

ON SUPERINTEGRAL KLEINIAN SPHERE PACKINGS, BUGS, AND ARITHMETIC GROUPS

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ABSTRACT. We develop the notion of a Kleinian Sphere Packing, a generalization of “crystallographic” (Apollonian-like) sphere packings defined by Kontorovich-Nakamura [KN19]. Unlike crystallographic packings, Kleinian packings exist in all dimensions, as do “superintegral” such. We extend the Arithmeticity Theorem to Kleinian packings, that is, the superintegral ones come from \mathbb{Q} -arithmetic lattices of simplest type. The same holds for more general objects we call Kleinian Bugs, in which the spheres need not be disjoint but can meet with dihedral angles π/m for finitely many m . We settle two questions from [KN19]: (i) that the Arithmeticity Theorem is in general false over number fields, and (ii) that integral packings only arise from non-uniform lattices.

1. INTRODUCTION

The classical Apollonian packing in the plane, usually described by an ad hoc construction involving inscribing tangent circles, exhibits a number of thereafter surprising arithmetic and dynamical properties; see, e.g., [K13]. In this paper, we complete the program initiated in [KN19] to understand the relationship between such packings and the theory of arithmetic groups in hyperbolic space.

1.1. Kleinian (and Crystallographic) Packings.

A **sphere packing** (or just “packing”) \mathcal{P} of $\mathbb{S}^n \cong \partial_\infty \mathbb{H}^{n+1}$ ($n \geq 2$) is an infinite collection of round balls in \mathbb{S}^n with pairwise disjoint interiors, such that the union of the balls is dense in \mathbb{S}^n . We identify the ideal boundary $\mathbb{S}^n = \partial_\infty \mathbb{H}^{n+1}$ with the one-point compactification of Euclidean n -space $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$. By abuse of terminology, we will

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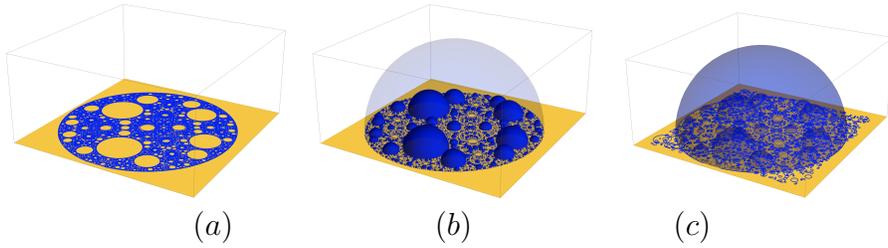


FIGURE 1. (a) A packing \mathcal{P} , (b) its reflection group $\Gamma_{\mathcal{P}}$, and (c) its superpacking $\widetilde{\mathcal{P}}$.

conflate the collection of balls \mathcal{P} with the collection of round spheres bounded by these balls. In view of the density condition, a packing contains balls with arbitrarily small Euclidean radii. (Of course, the radii are only defined once we choose an identification $\mathbb{S}^n \cong \partial_{\infty}\mathbb{H}^{n+1}$, in particular, we choose a point at ∞ .) The **bend**¹ of a sphere is the reciprocal of its radius, with the convention that a sphere containing ∞ in its interior has negative radius and bend. A packing is **integral** if all its spheres have integer bends.

Next we attach to any sphere packing its “superpacking.” For an $(n-1)$ -sphere $S \subset \mathbb{S}^n$, denote by R_S reflection through S acting on \mathbb{H}^{n+1} . Given a packing \mathcal{P} , let

$$(1) \quad \Gamma_{\mathcal{P}} = \langle R_S : S \in \mathcal{P} \rangle < \text{Isom}(\mathbb{H}^{n+1})$$

be the **reflection group of \mathcal{P}** , generated by reflections through the spheres in \mathcal{P} . The **superpacking**

$$(2) \quad \widetilde{\mathcal{P}} := \Gamma_{\mathcal{P}} \cdot \mathcal{P}$$

is defined as the orbit of the packing under the action of its reflection group, see [Figure 1](#). A packing is **superintegral** if its superpacking has all integer bends.² Note that no tangency conditions are imposed on the spheres; indeed the circles in the packing shown in [Figure 1\(a\)](#) are all disjoint (and this packing is superintegral).

As we will show, superintegrality, even in the absence of any other structure imposed on the packing \mathcal{P} , is already related to (sub)arithmeticity, as follows. Recall that a group of hyperbolic isometries is called

¹Note that in the theory of Kleinian groups, “bend” more often refers to dihedral angle; but for integral sphere packings, bend is used for inversive radii. For circles, bend is the curvature, but for higher dimensional spheres, (Gaussian) curvature is inverse square-radius.

²It turns out that this condition is strictly stronger than integrality; that is, there exist packings, even crystallographic ones, which are integral but not superintegral [[KN19](#)].

“ k -arithmetic” (of simplest type³) if, possibly after conjugation, it is commensurable with the group $O_{\mathcal{F}}(\mathfrak{o})$ of \mathfrak{o} -integral automorphs of a hyperbolic⁴ quadratic form \mathcal{F} defined over a totally real number field k with ring of integers \mathfrak{o} (see, e.g., [VS93]). We call a Zariski-dense, discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$ **k -subarithmetic** if Γ is contained in a k -arithmetic lattice.

Theorem 3 (Subarithmeticity Theorem). *If an orbit*

$$\mathcal{O} = \Gamma \cdot S_0$$

of a fixed sphere $S_0 \subset \partial_{\infty}\mathbb{H}^{n+1}$ under a Zariski dense subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$ has all integer bends, then Γ is \mathbb{Q} -subarithmetic. More precisely, there exists an isotropic rational hyperbolic quadratic form \mathcal{F} so that Γ is contained in $O_{\mathcal{F}}(\mathbb{Z})$.

Corollary 4. *In particular, if a general packing \mathcal{P} happens to be superintegral, then its reflection group $\Gamma_{\mathcal{P}}$ is necessarily \mathbb{Q} -subarithmetic.*

Remark 5. These conclusions need not hold for \mathfrak{o} -(super)integral packings, that is, ones with all bends in a ring of integers \mathfrak{o} ; see §1.2.

In general, there is not much more one can say about integral or superintegral packings without assuming more structure.

Definition 6. *A sphere packing \mathcal{P} is **Kleinian** if its set of limit points also arises as the limit set of a geometrically finite⁵ group $\Gamma_S < \text{Isom}(\mathbb{H}^{n+1})$. We call Γ_S a **symmetry group** of the packing.*

In the special case that Γ is generated by finitely many reflections in hyperplanes, \mathcal{P} is called **crystallographic**; such were defined and studied in [KN19]. It is well-known and easy to see (e.g., from the Poincaré Fundamental Polyhedron Theorem) that discrete, finitely generated, hyperbolic *reflection* groups are geometrically finite, so every crystallographic packing is also Kleinian.

The following statement is both standard and deserves to be stated explicitly.

Theorem 7. *Let $\Gamma_S < \text{Isom}(\mathbb{H}^{n+1})$ be geometrically finite. Then its domain of discontinuity is a disjoint union of open round balls (that is, is a Kleinian sphere packing) if and only if the boundary of the convex core M^* of the orbifold $M = \mathbb{H}^{n+1}/\Gamma_S$ is totally geodesic.*

³If an arithmetic subgroup of $O_{n,1}$ contains a hyperplane reflection, then it is necessarily of simplest type, see [Vin67, Lemma 7].

⁴See Definition 47.

⁵See Definition 28.

An important role is played by the “supergroup” of a Kleinian packing. If \mathcal{P} has symmetry group Γ_S , then the **supergroup** $\tilde{\Gamma}$ is defined by

$$\tilde{\Gamma} := \langle \Gamma_S, \Gamma_{\mathcal{P}} \rangle.$$

That is, the supergroup is the group generated by both Γ_S and $\Gamma_{\mathcal{P}}$. Note that both the symmetry group Γ_S and supergroup $\tilde{\Gamma}$ are not uniquely determined by the packing \mathcal{P} ; indeed, any nontrivial normal subgroup of Γ_S will have the same limit set. *A priori* it is not even obvious that $\tilde{\Gamma}$ acts discretely, but it is in fact a lattice, acting on \mathbb{H}^{n+1} with finite covolume; see the Structure [Theorem 22](#).

We turn our attention now to the integral and superintegral Kleinian packings. An immediate corollary of the Subarithmeticity [Theorem 3](#) is that, if a Kleinian packing \mathcal{P} is integral, then any symmetry group Γ_S is \mathbb{Q} -subarithmetic. If \mathcal{P} is moreover superintegral, then its supergroup $\tilde{\Gamma}$ is itself \mathbb{Q} -arithmetic. This is because the superpacking $\tilde{\mathcal{P}}$ can also be given as the orbit of the packing \mathcal{P} under the action of $\tilde{\Gamma}$, together with the obvious fact that, if a group is subarithmetic and a lattice, then it is arithmetic!

In the case of crystallographic packings, the main result of [[KN19](#), Thm 18] was the following finiteness theorem. Before stating the theorem, note that it is shown in [[KN19](#), Thm 3] that there exist infinitely many *conformally inequivalent* superintegral crystallographic packings in certain dimensions up to $n = 18$. We say that two Kleinian packings are **commensurable** if (conjugates of) their supergroups are.

Theorem 8 (Finiteness Theorem [[KN19](#)]). *Superintegral crystallographic packings exist in only finitely-many dimensions, and there are finitely many in each dimension, up to commensurability.*

Indeed, if a crystallographic packing is superintegral, then its supergroup is a \mathbb{Q} -arithmetic *reflective* lattice. Arithmetic reflective lattices are known (see, e.g., the discussion in [[Bel16](#)]) to lie in finitely many commensurability classes in finitely many dimensions, which implies [Theorem 8](#). Moreover, there are no \mathbb{Q} -arithmetic reflective lattices acting on \mathbb{H}^{n+1} with $n + 1 = 20$ or $n + 1 \geq 22$, see [[Ess96](#)], so there are no superintegral crystallographic packings in $n = 19$ or $n \geq 21$ dimensions. Worse yet, crystallographic packings are not yet known to exist (nevermind integrality) in dimensions $n = 14, 15, 16, 18$ and 20 , although reflective lattices are known in one more than these dimensions [[KN19](#)].

While the classification of commensurability types of superintegral crystallographic packings awaits first that of arithmetic hyperbolic reflection groups, one can completely classify superintegral Kleinian packings in terms of arithmetic groups. The following is the first main theorem of this paper.

Theorem 9 (Classification Theorem). *A hyperbolic lattice is commensurable to a supergroup of a superintegral Kleinian packing if and only if it is a non-uniform \mathbb{Q} -arithmetic lattice of simplest type.*

So in contradistinction with the Finiteness [Theorem 8](#) for crystallographic packings, more general Kleinian packings exist in every dimension. Moreover, the Classification [Theorem 9](#) answers a question posed in [\[KN19\]](#) on whether there exist superintegral crystallographic packings with *cocompact* supergroups: there do not! We emphasize again that the spheres in a superintegral Kleinian (or even just crystallographic) packing could all be disjoint, but the supergroup $\tilde{\Gamma}$ must have cusps. See [Example 66](#).

Remark 10. The precise role of \mathbb{Q} -isotropy of the corresponding quadratic form leading to (super)integrality of packings is elucidated in [Lemma 52](#), which shows that the “covector” corresponding to the bend is itself isotropic in the dual form.

Remark 11. Note that even if a symmetry group Γ_S is geometrically finite and the convex core M^* of $M = \mathbb{H}^{n+1}/\Gamma_S$ has totally geodesic boundary (so its limit set gives rise to a Kleinian packing \mathcal{P}), and if the supergroup $\tilde{\Gamma}$ of \mathcal{P} is non-uniform and \mathbb{Q} -arithmetic of simplest type, it still need *not* be the case that the packing \mathcal{P} is necessarily integral, for any conformal choice of coordinates on \mathbb{S}^n , see [Example 67](#). But the next theorem states that, if M^* has only one boundary component (that is, Γ_S acts transitively on the spheres in the packing), then the packing *is* necessarily superintegral.

Theorem 12. (i) *Suppose that a supergroup $\tilde{\Gamma}$ of a packing \mathcal{P} is a non-uniform \mathbb{Q} -arithmetic of simplest type (i.e. is commensurable to $O_{\mathcal{F}}(\mathbb{Z})$, where \mathcal{F} is a rational hyperbolic quadratic form) and, moreover, a symmetry group $\Gamma_S < \tilde{\Gamma}$ acts transitively on the set of spheres in the packing \mathcal{P} . Then there is always a conformal change of coordinates such that \mathcal{P} is superintegral.*

(ii) *More generally, let Γ be any non-uniform \mathbb{Q} -arithmetic group of the simplest type and let S be a sphere for which the reflection R_S lies in Γ . Then the Γ -orbit of S can be made integral by a suitable conformal change of coordinates.*

There are two main ingredients leading to the proof of the Classification [Theorem 9](#), one geometric (constructing a Kleinian packing from the Structure [Theorem 22](#) below), and the second arithmetic, to ensure that the packing thus constructed is indeed superintegral; it is the second stage that forces \mathbb{Q} -arithmeticity and non-uniformity of the lattice.

A key step in the geometric argument relies on Millson’s theorem [\[Mi76\]](#) that every arithmetic hyperbolic lattice of simplest type is commensurable to a lattice $\tilde{\Gamma}$ such that $\mathbb{H}^{n+1}/\tilde{\Gamma}$ contains a nonseparating totally geodesic complete hypersurface of finite volume.

We will also show that, in each commensurability class, there are not only infinitely many conformally inequivalent packings (as shown for crystallographic packings in [\[KN19\]](#)), but their Hausdorff dimensions can be made arbitrarily close to maximal.

