tilde{\mathscr{P}_i}$ admits a regular cyclic cover η_i: $\Sigma_i \to \tilde{\mathscr{P}_i}$ of arbitrary multiplicity q_i, where the number of boundary components of Σ_1 is, as before, r_i. The genus k_i of Σ_i is $1 + r_i(q_i - 1)/2 + q_i(g_i - 1)$, and we shall choose the numbers q_i to be the same prime number q (for all i). Moreover, we choose q so large that $k_i > g_0(e_i, r_i)$, where $g_0(e, m)$ is the same function as in Sec. 2.7 [the condition $k_i > g_0(e_i, r_i)$ guarantees the existence of the modified group $H_{r_i}(k_i, e_i)$; see Sec. 2.7]. Finally, consider the covers ζ_i: $X_i=S^1\times\Sigma_i \to Y_i=S^1\times\tilde{\mathscr{P}_i}$, where θ_i: $S^1 \to S^1$ is a q-sheeted cover. Then the homeomorphisms by means of which M_2 is glued together from the manifolds Y_i lift to homeomorphisms $\tilde{f}_{ij}^{hn}$ of the boundaries X_i, with the same gluing matrix $\tilde{A}$. The components X_i are now glued together to get a manifold M_0 which is a finite-sheeted cover of M. Our next goal is to construct a Kleinian group G uniformizing M_0.

3.5. Let G_i denote the groups $H_{r_i}(k_i, e_i)$ (see Sec. 3.4). These groups (and their conjugates in $\mathscr{M}_3$) will be combined in the Klein–Maskit sense (see [14, 15]) to construct the required group G. We begin the operation with the group $G_1^* = G_1$. The boundary of the fundamental region of G_1 is in the interior of the torus $T(10(r_1 + 1), 8)$ (see Sec. 2.7). It is readily seen that, together with B(P, 8), the interior of this torus also contains r_1 disjoint balls $B(P_j, 8)$ of the same radius, whose centers P_j lie at the same distance $8 + 10(r_1 + 1)$ from the axis of rotation l as the point P ($j = 1,\ldots,r_1$).

Let π_j be the plane through l and P_j, and $l_j\subset\pi_j$ the straight line parallel to l at a distance 2 from P_j. Construct a torus T(1,1) with axis of rotation l_j and take its image under inversion with respect to the sphere of radius 1 about P_j (Fig. 5). Let $T_*(1,1)$ be the image of the resulting torus after dilation with center P_j and coefficient 7.5. We shall call P_j the <u>center</u> of this torus. It is readily verified that $T_*(1,1)$ is contained in the ball $B(P_j, 8)$, and if $\mathscr{T}_*(1,1)$ denotes the solid torus bounded by $T_*(1,1)$ and not containing the point ∞, then $\mathscr{T}_*(1,1)$ and the solid torus T_{r_1} (see Sec. 2.7 and Fig. 6) form a link in $\mathbf{R}^3$ of index 1.

We now place tori $\mathscr{T}_{1j}\simeq T_*(1,1)$, as well as $T_*(1,1)$ in the interior of each ball $B(P_j, 8)\subset \mathrm{int}(T_{(1)}=T(10(r_1+1), 8))$.