Theorem 13 (Abundance Theorem). *In every dimension $n \geq 2$, there exist quasiconformally inequivalent superintegral Kleinian packings, whose limit sets have Hausdorff dimension approaching n .*

Note the contrast to Phillips-Sarnak’s and Doyle’s work on Schottky groups [\[PS85, Theorem 5.4\]](#), [\[Do88\]](#), showing that their limit sets have Hausdorff dimensions bounded strictly away from the ambient dimension n . Also note that the conformally inequivalent superintegral Polyhedral Circle Packings in $\partial_\infty \mathbb{H}^3$ constructed in [\[KN19, Thm 7\]](#) are all Schottky, so their dimensions cannot approach $n = 2$.

1.2. Packings over Number Fields.

To complete the discussion of Kleinian packings (before we turn to Kleinian “bugs”), we show that the theory breaks down over number fields, answering another question posed in [\[KN19\]](#). For a totally real number field k with ring of integers \mathfrak{o} , a packing is \mathfrak{o} -integral (resp. \mathfrak{o} -superintegral) if every bend in its packing (resp. superpacking) lies in \mathfrak{o} .

Proposition 14. *The Subarithmeticity Theorem does not hold in number fields.*

Indeed, already for the simplest case of the golden ring $k = \mathbb{Q}(\sqrt{5})$, there exist \mathfrak{o} -superintegral crystallographic packings whose supergroups are non-arithmetic. (See [Example 71](#).) But this is not the whole story, as we also have the following.

Theorem 15. *For any k -arithmetic hyperbolic lattice of simplest type $O_{\mathcal{F}}(\mathfrak{o})$, there is a totally real, quadratic extension $k' \supset k$, with ring of*

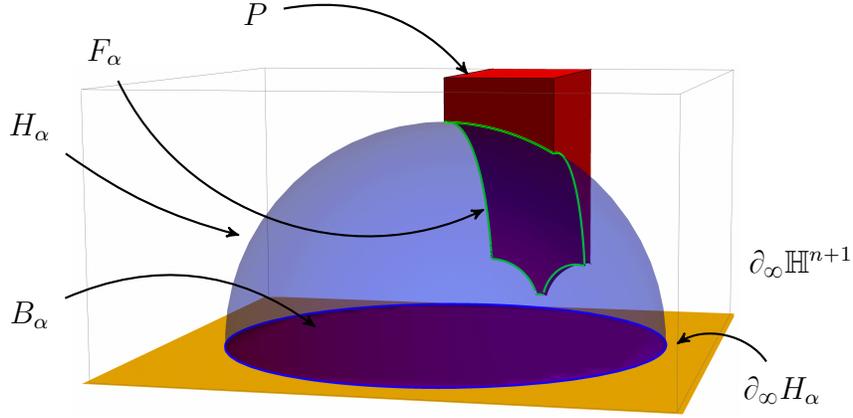


FIGURE 2. A portion of a polyhedron P having bounding facet F_α lying in the hyperplane H_α . Also shown is the ideal boundary $B_\alpha \subset \partial_\infty \mathbb{H}^{n+1}$ of the complementary half-space H_α^- , and the cooriented round sphere $\partial_\infty H_\alpha$ bounding B_α .

integers \mathfrak{o}' , such that $O_{\mathcal{F}}(\mathfrak{o})$ is commensurable to the supergroup of an \mathfrak{o}' -superintegral Kleinian packing.

Note that there is no condition on being non-uniform here, unlike [Theorem 9](#). Thus, even the \mathbb{Q} -anisotropic form $x_1^2 + x_2^2 + x_3^2 - 7x_4^2$ has \mathfrak{o}' -superintegral Kleinian packings, for certain quadratic rings \mathfrak{o}' . This leaves open the problem of giving a proper formulation for how to characterize which \mathfrak{o}' -superintegral packings come from arithmetic groups.

1.3. Kleinian Bugs.

To further extend the notion of a packing, we will allow spheres to meet at a finite set of dihedral angles, as follows.

A *cooriented* round sphere in \mathbb{S}^n is a round sphere together with a choice of a nowhere vanishing normal vector field. Cooriented spheres are in a natural bijective correspondence with cooriented hyperbolic hyperplanes in \mathbb{H}^{n+1} . A *convex polyhedron*, P , in \mathbb{H}^{n+1} is the intersection of a locally finite family of hyperbolic half-spaces H_α^+ in \mathbb{H}^{n+1} . It suffices to consider only half-spaces H_α^+ such that the hyperplanes H_α intersect ∂P along facets F_α . We coorient these hyperplanes H_α so that the normal vector fields point into the complementary half-spaces H_α^- . The ideal boundary $\partial_\infty H_\alpha^-$ of the half-space H_α^- is a round ball B_α bounded by the cooriented round sphere $\partial_\infty H_\alpha$, see [Figure 2](#).

Definition 16. A **bug** \mathcal{B} is an infinite collection of cooriented spheres in $\mathbb{S}^n \cong \partial_\infty \mathbb{H}^{n+1}$ containing facets of a convex polyhedron $P = P_{\mathcal{B}} \subset \mathbb{H}^{n+1}$, with dihedral angles lying in a finite subset of

$$\frac{\pi}{\mathbb{N}} = \left\{ \frac{\pi}{m} : m \in \mathbb{N} \right\},$$

such that the union of round balls bounded by these spheres is dense in \mathbb{S}^n . As before, a bug is **integral** if its spheres all have integer bends.

In plainer terms (forgetting coorientations), a bug is a collection of spheres, any pair of which is either disjoint, tangent or which intersect at angles π/m , for finitely many $m \in \mathbb{N}$. See [Figure 3](#) for an integral bug, and [Figure 4](#) for the corresponding hyperbolic polyhedron P , which should justify our calling these objects “bugs.”

1.3.1. Geometric Aspects of Bugs.

Before turning to any arithmetic properties of bugs, we discuss purely geometric aspects, culminating in the Structure [Theorem 22](#). As in [\(1\)](#), the *reflection group* generated by reflections through the spheres in \mathcal{B} , denoted $\Gamma_{\mathcal{B}}$, is discrete and has as its fundamental domain the hyperbolic convex polyhedron $P = P_{\mathcal{B}}$ bounded by the hemispheres.

In this setting, we need a substitute for the limit set of the packing, which will no longer necessarily contain spheres, see [Figure 5\(b\)](#).

Definition 17. The **accumulation set** $\mathcal{A}(\mathcal{B})$ of a bug \mathcal{B} consists of those points $\xi \in \partial_\infty \mathbb{H}^{n+1}$ such that every neighborhood of ξ contains infinitely many spheres in \mathcal{B} .

As a first (it turns out, insufficient) step, we define “geometric” bugs in terms of symmetry groups as before:

Definition 18. A bug \mathcal{B} is **geometric** if there exists a geometrically finite subgroup $\Gamma_S < \text{Isom}(\mathbb{H}^{n+1})$ such that

- (1) the accumulation set of the bug is the limit set of Γ_S , and
- (2) Γ_S preserves the bug polyhedron $P = P_{\mathcal{B}}$.

Such a Γ_S is called a **symmetry group** of the bug.

Recall again that a symmetry group of a bug is not unique. But its commensurability class is uniquely determined by the bug, provided that the accumulation set of \mathcal{B} consists of more than two points; see [Lemma 41](#).

The supergroup is defined in the now-familiar way:

Definition 19. Given a bug \mathcal{B} and a symmetry group Γ_S , the **supergroup**

$$\tilde{\Gamma} := \langle \Gamma_S, \Gamma_{\mathcal{B}} \rangle < \text{Isom}(\mathbb{H}^{n+1}),$$

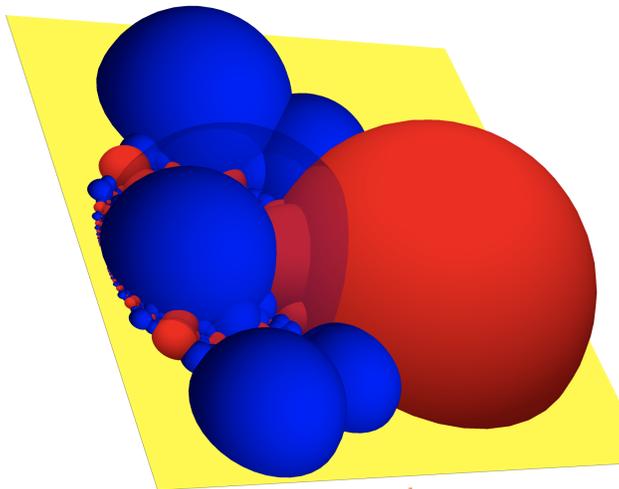


FIGURE 4. View from \mathbb{H}^3 of the bug polyhedron P from Figure 3. The polyhedron is exterior to all solid hemispheres shown, and interior to the one opaque hemisphere, which has bend -76 .

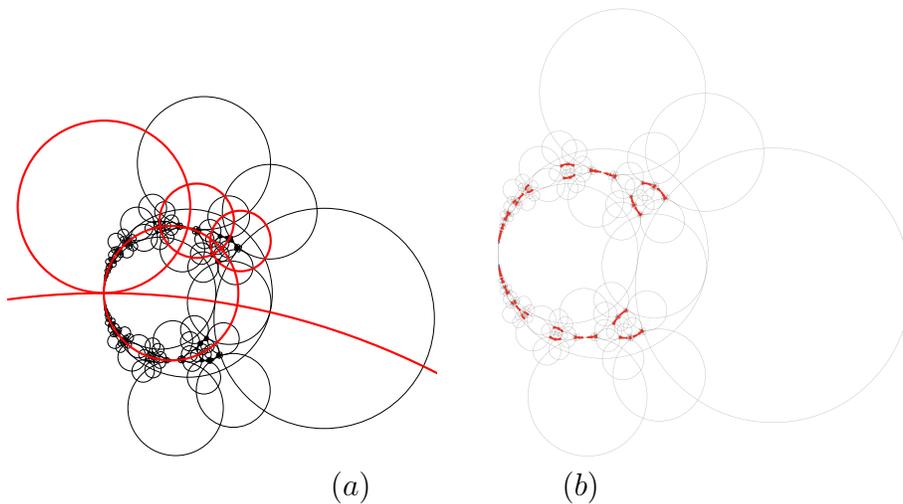


FIGURE 5. (a) The bug is Kleinian with symmetry group generated by reflections in the thick red circles. (b) The accumulation set in $\partial_\infty \mathbb{H}^{n+1}$ does not in general contain spheres.

there exists a subgroup $\Pi_1 < \Gamma_1$ such that the subgroup Π generated by Π_1, Π_2 is virtually abelian of virtual rank n .

Note that cusp compatibility depends only on the commensurability classes of Γ_1, Γ_2 .

Definition 21. A bug \mathcal{B} is called **Kleinian** if it is geometric with symmetry group Γ_S , such that the ordered pair $\Gamma_\emptyset, \Gamma_S$ is cusp-compatible.

We remark that, if \mathcal{B} is not just a bug but also a packing, then the cusp-compatibility condition is automatic, see [Lemma 44](#).

In [§2](#) we will prove the second main theorem of this paper, which is the following extension of the Crystallographic Structure Theorem in [[KN19](#), Thm 28].

Theorem 22 (Structure Theorem for Kleinian Bugs and Packings).

1. If \mathcal{B} is a Kleinian bug, then its supergroup $\tilde{\Gamma}$ is a lattice.
2. (i) Conversely, suppose we are given a lattice $\tilde{\Gamma} < \text{Isom}(\mathbb{H}^{n+1})$, a convex fundamental polyhedron D of $\tilde{\Gamma}$, and a finite set S' of elements $\gamma_j \in \tilde{\Gamma}$ pairing the facets of D . Let $R \subset S'$ be a nonempty subset consisting of reflections (assuming such exist) in some facets $F_\alpha \subset \partial D$, with α in an indexing set A . As before, let H_α be the cooriented hyperbolic hyperplanes containing the facets F_α . Let $S := S' \setminus R$ and $\Gamma_S := \langle S \rangle < \tilde{\Gamma}$. Then the orbit of the set of cooriented spheres in $\{\partial_\infty H_\alpha; \alpha \in A\}$ under the group Γ_S ,

$$\mathcal{B} := \Gamma_S \cdot \{\partial_\infty H_\alpha; \alpha \in A\},$$

is a Kleinian bug with a symmetry group Γ_S .

- (ii) Suppose, moreover, that: (a) the facets F_α , $\alpha \in A$, have pairwise disjoint or tangent cooriented spheres $\partial_\infty H_\alpha$, (b) the hyperplanes H_α meet the other bounding walls of D either tangentially or orthogonally (or not at all). Then \mathcal{B} is in fact a Kleinian **packing**, that is, if two spheres of \mathcal{B} intersect, they do so tangentially.

In particular, every lattice $\tilde{\Gamma} < \text{Isom}(\mathbb{H}^{n+1})$ containing a reflection yields a Kleinian bug.

Remark 23. While it is not hard to show that the supergroup of a crystallographic or Kleinian packing is a lattice, the proof of [Theorem 22](#) is rather more involved, see [§2](#).

Remark 24. We applied the above construction to create the bug in [Figures 3–4](#) from the extended Bianchi group $\widehat{Bi}(19)$ ⁷; namely, two

⁷Here and throughout, $\widehat{Bi}(D)$ denotes the extended Bianchi group, that is, the maximal discrete subgroup of $\text{Isom}(\mathbb{H}^3)$ containing the group $\text{PSL}_2(\mathcal{O}_D)$, where \mathcal{O}_D is the ring of integers of $\mathbb{Q}(\sqrt{-D})$.

reflective walls were chosen for $R \subset S'$ from this lattice, and the bug was created as the orbit of these two walls under the remaining generators. The orbit of one wall was colored blue and the other red.

1.3.2. Arithmetic Aspects of Bugs.

Turning our attention to arithmetic properties of bugs, we mimic (2) with the following.