3.6. Suppose the manifold X_2 is glued to X_1 along several boundary components $\tilde{f}_{11}^{21}$: $\tilde{\mathscr{T}}_{11}\subset \partial X_1 \to \tilde{\mathscr{T}}_{21}\subset\partial X_2, \ldots, \tilde{f}_{1q}^{2q}$: $\tilde{\mathscr{T}}_{1q}\subset\partial X_1\to\tilde{\mathscr{T}}_{2q}\subset\partial X_2$. Working with X_2, construct a torus $T_{(2)}=T(10(r_2+1), 8)$, group $G_2 = H_{r_2}(k_2, e_2)$ and system of q tori $\mathscr{T}_{2i}$, isometric to T(1,1), situated in balls of radius 8 and forming with T_{r_2} a link of index 1 (as done previously inside the torus $T_{(1)}$). The remaining $r_2 - q$ disjoint balls inside $T_{(2)}$ will be filled with tori of the form T(1,1) or $T_*(1,1)$ at the end of this subsection. Let $\mathscr{T}_{11}$ and $\mathscr{T}_{21}$ be any two tori in the interior of $T_{(1)}$ and $T_{(2)}$, respectively. There exists a Möbius transformation γ_{21}^{11}: ext$\mathcal{T}_{21} \to$ int$\mathcal{T}_{11}$ [see the definition of T(1,1) and $T_*(1,1)$]. It is not hard to see that the groups $H_{r_1}(k_1, e_1) = G_1^*$ and $G_2^* = \gamma_{21}^{11} G_2 \gamma_{11}^{21}$ form exactly the same "link" as described in Sec. 3.1. The elements γ_{21}^{11} are clearly not uniquely determined. However, if we confine attention to the induced isomorphism $(\gamma_{21}^{11})_*$: $\pi_1(\mathcal{T}_{21}) \to \pi_1(\mathcal{T}_{11})$, there exist exactly two possible choices for the map γ_{21}^{11} (differing from one another by a Euclidean axial symmetry of $\mathcal{T}_{11}$). We shall see later how to choose γ_{21}^{11}.

Let γ_{22}^{12}: ext $\mathcal{T}_{22} \to$ int $\mathcal{T}_{12}$, ..., γ_{2q}^{1q}: ext$\mathcal{T}_{2q} \to$ int$\mathcal{T}_{1q}$ be Möbius transformations. We construct a successive HNN-extension of the group $G_1^* * G_2^*$ by the elements $\gamma_{22}^{12} \circ \gamma_{11}^{21}, \ldots, \gamma_{2q}^{1q} \circ \gamma_{11}^{21}$. It is easy to see that under these conditions the conditions of Maskit's combination theorem (see [14]) are fulfilled, since the solid tori int$\mathcal{T}_{1i}$, int$\gamma_{21}^{11}(\mathcal{T}_{2i})$ are strictly invariant (with respect to the identity subgroup).

This process can be continued, considering the Klein–Maskit combinations of the groups $G_i = H_{r_i}(k_i, e_i)$ (and their conjugates) in accordance with the way in which M_0 is glued together from components X_i. When this is done, if manifolds X_i and X_j are to be glued together, we place in each of the unfilled balls of radius 8 in int$T_{(i)}$, int$T_{(j)}$ one torus, interlinked with T_{r_i} (resp., T_{r_j}) if the torus placed in $T_{(i)}$ was of type $T_*(1,1)$, that placed in a ball of T(j) will be of type T(1,1). The group G resulting from this combination procedure is the required group.

3.7. In this section we shall indicate how to choose the Möbius transformations γ_{ij}^{mn} and explain why G uniformizes the manifold M_0.

We consider the natural orientation of the curve $\eta \subset \partial\Phi$, defined by the ordering α_1, $\gamma_1', \alpha_1', \gamma_1, \ldots$ (see Fig. 1, Sec. 2.2, and Fig. 2, Sec. 2.4), where Φ is the fundamental polyhedron of the group H(g, e). The very same orientation can be considered on the loop $\varkappa \subset \partial\Phi$, parallel to the axis of the solid torus $S^3 \setminus \Phi$ (see Fig. 2). The orientation of the loop $t \subset \partial\Phi$, $t = Q_1 \cap R_g$ (see Sec. 2.2) is defined by the condition $\eta \sim |e| t + \varkappa$.

In a similar manner we orient the loops η_i, $\varkappa_i$, $t_i \subset \partial\Phi_{r_i}$, where Φ_{r_i} is the fundamental polyhedron of the group $H_{r_i}(k_i, e_i)$. The loop $\varkappa_i$ generates the kernel of the homomorphism $\pi_1(\partial\Phi_{r_i}) \to \pi_1(\Phi_{r_i})$, and the loop t_i the kernel of $\pi_1(\partial\Phi_{r_i}) \to \pi_1(S^3 \setminus \Phi_{r_i})$. Let $\mathcal{T}_{ij} \subset \text{int}(T_{(i)})$, on this torus we then obtain a pair of basis loops τ_{ij}, $\varkappa_{ij}$, parallel in $\Phi_{r_i} \setminus \text{int}\mathcal{T}_{ij}$ to t_i and $\varkappa_i$, respectively. We now choose the Möbius transformation γ_{ij}^{mn}: ext$\mathcal{T}_{ij} \to$ int$\mathcal{T}_{mn}$ subject to the condition

$$(\gamma_{ij}^{mn})_*(\tau_{ij}) = \varkappa_{mn} \in \pi_1(\mathcal{T}_{mn}), \qquad (\gamma_{ij}^{mn})_*(\varkappa_{ij}) = \tau_{mn} \in \pi_1(\mathcal{T}_{mn}).$$

Now put $\lambda_{ij} = \tilde{a}_{22}(i, j)\tau_{ij} + \varkappa_{ij} \in \pi_1(\mathcal{T}_{ij})$ (see Secs. 3.3, 3.4); the same symbol λ_{ij} will denote a simple loop on $\mathcal{T}_{ij}$, representing this element of $\pi_1(\mathcal{T}_{ij})$. A direct check now shows that $(\gamma_{ij}^{mn})_* \times (\tau_{ij}) = \tilde{a}_{11}(i, j)\tau_{mn} + \lambda_{mn}$, $(\gamma_{ij}^{mn})_*(\lambda_{ij}) = \tilde{a}_{12}(i, j)\tau_{mn} + \lambda_{mn} \cdot \tilde{a}_{22}(i, j)$, where $\tilde{a}_{11}(i, j) = -\tilde{a}_{22}(m, n)$, $\tilde{a}_{12}(i, j) = \tilde{a}_{12}(m, n) = \tilde{a}_{11}(i, j)\tilde{a}_{22}(i, j) + 1$.

On the other hand, we recall that $e_i = |\tilde{a}_{22}(i, 1) + \ldots + \tilde{a}_{22}(i, r_i)|$ (see Sec. 3.4). Therefore, in the manifold

$$X_i = \left(R(G_i) \setminus \bigcup_{g \in G_i} g\left(\bigcup_{j=1}^{r_i} \text{int}\mathcal{T}_{ij}\right)\right) / G_i$$

the sum of projections of the loops λ_{ij} bounds a surface Σ_i [recall that $G_i = H_{r_i}(k_i, e_i)$, and $R(G_i)$ is the region of discontinuity of G_i]. Denoting the projections of λ_{ij} in X_i by $\tilde{\sigma}_{ij}$ and the projections of τ_{ij} by $\tilde{t}_{ij}$, we see that the pairs $(\tilde{\sigma}_{ij}, t_{ij})$ are natural bases of ∂X_i, and the gluing matrix of the homeomorphism $\tilde{f}_{ij}^{mn}$, obtained when γ_{ij}^{mn} descends to ∂X_i and ∂X_j, coincides with $\tilde{A}(i, j)$ (see Secs. 3.3, 3.4). In sum, the manifold M(F) = R(G)/G (obtained from $M(G) = R(G)/G$ by gluing together at boundary points which are equivalent relative to G_i and the elements γ_{ij}^{mn}) is homeomorphic to M_0. Thus M_0, which finitely covers M, is uniformized by the Kleinian group G. This completes the proof of Theorem B.

3.8. As an application of Theorem B, we shall construct an example of a 3-manifold M which does not admit a CFS, but M has a uniformizable finite-sheeted cover.

Let $\mathcal{O}$ be an orbifold whose support is the annulus $S^1 \times [0, 1]$ and its singular set a conical point with angle π. Let N be a Seifert fiber space over $\mathcal{O}$ whose fundamental group has the corepresentation $\langle a, b, c, t\colon c^2 = t,\ abc = 1,\ [a, t] = [b, t] = 1 \rangle$. The boundary of N consists of two toric components whose fundamental groups are generated by the elements a and t, b and t, respectively. Let f be a homeomorphism mapping one boundary component onto the other, $f_*(a)=t$, $f_*^{-1}(b)=t$, where f_* is the induced homomorphism of the fundamental groups [the generators of $\pi_1(M)$ can be so chosen that f reserves the induced orientation of the boundary]. Let M denote the manifold obtained by identifying points x, $f(x)\in\partial N$.