Definition 25. The **superbug** $\tilde{\mathcal{B}}$ of an (arbitrary) bug \mathcal{B} is defined as the orbit of the bug under its reflection group, $\Gamma_{\mathcal{B}}$:

$$\tilde{\mathcal{B}} := \Gamma_{\mathcal{B}} \cdot \mathcal{B}.$$

A bug is **superintegral** if every sphere in its superbug has integer bend.

The Subarithmicity [Theorem 3](#) applies just as well to bugs, giving the following:

Theorem 26.

- If a bug (with no further structure imposed) is superintegral, then its reflection group $\Gamma_{\mathcal{B}}$ is \mathbb{Q} -subarithmetic.
- If a bug is Kleinian and integral, then its symmetry group Γ_S is \mathbb{Q} -subarithmetic.
- If a bug is Kleinian and superintegral, then its supergroup $\tilde{\Gamma}$ is \mathbb{Q} -arithmetic.

In the opposite direction, we already have from the Classification [Theorem 9](#) that every non-uniform \mathbb{Q} -arithmetic lattice (of simplest type) is commensurable to the supergroup of a superintegral Kleinian packing (and hence bug). Similarly, we have from [Theorem 12](#) that if a symmetry group Γ_S (resp. supergroup $\tilde{\Gamma}$) of a Kleinian bug is non-uniform and \mathbb{Q} -subarithmetic (resp. \mathbb{Q} -arithmetic), then there is a conformal choice of coordinates for which the bug can be made integral (resp. superintegral).

Remark 27. When a Kleinian bug (or just packing) is integral but not superintegral, we have the following curious situation: its symmetry group Γ_S is a subgroup of some integer orthogonal group $O_{\mathcal{F}}(\mathbb{Z})$, but it is also contained in a (non-arithmetic) lattice, the supergroup $\tilde{\Gamma}$.

Note further that there are essentially only three ways that a lattice $\tilde{\Gamma} < \text{Isom}(\mathbb{H}^{n+1})$ (with $n \geq 2$) can be non-arithmetic. By Selberg's Rigidity [[Sel60](#)], after conjugation, $\tilde{\Gamma}$ is contained in some $O_{n+1,1}(k)$ where k is a number field with a given embedding to \mathbb{R} . For $\tilde{\Gamma}$ to be non-arithmetic, either:

- (i) k is not totally real, or

- (ii) k is totally real but for at least one non-identity embedding ι , the orthogonal group $O_{n+1,1}(\iota(k))$ is non-compact (that is, the quadratic form is not hyperbolic), or
- (iii) k is totally real and the quadratic form is hyperbolic, but the entries of $\tilde{\Gamma}$ as elements of k have unbounded denominators.

Lattices satisfying the last condition (iii) are called *quasi-arithmetic* by Vinberg. The only currently known integral but non-superintegral bugs (e.g. the crystallographic packings constructed in [KN19, Lemma 20]) all come from this last situation (with $k = \mathbb{Q}$). It is interesting to investigate whether this is the only non-arithmeticity type possible for integral but not superintegral bugs. (Note that for \mathfrak{o} -superintegrality, the non-arithmeticity in Proposition 14 is of type (ii); see Example 71.)

1.4. Outline of the paper.

In §2 we prove geometric results on bugs, including the key the Structure Theorem 22. We spend §3 discussing the arithmetic properties of bugs, including the main Subarithmeticity Theorem 3. Of particular interest to the reader may be §3.3 where a number of archetypal examples are constructed. We conclude in §4 with a discussion of Hausdorff dimensions of accumulation sets of superintegral bugs, proving the Abundance Theorem 13.

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2. PROOF OF THE STRUCTURE THEOREM 22

In what follows, d denotes the hyperbolic distance on \mathbb{H}^{n+1} . For a subset $A \subset \mathbb{H}^{n+1}$ the *ideal boundary* of A , denoted $\partial_\infty A$, is the accumulation set of A in the ideal boundary sphere $\mathbb{S}^n = \partial_\infty \mathbb{H}^{n+1}$ of \mathbb{H}^{n+1} .

Recall that for a discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$, the *convex hull* of the limit set Λ of Γ , denoted $C = C_\Gamma$, is the intersection of all Γ -invariant closed convex nonempty subsets in \mathbb{H}^{n+1} . The convex hull has the property that $\partial_\infty C = \Lambda$ unless Λ is a singleton (in which case $C = \emptyset$). The *convex core* M^* of the orbifold $M = \mathbb{H}^{n+1}/\Gamma$ is the quotient C/Γ .

Definition 28. *A discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$ is geometrically finite if Γ is virtually torsion free and there is an ε -thickening of the convex core M^* of the orbifold $M = \mathbb{H}^{n+1}/\Gamma$ that has finite volume.*

A sufficient condition for geometric finiteness is the existence of a finitely-sided fundamental polyhedron of Γ in \mathbb{H}^{n+1} .

Definition 29. *A discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$ is called convex-cocompact if its limit set is not a singleton and the convex core M^* of the orbifold $M = \mathbb{H}^{n+1}/\Gamma$ is compact. Equivalently, there exists a closed convex nonempty Γ -invariant subset of \mathbb{H}^{n+1} on which Γ acts cocompactly.*

Convex-cocompact subgroups have the property that

$$(30) \quad \inf\{d(gx, x) : g \in \Gamma^*, x \in C_\Gamma\} > 0,$$

where $\Gamma^* \subset \Gamma$ consists of elements of infinite order.

We refer to [Bo93, Ra06] for more background on geometrically finite and convex-cocompact isometry groups of hyperbolic space.

Part 1. Let \mathcal{B} be a Kleinian bug and let $P \subset \mathbb{H}^{n+1}$ be the corresponding convex polyhedron. We wish to show that the supergroup $\tilde{\Gamma}$ is a lattice.

If $\Phi \subset \mathbb{H}^{n+1}$ is a convex fundamental domain of an arbitrary discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$, then the Γ -orbit of the relative interior of $\partial_\infty \Phi$ (with respect to \mathbb{S}^n) is dense in the domain of discontinuity of Γ in \mathbb{S}^n . Since \mathcal{B} is a bug, for $\Phi = P$, this relative interior of $\partial_\infty P$ is empty and, hence the limit set of $\Gamma_{\mathcal{B}}$ is equal to the entire sphere \mathbb{S}^n .

The ideal boundary $\partial_\infty P \subset \partial_\infty \mathbb{H}^{n+1} = \mathbb{S}^n$ is the disjoint union

$$(31) \quad \partial_\infty P = \mathcal{A}(P) \sqcup \mathcal{V}(P),$$

where each point of $\mathcal{V}(P)$ is isolated in the accumulation set $\mathcal{A}(P) := \mathcal{A}(\mathcal{B})$ of \mathcal{B} ; elements of $\mathcal{V}(P)$ are the ‘‘ideal vertices’’ of P . For each $\eta \in \mathcal{V}(P)$ and every horoball $B \subset \mathbb{H}^n$ centered at η , the intersection $P \cap B$ has finite volume.

The convex polyhedron $P = P_{\mathcal{B}}$ is the fundamental domain of the reflection group $\Gamma_{\mathcal{B}} < \text{Isom}(\mathbb{H}^{n+1})$ generated by isometric reflections in facets of P , if and only if each dihedral angle of P belongs to $\frac{\pi}{\mathbb{N}}$, see [Ma88, Ra06].

We start the proof of Part 1 with two auxiliary lemmata.

For each nonempty closed convex subset $C \subset \mathbb{H}^{n+1}$ we have the *nearest-point projection* $\pi_C : \mathbb{H}^{n+1} \rightarrow C$. This projection is a continuous (actually, 1-Lipschitz) map.

Lemma 32. *Suppose that C is a closed convex subset of \mathbb{H}^{n+1} and $C \subset A$, where A is a closed subset of \mathbb{H}^{n+1} such that $\partial_\infty C = \partial_\infty A$. Then the restriction of π_C to A is a proper map, i.e. preimages of compact subsets of C are compact.*

Proof. Assume that this is not the case. Then there exist sequences $y_i \in C$ and $x_i \in A$ such that $\pi(x_i) = y_i$, $\lim y_i = y \in C$, while $\lim x_i =$

$\xi \in \partial_\infty A = \partial_\infty C$. Let $\rho = y\xi$ denote the geodesic ray in \mathbb{H}^{n+1} emanating from y and asymptotic to ξ . By convexity of C , ρ is contained in C . Since $\lim x_i = \xi$, the angles $\angle x_i y \xi$ converge to zero as $i \rightarrow \infty$, since $x_i \rightarrow \xi$. It follows that the points $z_i := \pi_\rho(x_i)$ satisfy

$$\lim_{i \rightarrow \infty} \frac{d(x_i, z_i)}{d(x_i, y_i)} = \lim_{i \rightarrow \infty} \frac{d(x_i, z_i)}{d(x_i, y)} = 0.$$

This is a contradiction since $z_i \in C$ and y_i is the nearest-point projection of x_i to C . \square

Lemma 33. *Suppose that $\Gamma' \triangleleft \Gamma < \text{Isom}(\mathbb{H}^{n+1}) \cong PO(n+1, 1)$, where Γ' is a Zariski dense discrete subgroup. Then Γ is also discrete.*⁸

Proof. Let $\gamma_i \in \Gamma$ be a sequence converging to $1 \in PO(n+1, 1)$. Then for every $\gamma \in \Gamma'$ we have

$$\lim_{i \rightarrow \infty} \gamma_i \gamma \gamma_i^{-1} = \gamma.$$

Since $g_i := \gamma_i \gamma \gamma_i^{-1} \in \Gamma'$ and the latter is discrete, it follows that $g_i = \gamma$ for all sufficiently large i (depending on γ). Since $\Gamma < PO(n+1, 1)$ is Zariski dense, it contains a Zariski dense Schottky subgroup Σ of finite rank $r > 1$ with free generators $\sigma_1, \dots, \sigma_r$. Hence there exists i_0 such that for all $i \geq i_0$ we have that

$$\gamma_i \sigma_k \gamma_i^{-1} = \sigma_k, \quad k = 1, \dots, r.$$

By the Zariski density of Σ , we obtain that γ_i belongs to the center of $PO(n+1, 1)$ for all $i \geq i_0$. Since $PO(n+1, 1)$ is centerless, $\gamma_i = 1$ for all $i \geq i_0$. Thus, Γ is discrete. \square

Recall that $\Gamma_1 := \Gamma_{\mathcal{B}} < \text{Isom}(\mathbb{H}^{n+1})$ is the discrete reflection group with the convex fundamental polyhedron $P \subset \mathbb{H}^{n+1}$. As we noted earlier, the limit set of $\Gamma_{\mathcal{B}}$ is the entire sphere at infinity $\mathbb{S}^n = \partial_\infty \mathbb{H}^{n+1}$; in particular, $\Gamma_{\mathcal{B}}$ is Zariski dense in $PO(n+1, 1)$. Since \mathcal{B} is a bug, the polyhedron P has infinitely many faces, equivalently, $\mathcal{A}(P)$ is nonempty. Thus, if we put an accumulation point in $\mathcal{A}(P)$ at infinity (in the upper half-space model), then the spheres of \mathcal{B} are cooriented so that they bound round balls in \mathbb{R}^n . In this setting, the density condition simply means that the union of these round balls is dense in \mathbb{R}^n .

The group $\Gamma_2 := \Gamma_S$ is a *symmetry group* of the bug \mathcal{B} , i.e. a group of isometries of \mathbb{H}^{n+1} preserving P . Then supergroup of the bug \mathcal{B} is the subgroup $\tilde{\Gamma} < \text{Isom}(\mathbb{H}^{n+1})$ generated by Γ_1, Γ_2 .

⁸The same holds for subgroups of arbitrary algebraic Lie groups with discrete center.

Since Γ_2 preserves P , it normalizes Γ_1 ; hence, Γ_1 is a normal subgroup of $\tilde{\Gamma}$. In view of [Lemma 33](#) and Zariski density of Γ_1 , the subgroup $\tilde{\Gamma} < \text{Isom}(\mathbb{H}^{n+1})$ is discrete. Our goal is to show that $\tilde{\Gamma}$ is not just Zariski dense and discrete, but is also a lattice. We build up to the general case from some initial assumptions which simplify the exposition.

Theorem 34. *If, in addition to the above assumptions, Γ_2 is convex-cocompact and $\mathcal{A}(P) = \partial_\infty P$ (i.e. $\mathcal{V}(P) = \emptyset$), then the supergroup $\tilde{\Gamma}$ is a uniform lattice in $\text{Isom}(\mathbb{H}^{n+1})$.*

Proof. Since $\mathbb{H}^{n+1}/\tilde{\Gamma}$ is homeomorphic (actually, isometric) to P/Γ_2 , it suffices to show that the quotient P/Γ_2 is compact. Since Γ_2 is convex-cocompact, it acts cocompactly on the convex hull C of its limit set $\Lambda = \partial_\infty P$. Let $K \subset C$ be a compact fundamental set of the action of Γ_2 on C , i.e. $\Gamma_2 \cdot K = C$. Then, by [Lemma 32](#) (applied with $A = P$), the preimage $K' := \pi_C^{-1}(K) \cap P$ is also compact. Since π_C is equivariant with respect to the action of Γ_2 , it follows that K' is a compact fundamental set of the action of Γ_2 on P . Hence, P/Γ_2 is compact and $\tilde{\Gamma}$ is a uniform lattice. \square

Next we relax the assumption that the ideal boundary $\partial_\infty P$ coincides with the accumulation set $\mathcal{A}(P)$ (that is, we allow $\mathcal{V}(P)$ in [\(31\)](#) to be nonempty), while keeping convex-cocompactness of Γ_2 . We let μ_{n+1} denote the *Margulis constant* of \mathbb{H}^{n+1} (see e.g. [\[Ka\]](#) or [\[Ra06\]](#)).

Theorem 35. *If Γ_2 is convex-cocompact, then the supergroup $\tilde{\Gamma}$ is a lattice in $\text{Isom}(\mathbb{H}^{n+1})$.*

Proof. Let $C = C_{\Gamma_2}$ again denote the closed convex hull of the limit set of Γ_2 .

Each ideal vertex $\eta_j \in \mathcal{V}(P)$, $j \in J$, represents a finite volume cusp of the orbifold $\mathcal{O} = \mathbb{H}^{n+1}/\Gamma_1 \cong P$; its stabilizer $\Gamma_{1,j}$ in Γ_1 is generated by reflections in the faces of P asymptotic to η_j . We will be using certain open horoballs $B_{\eta_j} \subset \mathbb{H}^{n+1}$ centered at the points η_j . We call such a horoball $B \subset \mathbb{H}^{n+1}$ an ϵ -*Margulis horoball*, where $0 < \epsilon \leq \mu_{n+1}$, if it is the maximal horoball such that for each $x \in B$, there exists a parabolic element $g \in \Gamma_{1,j}$ satisfying

$$d(x, g(x)) < \epsilon.$$

The Margulis lemma implies that Margulis horoballs B_{η_j} are pairwise disjoint. In view of [\(30\)](#), we can choose ϵ such that each ϵ -Margulis horoball is disjoint from C .⁹ From now on, we fix a collection B_{η_j} , $j \in J$,

⁹This is not really necessary but provides a clearer picture of the situation.

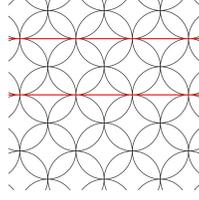


FIGURE 6. A Kleinian bug which is not cusp-compatible.

of such ϵ -Margulis horoballs. This collection is necessarily Γ_2 -invariant. Since each intersection $B_{\eta_j} \cap P$ has finite volume and $\mathbb{H}^{n+1}/\tilde{\Gamma}$ has the same volume as P/Γ_2 , in order to prove that $\tilde{\Gamma}$ is a lattice it suffices to show that the quotient

$$(P \setminus \bigcup_{j \in J} B_{\eta_j})/\Gamma_2$$

is compact. (This will also force finiteness of the number of Γ_2 -orbits of the horoballs B_{η_j} .) Compactness of the quotient again follows by applying [Lemma 32](#) to the set

$$A := P \setminus \bigcup_{j \in J} B_{\eta_j}$$

as in the proof of [Theorem 34](#). \square

Lastly, we would like to relax the convex-cocompactness assumption on Γ_2 , replacing it with geometric finiteness. Unfortunately, in general, the subgroup $\tilde{\Gamma} < \text{Isom}(\mathbb{H}^{n+1})$ will not be a lattice, already in the case $n = 3$. This can happen even if the limit set $\mathcal{A}(P)$ is Zariski-dense in \mathbb{S}^n .

Example 36. A concrete example is the bug consisting of circles of unit diameter centered at the set $\mathbb{Z}[i] \cup (\frac{1}{2} + \frac{1}{2}i + \mathbb{Z}[i])$, that is, the Gaussian integers and their translates by the vector $\frac{1}{2} + \frac{1}{2}i$. The group Γ_1 is generated by reflections in these circles. The group Γ_2 is generated by reflections in the lines $\{\Re(z) = 0\}$ and $\{\Re(z) = 1\}$. The situation is illustrated in [Figure 6](#). The accumulation point is ∞ , and the group $\tilde{\Gamma} = \langle \Gamma_1, \Gamma_2 \rangle$ is a non-lattice. On the other hand, if we take Γ'_2 generated by the reflections in the lines $\{\Re(z) = 0\}$, $\{\Re(z) = 1\}$, $\{\Im(z) = 0\}$, $\{\Im(z) = 1\}$, then the group $\tilde{\Gamma}' = \langle \Gamma_1, \Gamma'_2 \rangle$ is a lattice.

Example 37. More generally, let Γ_2 be a geometrically finite subgroup of $\text{Isom}(\mathbb{H}^3)$ which has at least one rank-one cusp. Then (by either using the Andreev–Thurston theorem, see [[T81](#), §§13.6–13.7], or Brooks’s theorem, see [[Br86](#)] or [[Ka](#)]) there exists a quasiconformal deformation

Γ'_2 of Γ_2 and a reflection group Γ_1 such that the fundamental domain of Γ_1 is invariant under Γ'_2 and its accumulation set equals the limit set of Γ'_2 , but the groups Γ_1, Γ'_2 are not cusp-compatible.

A necessary condition for $\tilde{\Gamma}$ to be a lattice is that the $\tilde{\Gamma}$ -stabilizer of each parabolic fixed point of Γ_2 is virtually abelian of virtual rank n . (Recall that a group G is a virtually abelian if it contains a free abelian subgroup of finite index. The virtual rank of G is defined to be the rank of any maximal free abelian subgroup of G . By Bieberbach's Theorem, a discrete parabolic subgroup of $\text{Isom}(\mathbb{H}^{n+1})$ is necessarily virtually abelian of virtual rank $\leq n$.) We thus arrive at the following definition of cusp-compatibility, for the pair of discrete groups $\Gamma_1 = \Gamma_{\mathcal{P}}, \Gamma_2 = \Gamma_S$, repeated from [Definition 20](#).

Definition 38. *The subgroups $\Gamma_1, \Gamma_2 < \text{Isom}(\mathbb{H}^{n+1})$ are cusp-compatible if for every maximal parabolic subgroup $\Pi_2 < \Gamma_2$, there exists a subgroup $\Pi_1 < \Gamma_1$ such that the subgroup Π generated by Π_1, Π_2 is virtually abelian of virtual rank n .*

Observe that the subgroup Π_1 in this definition necessarily fixes the limit set (the single fixed point at infinity) $\{\lambda\} \subset \mathbb{S}^n$ of Π_2 . Thus, Π fixes λ as well. In view of the discreteness of $\tilde{\Gamma}$ (and, hence, of Π), we can (and will) as well assume that Π_1 is the full stabilizer of λ in Γ_1 . (The group Π_1 might be finite.) Since Γ_1 is normal in $\tilde{\Gamma}$, the subgroup Π_1 is normal in Π and Π splits as the semidirect product $\Pi_1 \rtimes \Pi_2$.

Lemma 39. *1. The group Π_1 is generated by reflections in the facets of P asymptotic to λ and, hence, has a fundamental domain P_λ in \mathbb{H}^{n+1} equal to the intersection of all half-spaces defined by facets of P asymptotic to λ .*

3. The fundamental domain P_λ is invariant under the action of Π_2 .

Proof. 1. Let s_α denote the isometric reflections in the facets F_α of P generating the reflection group Γ_1 . Every element $\gamma \in \Gamma_1$ is represented by a reduced word w in the generators s_α . Suppose that w has the form $u \cdot s_\alpha \cdot v$, where v is the product of reflections in the facets of P_λ , while s_α is the reflection in a facet F_α not asymptotic to λ . Then $s_\alpha v(P)$ is contained in the half-space H_α^- whose closure in $\mathbb{H}^{n+1} \cup \partial_\infty \mathbb{H}^{n+1}$ does not contain λ . It follows that $u \cdot s_\alpha \cdot v(P)$ is also contained in H_α^- and, hence, γ cannot possibly fix λ .

2. Since P is preserved by Γ_2 , and, hence, by its subgroup Π_2 , and Π_2 fixes λ , the elements of Π_2 send facets of P asymptotic to λ to facets of P asymptotic to λ . \square

The following lemma provides a list of equivalent algebraic and geometric characterizations of cusp-compatibility in the context of the pair of groups $\Gamma_1 = \Gamma_{\mathcal{P}}$, $\Gamma_2 = \Gamma_S$:

Lemma 40. *The following are equivalent for subgroups Π_1, Π_2 of Γ_1, Γ_2 as above:*

- (1) *The subgroup Π generated by Π_1, Π_2 is virtually abelian of virtual rank n .*
- (2) *Virtual ranks of Π_1, Π_2 add up to n .*
- (3) *Π acts cocompactly on $\partial_{\infty}\mathbb{H}^{n+1} \setminus \{\lambda\}$.*
- (4) *Π_2 acts cocompactly on the intersection $P_{\lambda} \cap \partial B$, for every horoball $B \subset \mathbb{H}^{n+1}$ centered at λ .*
- (5) *Π_2 acts cocompactly on the intersection $P \cap \partial B$, for every horoball $B \subset \mathbb{H}^{n+1}$ centered at λ .*
- (6) *Π_2 acts with finite covolume on the intersection $P_{\lambda} \cap B$, for every horoball $B \subset \mathbb{H}^{n+1}$ centered at λ .*
- (7) *Π_2 acts with finite covolume on the intersection $P \cap B$, for some horoball $B \subset \mathbb{H}^{n+1}$ centered at λ .*

Proof. The equivalence (1) \Leftrightarrow (2) follows from the semidirect product decomposition $\Pi = \Pi_1 \rtimes \Pi_2$. The equivalence (1) \Leftrightarrow (3) follows from the fact that a discrete subgroup of \mathbb{R}^n is a uniform lattice if and only if this subgroup has rank n .

To prove the equivalence (3) \Leftrightarrow (4) observe that, by Lemma 39, $P_{\lambda} \cap \partial B$ is the fundamental domain for the action of Π_1 on the horosphere ∂B and this intersection is invariant under Π_2 . Therefore, cocompactness of the action of Π on the horosphere is equivalent to the cocompactness of the action of Π_2 on $P_{\lambda} \cap \partial B$.

The proofs of equivalences (3) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) are similar to the proof of (3) \Leftrightarrow (4) and are left to the reader. \square

Subgroups Π_1, Π_2 are called **compatible** if they satisfy one of the equivalent conditions in this lemma. Note that since $P \subset P_{\lambda}$, compactness of $(P_{\lambda} \cap \partial B)/\Pi_2$, implies that only finitely many Π_2 -orbits of faces of P might intersect B . Moreover, the compatibility of Π_1, Π_2 (in the form of the 3rd condition) implies that any sequence of faces of P converging to λ (possibly outside of a horoball B) is contained in finitely many Π_2 -orbits of faces.

Lemma 41. *Suppose that $\mathcal{A}(P)$ has cardinality ≥ 2 . Then any two symmetry groups of \mathcal{B} are commensurable and, in particular, the cusp-compatibility of the subgroups Γ_1, Γ_2 depends only on P .*

Proof. Suppose that $\Gamma_2, \Gamma'_2 < \text{Isom}(\mathbb{H}^{n+1})$ are geometrically finite subgroups preserving P and having $\mathcal{A}(P)$ as their limit sets. Since $|\mathcal{A}(P)| \geq 2$, by the definition of geometric finiteness, the groups Γ_2, Γ'_2 act with finite covolume on the ϵ -neighborhood Y of the convex hull of $\mathcal{A}(P)$ (which is their common limit set). Since both Γ_2, Γ'_2 preserve the polyhedron P , they generate a discrete subgroup $\Gamma_3 < \text{Isom}(\mathbb{H}^{n+1})$, which, therefore, acts properly discontinuously, with finite covolume on Y . It follows that $|\Gamma_3 : \Gamma_2| < \infty$, $|\Gamma_3 : \Gamma'_2| < \infty$, which implies commensurability of Γ_2, Γ'_2 . Therefore, the pair (Γ_1, Γ_2) is cusp-compatible if and only if (Γ_1, Γ'_2) is. \square

Remark 42. [Example 36](#) shows that, when $|\mathcal{A}(P)| = 1$, cusp-compatibility depends not only on P but also on Γ_2 .

Theorem 43. *If Γ_2 is geometrically finite and the pair Γ_1, Γ_2 is cusp-compatible, then the subgroup $\tilde{\Gamma}$ generated by Γ_1, Γ_2 is a lattice in $\text{Isom}(\mathbb{H}^{n+1})$.*

Proof. Similarly to the proof of [Theorem 35](#), we define open horoballs $B_{\eta_j}, j \in J$. Since each η_j does not belong to $\mathcal{A}(P)$, it follows that all η_j 's lie in the discontinuity domain of the group Γ_2 . This, together with geometric finiteness of Γ_2 implies that there exists a collection $\{\bar{B}_{\lambda_i}; i \in I\}$ of closed (Margulis) horoballs centered at parabolic fixed points of Γ_2 , such that:

- (1) The collection of horoballs is Γ_2 -invariant.
- (2) The horoballs are pairwise disjoint and disjoint from the horoballs \bar{B}_{η_j} .
- (3) Each horoball \bar{B}_{λ_i} intersects only those faces of P which are asymptotic to λ_i .

Set

$$A := P \setminus \left(\bigcup_{j \in J} B_{\eta_j} \cup \bigcup_{i \in I} B_{\lambda_i} \right).$$

There are two cases to consider. First, suppose that $\mathcal{A}(P) = \Lambda(\Gamma_2)$ is a singleton $\{\lambda\}$. Then $\{B_{\lambda_i} : i \in I\}$ consists of a single horoball B . Consider the nearest-point projection $\pi_{\bar{B}} : \mathbb{H}^{n+1} \rightarrow \bar{B}$. For $x \notin \bar{B}$, this projection is given by the unique intersection point of the geodesic ray $x\lambda$ with the horosphere ∂B . As in the proof of [Theorem 35](#), the restriction of $\pi_{\bar{B}}$ to A is a proper map. The image $\pi_{\bar{B}}(A)$ is contained in $\partial B \cap P$: By the convexity of P , for each $x \in A$ the ray $x\lambda$ is contained in P , hence, the intersection point $x\lambda \cap \partial B$ belongs to $\partial B \cap P$.