It is easy to see that M satisfies the assumptions of Theorem B (since the base orbifold σ is not Euclidean). Hence there exists a finite-sheeted cover over M that admits a uniformizable conformally flat structure.

THEOREM D. There exist no conformally flat structure on M.

Proof. Let us suppose that there exists a conformally flat structure K on M, and let $d_*\colon \pi_1(M)\to\mathcal{M}_3$ be the holonomy homomorphism (for the definition see [1, 2, 7]). If $g\in\pi_1(M)$, we let g^* denote $d_*(g)$. The fundamental group of M has a corepresentation $\langle a, b, c, abc=1, [a, t]=[b, t]=1, \varphi^{-1}a\varphi=t, \varphi^{-1}t\varphi=b\rangle$. We claim that the group $H=d_*(\pi_1(M))$ must satisfy one of the following conditions: it is conjugate to a subgroup of $SO(4)\subset\mathcal{M}_3$, it has two fixed points in $\overline{\mathbf{R}}^3$, it is Abelian; it is polycyclic of rank $r\leqslant 3$, it is nilpotent. Since $|\pi_1(M)|=\infty$, the first possibility cannot occur (cf. [26]); that the second case is impossible follows from [24, lemma and Theorem 1]. The group H can be neither nilpotent nor polycyclic of rank $r\leqslant 3$, in view of results of Kuiper [27] and Goldman [7] (see also [28]), since $\pi_1(M)$ is not Abelian. Thus verification of our claim will complete the proof.

(a) Suppose first that $t^*=1$. Then $a^*=b^*=1$, $c^*=1$, and therefore H is a cyclic group.

(b) Now let $1\neq t^*$ be an elliptic transformation. Then the elements a^*, b^*, c^* are also elliptic. If t^* has no fixed points in $\overline{\mathbf{R}}^3$, then its extension to $\mathbf{H}^4$ leaves exactly one point fixed there (denote this point by q). Clearly, q is also a fixed point of a^*, b^*. Thus the group $d_*(\pi_1(N))$ leaves q fixed. In addition, it follows at once from the condition $(\varphi^*)^{-1}\circ a^*\circ\varphi^*=t^*$ that $\varphi^*(q)=q$. Therefore $H(q)=q$ and H is conjugate to a subgroup of $SO(4)$.

Suppose now that t^* leaves a circle $l_t\subset S^3$ fixed point for point. Then the fixed sets of a^*, b^* are circles l_a, $l_b\subset S^3$. If at least one of these circles is l_t, then $l_t=l_a=l_b$ and H is Abelian. Note that for any $g\in\pi_1(N)$ $g^*(l_t)=l_t$. Hence there exists only one possibility in case (b): the pairs l_a and l_t, l_b and l_t, have linking number 1. But then, as is easily seen, $(c^*)^2=(a^*b^*)^{-2}\neq 1$ and this element cannot have a circle of fixed points l_t; consequently, $(c^*)^2\neq t^*$, which is false.

(c) Suppose that t^* is a loxodromic element with fixed points 0 and $\infty\in\overline{\mathbf{R}}^3$. Then a^* and b^* are also loxodromic transformations and their fixed points are 0 and ∞ (since $[a^*, t^*]=[b^*, t^*]=1$). Therefore $\varphi^*(0)=0$, $\varphi^*(\infty)=\infty$ and the entire group H leaves 0 and ∞ fixed.

(d) The last case: t^* is a parabolic transformation, $t^*(\infty)=(\infty)$. It is readily seen that then the group $d_*(\pi_1 N)$ leaves invariant either a straight line or a plane in $\mathbf{R}^3$. This invariant line (or plane) may be so chosen that it is also invariant to φ^* [note that $\varphi^*(\infty)=\infty$]. It follows at once that H is either polycyclic of rank $r\leqslant 3$ or nilpotent. This completes the proof.

COROLLARY. The manifold M just constructed does not admit a CFS, but it has a uniformizable finite-sheeted cover.

This settles Problem No. 41 in [8].

Remark. The author's preprint [29] contains a proof of Theorem A and a sketch of the proof of Theorem B.

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