Since the pair Γ_1, Γ_2 is cusp-compatible, by [Lemma 40\(5\)](#), the action of Γ_2 on $P \cap \partial B$ is cocompact. Combining this with the fact that the

map $\pi_{\bar{B}} : A \rightarrow \bar{B}$ is proper and its image is contained in $\partial B \cap P$, we conclude that the quotient A/Γ_2 is compact. Since $(P \cap B)/\Gamma_2$ has finite volume and the volume of each cusp $P \cap B_{\eta_j}$ is finite, we conclude that P/Γ_2 has finite volume and, therefore, $\tilde{\Gamma}$ is a lattice.

We now consider the “generic” case, when Γ_2 has at least two limit points. As before, let $C = C_{\Gamma_2}$ denote the closed convex hull in \mathbb{H}^{n+1} of the limit set $\Lambda = \Lambda(\Gamma_2)$. Since $\Lambda = \mathcal{A}(P) \subset \partial_{\infty} P$, convexity of P implies that $C \subset P$. We let $C' \subset C$ be the complement to the union of open horoballs $B_{\lambda_i}, i \in I$. The group Γ_2 acts cocompactly on C' since it is geometrically finite. As in the proof of [Theorem 35](#), the restriction of π_C to A is a proper map and $(A \cap \pi_C^{-1}(C'))/\Gamma_2$ is compact. Lastly, [Lemma 40](#) (and the discussion following it) implies that for every horoball $B_{\lambda_i}, i \in I$, its Γ_2 -stabilizer Π_{λ_i} acts with finite covolume on

$$P \cap \pi_C^{-1}(B_{\lambda_i}).$$

Hence, P/Γ_2 has finite volume and, since $\text{Vol}(P/\Gamma_2) = \text{Vol}(\mathbb{H}^{n+1}/\tilde{\Gamma})$, it follows that $\tilde{\Gamma}$ is a lattice. \square

The next lemma establishes the cusp-compatibility condition in the the special case of bugs which are packings.

Lemma 44. *If \mathcal{B} is a Kleinian packing then the cusp-compatibility condition always holds.*

Proof. Since \mathcal{B} is a Kleinian packing, the polyhedron P is the intersection of half-spaces H_{α}^+ , which are bounded by pairwise disjoint hyperbolic hyperplanes H_{α} . In particular, $\partial_{\infty} P$ has no isolated points and, hence, equals the accumulation set $\mathcal{A}(P)$, which is the limit set Λ of Γ_S , and $P = C_{\Gamma_S}$, the closed convex hull of Λ . Since Γ_S is geometrically finite, C_{Γ_S}/Γ_S has finite volume. But $C_{\Gamma_S}/\Gamma_S = P/\Gamma_S$ is naturally isometric to $\mathbb{H}^{n+1}/\tilde{\Gamma}$. We conclude that $\tilde{\Gamma}$ is a lattice, which is equivalent to the cusp-compatibility condition. \square

Part 2.

Proof of Part 2(i). Step 1. For each defining hyperplane H_{α} of D , $\alpha \notin A$, we let $H_{\alpha}^+ \subset \mathbb{H}^{n+1}$ denote the closed half-space bounded by H_{α} and containing D . We define D_S as the intersection of these half-spaces,

$$D_S = \bigcap_{\alpha \notin A} H_{\alpha}^+.$$

Each facet of D_S is contained in one of the facets of D which is not of the form $F_{\alpha}, \alpha \in A$. And conversely, each facet of D not of the form $F_{\alpha}, \alpha \in A$, is contained in one of the facets of D_S . The generators $\gamma \in S$

still pair the facets of D_S . We leave it to the reader to check that, since D is a fundamental polyhedron of Γ , the conditions of the Poincaré's Fundamental Domain Theorem (see [Ra06]) still hold for D_S and the face-pairing transformations $\gamma \in S$, and, hence, D_S is a fundamental polyhedron of Γ_S . Since, by the construction, D_S is finitely-sided, the group Γ_S is geometrically finite with the fundamental domain D_S .

Step 2. We let \mathcal{R} denote the collection of all $\tilde{\Gamma}$ -images of the hyperplanes $H_\alpha, \alpha \in A$. Since D is a fundamental domain of $\tilde{\Gamma}$, none of the hyperplanes in \mathcal{R} intersects the interior of D . The hyperplanes in \mathcal{R} define a partition of \mathbb{H}^{n+1} into (convex) connected components; the closure of one of these components, denoted P , contains D . We let $\Gamma_R < \tilde{\Gamma}$ denote the subgroup generated by reflections in the hyperplanes $H \in \mathcal{R}$. Thus, Γ_R is a nontrivial normal subgroup in the lattice $\tilde{\Gamma}$; hence, the limit set of Γ_R is the entire sphere \mathbb{S}^n . The polyhedron P is a fundamental domain of Γ_R and Γ_R is generated by reflections in those hyperplanes $H \in \mathcal{R}$ for which $H \cap P$ are facets of P . Since $\Gamma_R < \tilde{\Gamma}$ and the latter is a lattice, it follows that the dihedral angles of P come from a finite subset of $\frac{\pi}{\mathbb{N}}$. We conclude that P defines a bug \mathcal{B} .

Step 3. We next verify that Γ_S is a symmetry group of the bug \mathcal{B} . We first check that the generators $\gamma \in S$ of Γ_S preserve the polyhedron P . It suffices to show that each γ sends facets of P to facets of P . The element γ pairs a facet F_γ of D to a facet F'_γ of D . Let G_β be a facet of P and $H_\beta \in \mathcal{R}$ be the hyperbolic hyperplane containing G_β . We pick a generic base-point o in the interior of D and a generic point x on the facet G_β . Then, by the convexity of P and since $o \in D \subset P$, the geodesic segment ox is disjoint from all the hyperplanes $H \in \mathcal{R}$ except for the point $x \in G_\beta$. Similarly, the geodesic segment $o\gamma(o)$ is contained in $D \cup \gamma(D)$ and crosses their intersection at an interior point of F'_γ (since o was chosen generically). It follows that the segment $o\gamma(o)$ is also disjoint from all the hyperplanes in \mathcal{R} . Thus, the union

$$o\gamma(o) \cup \gamma(o)\gamma(x)$$

is a path connecting o to $\gamma(x)$ and disjoint from all the hyperplanes in \mathcal{R} except for the point $\gamma(x) \in \gamma H_\beta$. Hence, $\gamma(x)$ lies in a facet of P . Thus, we verified that the generators $\gamma \in S$ have the property that they send facets of P to facets of P and, moreover, respect their coorientation: The half-space H_α^+ determined by a facet F_α and containing o , maps to the half-space $H_{\alpha'}^+$ containing o . Therefore, the entire group Γ_S preserves P .

Step 4. Since, by the construction, $\tilde{\Gamma}$ is a lattice generated by Γ_R and Γ_S , the group Γ_S acts on the set of facets of P with finitely many orbits. Thus, if ξ is an accumulation point of the bug \mathcal{B} , there is a facet G_α of P and an infinite sequence $\gamma_i \in \Gamma_S$ such that $\gamma_i(G_\alpha)$ converges to ξ . In other words, $\mathcal{A}(\mathcal{B})$ is contained in the limit set of Γ_S . The opposite inclusion follows from the fact that Γ_S preserves the polyhedron P . We conclude, therefore, that Γ_S is a symmetry group of \mathcal{B} and \mathcal{B} is geometric with $\Gamma_R = \Gamma_{\mathcal{B}}$.

Step 5. Lastly, the cusp-compatibility of the pair of groups Γ_R, Γ_S follows from the fact that they generate a lattice $\tilde{\Gamma}$.

Thus, we proved that \mathcal{B} is a Kleinian bug, which concludes the proof of Part 2(i) of [Theorem 22](#).

Proof of Part 2(ii). We continue with the notation from the proof of Part 2(i). However, before starting the actual proof of (ii) we will have a discussion related to combinatorics of convex fundamental polyhedra. Each facet F of ∂D is *paired* with another facet F' by a unique generator $\gamma = \gamma_{F,F'} \in S'$; $\gamma(F) = F'$, where, possibly, $F = F'$.

Remark 45. Following Ratcliffe in [\[Ra06\]](#), we require that if a generator $\gamma \in S'$ preserves a facet F of D , then it fixes F pointwise, i.e. is a reflection in F . To achieve this, one performs, if necessary, a subdivision of geometric facets of D . We refer to [\[Ra06\]](#) for details. Accordingly, if F_1, F_2 are distinct facets of the same geometric facet of ∂D and $\gamma_i : F_i \rightarrow F'_i, i = 1, 2$, pair these facets, and it happens that $\gamma_1 = \gamma_2$ are equal as elements of $\tilde{\Gamma}$, then we still regard γ_1, γ_2 as distinct elements of S' .

A *ridge* of a convex polyhedron in \mathbb{H}^{n+1} is the $n - 1$ -dimensional intersection of two facets.

The pseudogroup \mathcal{G} . The pair (D, S') defines a pseudogroup \mathcal{G} acting on ∂D , which we discuss below. Each generator $\gamma \in S'$ is an element of \mathcal{G} ; it is a partially defined map between the two facets of D paired as $\gamma : F \rightarrow F'$. Then the unique generator sending $F' \rightarrow F$ is γ^{-1} . A composition $\gamma_2 \circ \gamma_1$ of two generators

$$\gamma_1 : F_1 \rightarrow F'_1, \gamma_2 : F_2 \rightarrow F'_2$$

is *admissible* if $F'_1 \cap F_2$ is a common ridge of these facets. A (possibly empty) word

$$w = \gamma_l \circ \dots \circ \gamma_1, \gamma_i : F_i \rightarrow F'_i, i = 1, \dots, l,$$

is *admissible* if each consecutive composition in it is admissible. Thus, each admissible word defines a map from one ridge E to another ridge

E' , the domain and the range of w (unless $l = 1$ in which case the domain and the range are facets). Here E is a boundary ridge of F_1 and E' is a boundary ridge of E'_l . The pseudogroup \mathcal{G} then consists of admissible compositions of the generators. Note that each admissible word is necessarily a reduced word in the alphabet S' : $\gamma_{i+1} \neq \gamma_i^{-1}$ for each i .

The pseudogroup \mathcal{G} defines an equivalence relation $\sim_{\mathcal{G}}$ on D : $x \sim_{\mathcal{G}} y$ (x is \mathcal{G} -equivalent to y) if and only if there exists an element $\gamma \in \mathcal{G}$ sending x to y . This equivalence relation is the one obtained by saturating the non-reflexive and non-transitive relation given by

$$x \sim y, x \in F, y \in F', y = \gamma_{F,F'}(x).$$

An important fact, coming from the assumption that D is a fundamental polyhedron of Γ is that the natural projection of quotient spaces

$$D/\sim_{\mathcal{G}} \rightarrow \mathbb{H}^{n+1}/\tilde{\Gamma}$$

is a homeomorphism; this implies that two points in D are \mathcal{G} -equivalent if and only if they are $\tilde{\Gamma}$ -equivalent, i.e. belong to the same $\tilde{\Gamma}$ -orbit.

Ridge-chains and cycles. Suppose that $E = E_1 = F_1 \cap F'_0$ is a ridge of D . Let $\gamma_1 \in S'$ denote the generator pairing the facet F_1 to another facet, F'_1 ; this yields a new ridge $E_2 := \gamma_1(E_1) = F'_1 \cap F_2$. Then let $\gamma_2 \in S'$ be the generator pairing F_2 to a facet F'_2 . The composition $\gamma_2 \circ \gamma_1$ is admissible (by the construction). This composition process continues (uniquely), until we return to the original ridge so that

$$\gamma_k : E_k \rightarrow E_1, \quad \text{and} \quad \gamma_k(F_k) = F'_k = F'_0.$$

(The process has to terminate since D has only finitely many faces; in the case of polyhedra with infinitely many faces, such termination is a consequence of one of the axioms of fundamental polyhedra.) There is an important caveat regarding this definition that applies in the special case when the *first return* to the initial ridge yields $F'_k = F_1$ instead of $F'_k = F'_0$; we discuss this in [Remark 46](#) below.

The finite sequence

$$c_E = (\gamma_1, \gamma_2, \dots, \gamma_k)$$

is called a *ridge-cycle*; it corresponds to the word

$$w_{c_E} = \gamma_k \circ \dots \circ \gamma_1,$$

which is an admissible composition. Its subwords

$$w_{c_{E,E'}} = \gamma_l \circ \dots \circ \gamma_1, \quad l \leq k,$$

correspond to *ridge-chains*

$$c_{E,E'} = (\gamma_1, \gamma_2, \dots, \gamma_l),$$

where $E' = E_{l+1} = \gamma_l(E_l)$. (The notation is slightly ambiguous since the chain $c_{E,E'}$ is not uniquely determined by E, E' ; the same, of course, applies to the notation c_E .) The element $\gamma_{c_{E,E'}} \in \Gamma$ corresponding to the composition $w_{c_{E,E'}}$ sends E_1 to E_{k+1} . We will refer to $w_{c_{E,E'}}$ as the *word of the ridge-chain* $c_{E,E'}$.

We let θ_i denote the interior dihedral angles of D along the ridges E_i ; the sum

$$\theta_{c_{E,E'}} = \sum_{i=1}^l \theta_i$$

is the *total angle* of the ridge-chain $c_{E,E'}$.

Remark 46. 1. In the sequence of facets given by a ridge-chain, we could have $F'_i = F_i$; this happens when γ_i is the reflection in the facet F_i . Accordingly, in the case $F'_i = F_i$ we will have $\gamma_i(E_i) = E_{i+1} = E_i$.

2. In this situation (i.e. $F'_i = F_i$), the word w_{c_E} contains a prefix subword which is a “palindrome”

$$u = (\gamma_{i-1} \circ \dots \circ \gamma_1)^{-1} \circ \gamma_i \circ (\gamma_{i-1} \circ \dots \circ \gamma_1).$$

The word u represents an element of Γ sending F_1 back to itself and the ridge E back to itself. In this case, of course, u does not send F_1 to F'_0 as required by the definition of a ridge-cycle. Thus, the actual cycle-word w will be longer than u and will equal $v \circ u$, where v is another palindromic composition, starting with a face-pairing $F'_0 \rightarrow F_0$. As a simple example of this situation, one can take the case when both F'_0, F_1 are reflective facets with the corresponding generating reflections γ_0, γ_1 respectively. Then $w = \gamma_0 \circ \gamma_1$.

3. Each ridge E of D defines exactly two ridge-cycles, which differ by swapping the facets F_1, F'_0 , reversing the order in the sequence (γ_i) and inverting the generators in the cycle. The total angle is, of course, independent of which of the two ridge-cycles is used.

Conversely, suppose we are given an admissible composition

$$w = \gamma_l \circ \dots \circ \gamma_1$$

representing $\gamma \in \tilde{\Gamma}$ sending a ridge $E = F'_0 \cap F_1$ to a ridge E' . This alone, however, does not guarantee that corresponding sequence of generators is a chain since for some i , $1 \leq i < k$, we may have that $\gamma_i \circ \dots \circ \gamma_1$ sends E to itself and $\gamma_i : F_i \rightarrow F'_i = F'_0$. Taking the minimal i with

this property we obtain a ridge-cycle c_E and the corresponding word w_{c_E} . Hence, we decompose w as

$$w = w' \circ (w_{c_E})^t,$$

where w' is a subword in w_{c_E} which represents an element γ' , $\gamma'(E) = E'$. In particular, w' is the word of a ridge-chain $c_{E,E'}$.

The fact that D is a convex fundamental polyhedron of $\tilde{\Gamma}$ implies that for every ridge E , the total angle θ_E is of the form $\frac{2\pi}{m}$, and γ_{c_E} has order m . Suppose now that the ridge-cycle c_E is such that the consecutive facets F_i, F'_i are equal (this happens when γ_i is a reflection). In this case the angle θ_i appears twice in the sum defining the total angle θ_E . In particular, if $\theta_1 = \theta_k = \frac{\pi}{2}$ and $\theta_s = \frac{\pi}{2}$ for some $1 \leq s \leq k$, and γ_s, γ_k are reflections, then both θ_1, θ_s contribute *twice* to the total angle and, thus, there are exactly two possibilities for the ridge-cycle c_E :

1. $k = 2, s = 1, E = E_1 = E_2, \theta_{c_E} = \pi$, and the ridge-cycle is as in the simple example in [Remark 46\(2\)](#) above. In particular, the reflections γ_1, γ_2 commute and $\gamma_i(H_j) = H_j, j = 1, 2$ is taken modulo 2.

2. $k = 4, s = 2, E_1 \neq E_2, \theta_{c_E} = 2\pi$, and, up to inversion,

$$w_{c_E} = \tau_0 \circ \gamma_1^{-1} \circ \tau_2 \circ \gamma_1$$

where $\tau_2 = \gamma_2$ is the reflection in the facet F_2 , while $\tau_0 = \gamma_4$ is the reflection in the facet $F'_0 = F_4$. The composition γ_{c_E} then has order 1, i.e. the word w_{c_E} represents the neutral element of the group $\tilde{\Gamma}$. In particular, in this case,

$$\tau_2 = \gamma_1 \circ \tau_0 \circ \gamma_1^{-1}.$$

In both cases 1 and 2, there are exactly four images of D (one of which is D itself) under the elements of $\tilde{\Gamma}$, sharing the ridge E , all with right dihedral angles at E . The hyperplanes H_0, H_1 bounding these images and containing the faces F'_0, F_1 are orthogonal to each other and divide the hyperbolic space in four quadrants, each containing one of the above images of D . The stabilizer of E in $\tilde{\Gamma}$ preserves both hyperplanes.

We are now ready for the proof of Part 2(ii). Suppose that the bug \mathcal{B} is not a packing and two walls in \mathcal{R} have nonempty transversal intersection in the hyperbolic space; the intersection necessarily has codimension 2. Since D is a fundamental polyhedron of $\tilde{\Gamma}$, this intersection comes from D in the following sense: There exists a pair of ridges E, E' (possibly equal!) formed, respectively, by pairs of facets F_1, F'_0 (the ridge E) and F_s, F'_s (the ridge E'). The facets F'_0, F'_s are

contained in hyperplanes $H = H_0, H' = H_s \in \mathcal{R}$ fixed by reflections τ_0, τ_s . There exists an element $g \in \tilde{\Gamma}$ which carries E to E' and sends H to a hyperplane $g(H)$ which meets H' orthogonally along E' . In particular, E, E' are in the same $\tilde{\Gamma}$ -orbit. It follows that there exists a chain $c_{E,E'}$ corresponding to a word $w_{E,E'}$ representing an element $\gamma = \gamma_{E,E'}$ which sends E to E' . The element γ need not be equal to g , but $g = \gamma' \circ \gamma$, where $\gamma' \in \tilde{\Gamma}$ is an elliptic isometry preserving E' . Since H, H' are both in \mathcal{R} , the dihedral angles of D along E, E' are both right angles. Thus, the discussion above regarding the ridge-chain $w_{E,E'}$ applies. In particular, both $H, g(H)$ define facets of D . Recall that by the assumptions of Part 2(ii), the hyperplanes in \mathcal{R} defining facets of D are pairwise disjoint, which implies that $g(H) = H'$, a contradiction.

This concludes the proof of [Theorem 22](#). \square

3. (SUB)ARITHMETICITY

In this section, we prove the various (sub)arithmeticity theorems. We will be repeatedly using the *Lorentzian model* of hyperbolic space \mathbb{H}^{n+1} and the corresponding parameterization of round spheres in $\mathbb{S}^n = \partial_\infty \mathbb{H}^{n+1}$ by unit vectors of the associated quadratic form.

Definition 47. *A quadratic form \mathcal{F} over a totally real number field k is called hyperbolic if it has signature $(n+1, 1)$ in the identity embedding $k \rightarrow \mathbb{R}$, and is definite in all others embeddings. A quadratic space is an $(n+2)$ -dimensional real vector space V together with a real quadratic form \mathcal{F} on V defining a bilinear form $\langle \cdot, \cdot \rangle$ in the usual way. A quadratic space is said to be hyperbolic if \mathcal{F} has signature $(n+1, 1)$.*

We let Q denote the *standard* hyperbolic quadratic form with half-Hessian:

$$(48) \quad Q = \begin{pmatrix} & & -\frac{1}{2} \\ & I & \\ -\frac{1}{2} & & \end{pmatrix}.$$

3.1. Inversive coordinates.

Before embarking on the proofs, we recall the very convenient (in this context) *inversive coordinate system* (see, e.g., [K17, LMW02, Wil82]). For convenience, we work here with the standard hyperbolic quadratic form Q , but the entire discussion applies to all quadratic forms after an appropriate change of coordinates.

To a cooriented round sphere S in the boundary $\mathbb{S}^n = \partial_\infty \mathbb{H}^{n+1} = \mathbb{R}^n \cup \{\infty\}$ having center $z = (x_1, \dots, x_n)$ and signed radius r , we associate the column vector

$$v_S := \left(\frac{1}{r}, \frac{z}{r}, \frac{1}{\hat{r}} \right)^t \in \mathbb{R}^{n+2}.$$

Here \hat{r} is the co-radius, defined to be the signed radius of the image of S under reflection through the unit sphere; more concretely,

$$(49) \quad \hat{r} = \frac{r}{|z|^2 - r^2}.$$

When S is a hyperplane, the inversive coordinates are those obtained from a limit of spheres. That is, $1/r = 0$, $1/\hat{r}$ is half the distance from the hyperplane to the origin, and z/r is the unit normal to the cooriented hyperplane.

Rewriting (49) as

$$|\frac{z}{r}|^2 - \frac{1}{r\hat{r}} = 1,$$

we see that $Q(v_S) = 1$.

The Möbius group Mob_n (the group of Möbius transformations of \mathbb{S}^n) acts on the space $Sph(n)$ of (cooriented) round spheres and the group O_Q of automorphisms of the form Q acts on the 1-sheeted hyperboloid $\{Q(v) = 1\}$.

Lemma 50. *The map $j : S \mapsto v_S$ is equivariant with respect to the actions of Mob_n and O_Q . More precisely, j conjugates the Möbius action on the space of cooriented spheres $S \in Sph(n)$ to the Lorentzian action on the inversive coordinate vectors $v_S \in \{Q_S = 1\}$.*

Proof. The proof is essentially contained in the proof of [Iv92, Theorem 7.5]. Iversen constructs an equivariant map ι of \mathbb{S}^n to the projectivization of the conic $\{Q = 0\}$. In the proof he verifies that

$$\iota R_S \iota^{-1} = \tau_{j(S)},$$

where R_S is inversion in $S \in Sph(n)$ and τ_v is the Lorentzian reflection in v^\perp , the Lorentzian orthogonal complement to the vector v . To prove equivariance of j it remains to observe that for every $g \in Mob_n$, $S \in Sph(n)$ and $v_S = j(S)$,

$$\iota \sigma_{gS} \iota^{-1} = \iota g R_S g^{-1} \iota^{-1} = g \iota R_S \iota^{-1} g^{-1} = g \tau_{v_S} g^{-1} = \tau_{g v_S}.$$

Hence, j sends the sphere gS to $g v_S$. □

Remark 51. Iversen uses the description of spheres S in \mathbb{R}^n by the quadratic equations

$$b \langle x, x \rangle - 2 \langle x, f \rangle + a = 0,$$

Assuming the normalization $|f|^2 - ab = 1$, the inversive coordinates then become (for $r < \infty$)

$$(b, f, a).$$

The natural invariant, the negative of the cosine of the angle between the spheres, is given by

$$\langle f_1, f_2 \rangle - \frac{1}{2}(a_1 b_2 + b_1 a_2),$$

see [Iv92, sect. I.8], i.e. the Lorentzian inner product for vectors in $\{Q = -1\}$. In the case when S is a cooriented hyperplane, $b = 0$, f is the unit vector normal to S and $a = \hat{r}$ is the coradius. This gives an alternative proof of the lemma.

Fixing one sheet of the two-sheeted hyperboloid $\{Q = -1\}$ as a model of \mathbb{H}^{n+1} , the original sphere S corresponds to the boundary at infinity of the intersection with $\{Q = -1\}$ of the plane Q -orthogonal to v_S . Then under the isomorphism j_* induced by j , $\text{Isom}(\mathbb{H}^{n+1}) \cong O_Q^+(\mathbb{R})$, where O_Q^+ is the ‘‘orthochronous’’ subgroup of O_Q which preserves the two sheets of $Q = -1$ (rather than allowing them to interchange).

The pair (\mathbb{R}^{n+2}, Q) defines a hyperbolic quadratic space. Write $(\mathbb{R}^{n+2})^*$ for the dual vector space, and Q^* for the induced dual form. The key observation elucidating the role of isotropic vectors in the study of the arithmetic of sphere packings is the following.

Lemma 52. *The ‘‘bend’’ covector $\mathbf{b} = (0, \dots, 0, -2) \in (\mathbb{R}^{n+2})^*$ is isotropic,*

$$(53) \quad Q^*(\mathbf{b}) = 0$$

and captures the bend of a sphere S with inversive coordinates $v_S = (\frac{1}{r}, \frac{z}{r}, \frac{1}{\hat{r}})^t$. That is,

$$(54) \quad \mathbf{b}(v_S) = \frac{1}{r}.$$

Similarly, the co-bend covector $\hat{\mathbf{b}} = (-2, 0, \dots, 0)$ is also isotropic, and has $\hat{\mathbf{b}}(v_S) = \frac{1}{\hat{r}}$. In the dual inner product $\langle \cdot, \cdot \rangle_*$ defined by Q^* , we have

$$(55) \quad \langle \mathbf{b}, \hat{\mathbf{b}} \rangle_* = -2.$$

Proof. Direct and elementary computation. □

We can also identify the hyperbolic space \mathbb{H}^{n+1} with a component of the two-sheeted hyperboloid $\{\mathcal{F} = -1\}$, where (V, \mathcal{F}) is another real hyperbolic quadratic space of the same dimension $n + 2$. A convenient way to choose coordinates here is as follows. Let $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ denote the

bilinear form on V corresponding to \mathcal{F} , and let \mathcal{F}^* be the dual form on the dual space V^* .

Lemma 56. *The dual space V^* admits an orthogonal splitting*

$$(57) \quad V^* = V_1^* \oplus V_2^*$$

so that $\dim V_1^* = 2$ with

$$(58) \quad \mathcal{F}^* \Big|_{V_1^*} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix},$$

and \mathcal{F}^* restricts to a definite form on the second factor.

Proof. Take two linearly independent light-like covectors $\alpha_0, \alpha_{n+1} \in V^*$ (i.e. $\mathcal{F}^*(\alpha_0) = \mathcal{F}^*(\alpha_{n+1}) = 0$), and rescale α_{n+1} to ensure that $\langle \alpha_0, \alpha_{n+1} \rangle_{\mathcal{F}^*} = -2$, as in (55). Let V_1^* be the span of these vectors. Defining V_2^* to be the orthogonal complement spanned by an arbitrary orthonormal system $\alpha_1, \dots, \alpha_n$ gives the required splitting. \square

We next transition to the number-theoretic discussion.

Addendum 59. *Suppose that \mathcal{F} is a hyperbolic quadratic form on an $(n+2)$ -dimensional real vector space V and there is a basis of V with respect to which the form is a hyperbolic over a (totally real) field k . Then:*

1. *There is field extension $k' \supset k$ of degree at most two with ring of integers \mathfrak{o}' , and a choice of covectors $\alpha_0, \dots, \alpha_{n+1}$ so that the splitting (57) is realized with V_1^* spanned by α_0, α_{n+1} and $V_2^* = \langle \alpha_1, \dots, \alpha_n \rangle$, where α_0 can be chosen to be defined over \mathfrak{o}' .*

2. *If $k = \mathbb{Q}$ and \mathcal{F} is isotropic over \mathbb{Q} , then we can take $k' = k = \mathbb{Q}$ and $\mathfrak{o}' = \mathbb{Z}$. Moreover, V contains a rational unit vector u (of \mathcal{F}).*

Proof. 1. For a general field k , we can find an isotropic covector α_0 over a suitable quadratic extension k' .¹⁰ We may clear denominators to ensure that α_0 is defined over \mathfrak{o}' . Then we proceed as in the proof of Lemma 56, finding a second isotropic vector α_{n+1} (again over k') and rescaling to ensure (58). Lastly, choose an orthonormal basis $\alpha_1, \dots, \alpha_n$ for the orthogonal complement V_2^* of V_1^* .

2. Since \mathcal{F} is isotropic over \mathbb{Q} , so is the dual form \mathcal{F}^* . We then let α_0, α_{n+1} denote linearly independent rational isotropic vectors of \mathcal{F}^* . As before, by rescaling we may assume that α_0 is an integer vector and

¹⁰As an aside, recall Godement's compactness criterion, that $O_{\mathcal{F}}(\mathfrak{o})$ is non-uniform if and only if \mathcal{F} is isotropic over k , see [BHC62]; and furthermore, if this is the case, then $k = \mathbb{Q}$. Indeed, if \mathcal{F} represents 0 nontrivially, then so does every Galois conjugate \mathcal{F}^σ . But \mathcal{F}^σ is definite since \mathcal{F} is hyperbolic, so there can be no other Galois conjugates, and $k = \mathbb{Q}$.

(58) is satisfied. In order to find a rational unit vector, we similarly find two rational isotropic vectors $v, w \in V$ such that $\langle v, w \rangle_{\mathcal{F}} = 1/2$. Then the vector $u := v + w$ has $\langle u, u \rangle_{\mathcal{F}} = 1$, as desired. \square

3.2. Constructing (Super)Integral Kleinian Bugs/Packings.

Proof of Theorem 12. We first prove part (ii). Let Γ be a non-uniform \mathbb{Q} -arithmetic group of simplest type, commensurable to $O_{\mathcal{F}}(\mathbb{Z})$, where \mathcal{F} is a hyperbolic quadratic form defined over \mathbb{Q} (with respect to some basis for V). Let $R_S \in \Gamma$ be a reflection with respect to a rational vector $u = w_S$:

$$R_S : x \mapsto I - 2 \frac{\langle x, w_S \rangle}{\langle w_S, w_S \rangle} w_S$$

Hence, $a = \langle w_S, w_S \rangle$ is rational. Dividing \mathcal{F} by a we obtain a new rational hyperbolic form $a^{-1}\mathcal{F}$, with respect to which w_S is a unit vector. Then we choose coordinates $\alpha_0, \dots, \alpha_{n+1}$ for V^* as in Addendum 59, adapted to the form $a^{-1}\mathcal{F}$.

Since Γ is commensurable to $O_{\mathcal{F}}(\mathbb{Z})$, all the vectors in the Γ -orbit of the vector u have uniformly bounded denominators, hence, their bends $\alpha_0(\Gamma u)$ also have uniformly bounded denominators. Under the correspondence, $\gamma(S) \mapsto \gamma(u)$ defined in §3.1 (with respect to the form $a^{-1}\mathcal{F}$), the radius of the sphere $\gamma(S)$ equals $\alpha_0(u)$. Hence, by a suitable rescaling in the Euclidean space \mathbb{R}^n (equivalently, rescaling the choice of α_0 to clear the denominators), all the bends can be made integral, as claimed.

The proof of part (i) is now clear. By rescaling the form \mathcal{F} , we get a new form rational hyperbolic form $a^{-1}\mathcal{F}$ such that the reflection $R_S \in \Gamma$ is given by a unit (with respect to $a^{-1}\mathcal{F}$) vector w_S . The group $\tilde{\Gamma}$, of course, preserves the new form. Since, by the assumption, the action of the symmetry group $\Gamma_S < \tilde{\Gamma}$ on the packing \mathcal{P} is transitive, so is the action of $\tilde{\Gamma}$ on the superpacking $\tilde{\mathcal{P}}$. Thus, by Part (ii), after a suitable rescaling, the superpacking $\tilde{\mathcal{P}}$ becomes integral. \square

We now prepare for the proof of Theorem 15. Before getting to questions of (super)integrality, we construct the necessary packings as follows.

Proposition 60. *Let $O_{\mathcal{F}}(\mathfrak{o}) < \text{Isom}(\mathbb{H}^{n+1})$ be a k -arithmetic lattice of simplest type. Then there is a sequence of conformally inequivalent Kleinian packings \mathcal{P}_j such that $O_{\mathcal{F}}(\mathfrak{o})$ is commensurable to a supergroup $\tilde{\Gamma}_j$ of \mathcal{P}_j which acts transitively on \mathcal{P}_j .*

Proof. While this result is stated in arithmetic terms, most of the proof is non-arithmetic.

Geometric setup. Suppose that $\Gamma < G = \text{Isom}(\mathbb{H}^{n+1})$ is a torsion-free lattice such that the hyperbolic manifold $M = \mathbb{H}^{n+1}/\Gamma$ contains a properly embedded complete connected nonseparating totally geodesic hypersurface of finite n -dimensional volume N . The hypersurface N lifts to a hyperplane H in \mathbb{H}^{n+1} . We will assume that the reflection R_H in H normalizes the group Γ and let $\tilde{\Gamma}$ denote the subgroup of G generated by R_H and Γ . The group $\tilde{\Gamma}$ contains Γ as an index 2 subgroup; hence, $\tilde{\Gamma}$ is again a lattice. Since R_H normalizes Γ , it descends to an isometric reflection $\sigma : M \rightarrow M$ fixing N pointwise.

Remark 61. Abundance of examples of this type (comping from arithmetic groups) was first established by John Millson in [Mi76]. More precisely, he proved that every k -arithmetic lattice $O_{\mathcal{F}}(\mathfrak{o})$ in $O(n+1, 1)$ of simplest type (where \mathfrak{o} is the ring of integers of the field k and \mathcal{F} is a hyperbolic quadratic form over k) is commensurable to a group $\tilde{\Gamma}$ as above.

Our goal is define a sequence of Kleinian packings \mathcal{P}_j with the supergroups $\tilde{\Gamma}_j$ commensurable to $\tilde{\Gamma}$ and acting transitively on \mathcal{P}_j .

The hypersurface N does not separate M and, hence, defines a non-trivial element ξ of $H^1(M)$, which is Poincaré-dual to the locally-finite fundamental class of the hypersurface N . Since σ fixes N pointwise, $\sigma^*(\xi) = -\xi$. The class ξ defines a homomorphism

$$\phi : \Gamma \rightarrow H_1(M) \rightarrow \mathbb{Z}.$$

Let $p : \hat{M} \rightarrow M$ denote the infinite cyclic covering corresponding to the kernel $\text{Ker}(\phi)$ of ϕ . Since $\sigma^*(\xi) = -\xi$, $\text{Ker}(\phi)$ is R_H -invariant; hence σ lifts to a reflection $\tau : \hat{M} \rightarrow \hat{M}$ fixing pointwise one of the components N_0 of the preimage of N in \hat{M} . We let D denote a component of $p^{-1}(M - N)$ whose boundary contains N_0 . Then \bar{D} is a fundamental domain of the action of the deck-transformation group $\mathbb{Z} = \langle \theta \rangle$ of the regular covering $p : \hat{M} \rightarrow M$. Furthermore, for each $j \in \mathbb{N}$,

$$D_j := \bigcup_{-j \leq i \leq j} \theta^i(\bar{D}) = \bigcup_{0 \leq i \leq j} \theta^i(\bar{D}) \cup \tau \left(\bigcup_{0 \leq i \leq j} \theta^i(\bar{D}) \right)$$

is a fundamental domain for the index $2j$ subgroup in \mathbb{Z} . Each domain D_j is τ -invariant, has finite volume ($2j$ times the volume of M) and two boundary component, both totally geodesic in \hat{M} and isometric to N via the restriction of the covering map p .

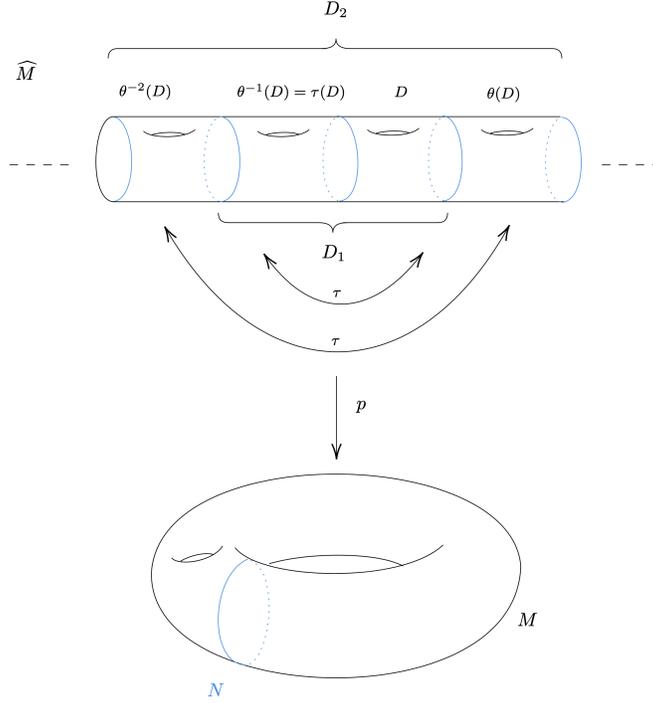


FIGURE 7. Infinite cyclic covering¹¹

Remark 62. For future reference, we record the following obvious properties of the domains D_j :

1. $cl(D_j) \subset D_{j+1}$ for each j .
2. $\bigcup_{j \geq 1} D_j = \hat{M}$.

The fundamental group Γ_j of D_j embeds in $\pi_1(\hat{M})$ (since D_j has totally-geodesic boundary) and, hence, $\pi_1(M)$. Since $\pi_1(M)$ is isomorphic to the lattice Γ , the group Γ_j will be identified with a subgroup (again denoted Γ_j) of Γ . Then the preimage of D_j in \mathbb{H}^{n+1} under the covering map $q : \mathbb{H}^{n+1} \rightarrow \hat{M}$ contains a (unique) Γ_j -invariant component $P_j = \tilde{D}_j$. Since ∂N is totally-geodesic, P_j is a convex domain with totally-geodesic boundary. By the construction, each boundary component of P_j is fixed by a reflection in $\tilde{\Gamma}$ conjugate to R_H . Thus, the collection of hyperplanes bounding P_j defines a sphere packing \mathcal{P}_j with the reflection group $\Gamma_{R,j}$, generated by reflections in the walls bounding P_j . The group Γ_j acts on P_j with quotient of finite volume (isometric to D_j); in particular, the limit set Λ_j of Γ_j equals $\partial_\infty P_j$ and

P_j is the closed convex hull of this limit set, $P_j = C_{\Gamma_j}$. We conclude that Γ_j is geometrically finite. Thus, \mathcal{P}_j is a Kleinian packing.

The supergroup of the packing generated by $\Gamma_{R,j}$ and Γ_j , however, acts on \mathcal{P}_j with *two orbits*: These two orbits correspond to the two connected components of ∂D_j . To fix this problem, we note that the reflection τ discussed above swaps these boundary components. Hence, we lift τ to a reflection $\tilde{\tau}$ in \mathbb{H}^{n+1} preserving P_j and let $\tilde{\Gamma}_j$ denote the subgroup of $\text{Isom}(\mathbb{H}^{n+1})$ generated by Γ_j and $\tilde{\tau}$. The group $\tilde{\Gamma}_j$, which is an index 2 extension of Γ_j , is still a symmetry group of the packing \mathcal{P}_j and acts transitively on the packing.

We are almost done with the proof of the proposition. It remains to show that the packings \mathcal{P}_j are conformally inequivalent for different j 's; equivalently, we claim that the convex subsets $P_j \subset \mathbb{H}^{n+1}$ are pairwise non-isometric.

To distinguish the sets P_j , we define the following invariant:

$$\rho_j = \sup_{\bar{x} \in D_j} d(\bar{x}, \partial D_j) = \sup_{x \in P_j} d(x, \partial P_j).$$

Here $d(x, \partial P_j)$ denotes the minimal distance from x to the points of ∂P_j , similar for $d(\bar{x}, \partial D_j)$. Since each P_j has finite volume, ρ_j is finite for each j . Then [Remark 62\(1\)](#) implies that the sequence ρ_j is strictly increasing with j and, hence P_j 's are pairwise non-isometric. \square

Remark 63. 1. In [§4](#) we will give a different argument for conformal inequivalence of packings (possibly after passing to a subsequence) using Hausdorff dimensions of the limit sets of the groups Γ_j .

2. Two Kleinian packings are conformally equivalent if and only if they are quasiconformally (more precisely, quasisymmetrically) equivalent, provided that $n \geq 2$, see [[Fr06](#), [BKM09](#)]. Hence, we obtain infinitely many quasiconformally inequivalent packings in every dimension $n \geq 2$.

Proof of [Theorem 15](#).

We are given $O_{\mathcal{F}}(\mathfrak{o})$, a k -arithmetic hyperbolic lattice of simplest type, with \mathfrak{o} the ring of integers of k . In [Proposition 60](#), we constructed a Kleinian packing $\mathcal{P} = \Gamma_S \cdot S_0$ with a symmetry group Γ_S which acts transitively on the spheres in the packing, and an arithmetic supersymmetry group $\tilde{\Gamma}$ (commensurable to $O_{\mathcal{F}}(\mathfrak{o})$). The group $\tilde{\Gamma}$ contains a reflection R through the sphere S_0 . As in the proof of [Theorem 12](#), the normal vector $w \in V$ to S_0 has all coordinates in k . By [Addendum 59](#), after a conformal change of coordinates on \mathbb{R}^n , the ‘‘bend’’ covector α_0 is defined over \mathfrak{o}' , the ring of integers of a quadratic extension k' of

¹¹Many thanks to Jules Flin for this drawing.

k . The supergroup $\tilde{\Gamma}$ of the packing is commensurable to $O_{\mathcal{F}}(\mathfrak{o})$, and so the orbit of w under the supergroup is defined entirely over k ; it is only when we measure the bends using the covector α_0 (after a suitable rescaling) that we obtain elements of \mathfrak{o}' . Regardless, the superpacking has all bends in \mathfrak{o}' , as claimed. \square

Proof of the Classification Theorem 9.

The same proof gives the forward direction of the Classification [Theorem 9](#); indeed, if \mathcal{F} is defined over \mathbb{Q} and isotropic, then by the same argument as above, the packing constructed in [Proposition 60](#) is superintegral by Part (i) of [Theorem 12](#). The backwards direction is a direct consequence of the Subarithmeticity [Theorem 3](#), which we turn to now. We will give two proofs, one that is basically identical to the proof of [[KN19](#), Thm 19], and another using somewhat different ideas. \square

First Proof of Theorem 3.

Let Q be the standard quadratic form of the signature $(n + 1, 1)$ as in equation (48), $\Gamma < G = O_Q^+(\mathbb{R})$ be a discrete, Zariski dense subgroup acting on the inversive coordinates v_{S_0} of a sphere S_0 , so that the bends, that is, first entries, in the orbit $\mathcal{O} = \Gamma v_{S_0}$ are all integers. The action of Γ is on the left (on column vectors) and involves all the entries of v_{S_0} ; we conjugate it to a right action just on (row vectors of) bends, as follows.

By the Zariski density of Γ , the orbit \mathcal{O} is also Zariski dense in the one-sheeted hyperboloid $Q = 1$, and hence contains $n + 2$ linearly independent vectors $\{v_1 = v_{S_0}, v_2, \dots, v_{n+2}\} \subset \mathcal{O}$. These vectors provide a coordinate system for the dual vector space $(\mathbb{R}^{n+2})^*$ so that if one applies these coordinates to the covector α_0 (that is, the first coordinate in \mathbb{R}^{n+2}), one gets integers. Moreover, for each $\gamma \in \Gamma$, the pairing of vectors and covectors $\langle \gamma v_i, \alpha_0 \rangle = \langle v_i, {}^* \gamma^{-1} \alpha_0 \rangle$ is also an integer. Hence, the vectors v_i evaluate to integers on the Γ -orbit of the covector α_0 in $(\mathbb{R}^{n+2})^*$. Hence Γ preserves the finite index sublattice L generated by the Γ -orbit of α_0 in $(\mathbb{Z}^{n+2})^*$; here $(\mathbb{Z}^{n+2})^*$ consists of covectors with integer coordinates with respect to the coordinate system given by v_1, \dots, v_{n+2} . Thus $\Gamma < G^L$ is a subgroup of $G^L = \{g \in O_Q^+(\mathbb{R}) : gL = L\}$. The latter group is easily seen to be (and is sometimes taken to be the very definition of) a congruence subgroup. Note also that the covector α_0 is rational with respect to the integral structure given by L and isotropic, see [Lemma 52](#). It follows that G^L , regarded as a lattice acting on the dual vector space V^* , is non-uniform, i.e. contains a unipotent element. Then G^L itself, regarded as a subgroup of G , contains a unipotent element, and, hence, is non-uniform. To conclude:

Γ is a subgroup of a non-uniform \mathbb{Q} -arithmetic hyperbolic group of simplest type, as claimed. \square

Second Proof of Theorem 3.

Let $\Gamma < O_{\mathbb{Q}}^+(\mathbb{R})$ be a discrete, Zariski dense subgroup acting on the inversive coordinates v_{S_0} of a sphere S_0 , so that the bends, that is, first entries, in the orbit $\mathcal{O} = \Gamma v_{S_0}$ are all integers. The first coordinate on \mathbb{R}^{n+2} is a (nonzero) linear functional α on the real vector space $V = \mathbb{R}^{n+2}$. The key is the following general lemma:

Lemma 64. *Let V be a finite-dimensional real vector space, $\Gamma < GL(V)$ an irreducible subgroup, i.e. a subgroup which has no proper invariant subspaces. Let $v \in V, \alpha \in V^*$ be nonzero vectors with the property that $\alpha(gv) \in \mathbb{Z}$ for all $g \in \Gamma$. Define the \mathbb{Z} -submodule L in V generated by the orbit $\Gamma \cdot v$. Then L is a free \mathbb{Z} -module of rank equal to the dimension of V .*

Proof. The group Γ obviously acts on L by automorphisms and α still takes only integer values on L . The irreducibility of Γ implies that L spans V as a real vector space. It remains to prove that L is a discrete subgroup of V regarded as an abelian Lie group. Let \bar{L} denote the closure of L in the classical topology on V and let $W := \bar{L}_0$ be the identity component of this Lie subgroup of V ; this component is a (real) linear subspace in V . The group Γ preserves this subspace. In view of irreducibility of Γ , the subspace W is either $\{0\}$ or the entire V . However, $\alpha \neq 0$ still takes only integer values on W , hence, $W \neq V$ and we conclude that $W = \{0\}$, i.e. L is a discrete subgroup of V . \square

We apply this lemma in our setting. The submodule $L \subset V = \mathbb{R}^{n+2}$ defines an integral structure on V . We claim that the quadratic form Q is rational with respect to this integral structure. Since $G_{\mathbb{Z}}$, the set of integer points in $G = O_{\mathbb{Q}}$, contains Γ , it is Zariski dense in G , hence, is an (arithmetic) lattice. It follows from Exercise 4 in [WM15, Sect. 5A] that G is defined over \mathbb{Q} . We claim that the form Q is also defined over \mathbb{Q} . The proof is the same as the one of Exercise 4 in [WM15, Sect. 5A]: Consider the vector space U of all quadratic forms on V . In view of Zariski density of Γ in G , there is a unique Γ -invariant line in $U \otimes \mathbb{C}$. Since Γ consists of integer matrices (with respect to the integer structure on V defined by L), for every Galois automorphism σ of \mathbb{C} , for every $q \in U$, we have

$$\gamma^*(q)^\sigma = \gamma^*(q^\sigma), \gamma \in \Gamma.$$

Therefore, since Q is Γ -invariant, so are the forms $Q^\sigma, \sigma \in Gal(\mathbb{C})$. Thus, the forms Q^σ belong to the line $\mathbb{C}Q$, i.e. for every $\sigma \in Gal(\mathbb{Q})$

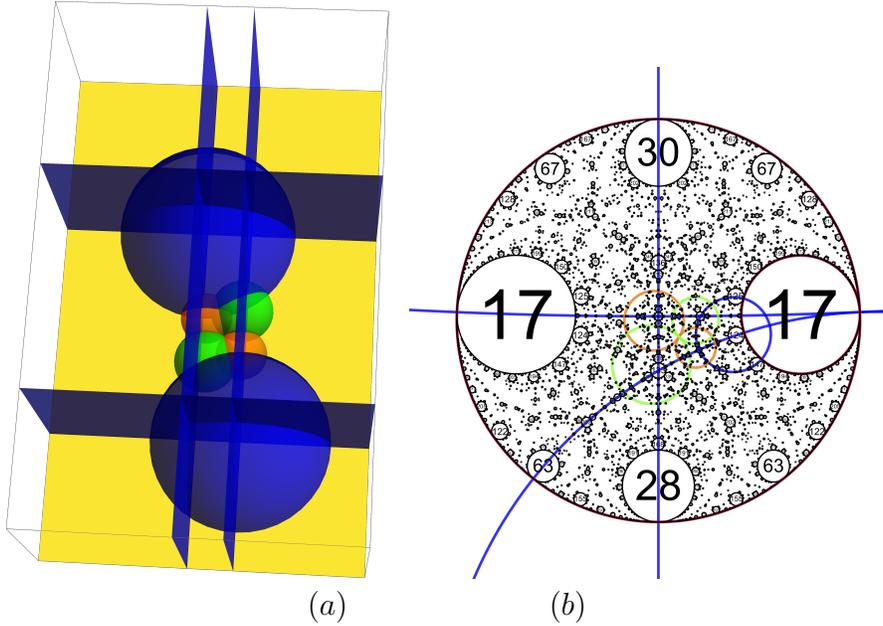


FIGURE 8. (a) A fundamental domain for the extended Bianchi group $\widehat{Bi}(23)$, and (b) a superintegral Kleinian packing attached to it.

there exists $z \in \mathbb{C}^\times$ such that

$$Q^\sigma = zQ.$$

Lastly, $Q(v) = 1$ and $v \in L$, hence,

$$z = zQ(v) = Q^\sigma(v) = Q^\sigma(v^\sigma) = (Q(v))^\sigma = 1.$$

It follows that Q itself is a rational form with respect to the rational structure on V defined by the lattice L .

Lastly, the argument that Γ is contained in a nonuniform lattice is the same as in the first proof: The covector α_0 is isotropic and rational with respect to the integral structure defined by the lattice L . \square

3.3. Caveats and Examples.

We collect here some examples that illustrate various caveats given in the introduction to the main theorems. We begin with an explicit example of a superintegral Kleinian packing which is not crystallographic.

Example 65. The extended Bianchi group $\widehat{Bi}(23)$ (see footnote 7) is not reflective (see [BM13]). Indeed, applying Vinberg's algorithm [Vin72] to the quadratic form $f = -2xy + 2z^2 + 2zw + \frac{23+1}{2}w^2$ shows that the subgroup of $O_f(\mathbb{Z})$ generated by all reflections has infinite

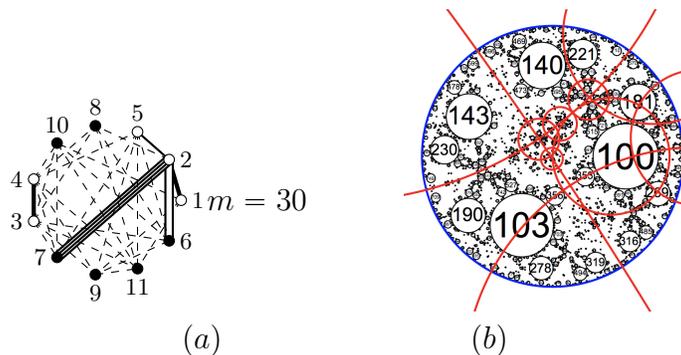


FIGURE 9. (a) The Coxeter diagram for $\widehat{Bi}(30)$, and (b) a superintegral packing with all spheres disjoint.

index in $O_f(\mathbb{Z})$. One can give a fundamental domain in \mathbb{H}^3 for $O_f(\mathbb{Z})$ as shown in Figure 8(a); here the blue walls act by reflections, and there is a pair of commuting unipotent elements, one identifying the green walls, and another identifying the orange ones. By Part 2(ii) of the Structure Theorem 22, we can construct a superintegral packing from this fundamental polyhedron; see Figure 8(b). So this is a superintegral packing which is Kleinian but not crystallographic.¹²

As mentioned above Remark 10, a superintegral Kleinian packing can have all spheres disjoint (as observed already in [KN19]); but its supergroup must still be non-uniform and \mathbb{Q} -arithmetic. One such is the following.

Example 66. Consider the extended Bianchi group $\tilde{\Gamma} = \widehat{Bi}(30)$. Equivalently, $\tilde{\Gamma} \cong O_{\mathcal{F}}(\mathbb{Z})$ is the integer orthogonal group preserving the form $\mathcal{F} = -xy + z^2 + 30w^2$. This group is reflective (see [BM13]), and applying Vinberg’s algorithm [Vin72] gives the Coxeter diagram¹³ shown in Figure 9. The node marked “8” is totally isolated from the others, being either orthogonal or some given distance apart from the other generating reflective walls. (So are nodes “9”, “10”, and “11” but we choose to use “8”.) Dropping this wall from the generators and letting the remaining generators act on it by reflections (as in the Structure

¹²Note that this particular packing appeared previously in work of Stange [St18] and Martin [Ma19] but was not recognized there as being dense (and hence wasn’t considered a “packing” by our definition), due to the significant distances between disjoint circles. The general theory given here makes this density apparent.

¹³The diagram has nodes for each facet, and facets that are orthogonal are not connected; the dihedral angle $\pi/3$ is denoted by a single line, angle $\pi/4$ is a double line, and angle $\pi/\infty = 0$ is a thick solid line. Nodes of separated facets are connected with a dotted line.

[Theorem 22](#)) gives the superintegral crystallographic packing shown in [Figure 9\(b\)](#).

As observed already in [\[KN19\]](#), if a packing cannot be realized as the orbit of a single sphere (that is, there is no symmetry group for which the action on the spheres in the packing is transitive), then the resulting packing (or bug) need *not* be superintegral (or even integral); cf. [Remark 11](#). Here is an explicit example of a non-integral bug with non-uniform, \mathbb{Q} -arithmetic supergroup.

Example 67. Let $\tilde{\Gamma}$ be the extended Bianchi group $\tilde{\Gamma} = \widehat{Bi}(6)$. Equivalently, $\tilde{\Gamma} \cong O_{\mathcal{F}}(\mathbb{Z})$ is isomorphic to the integer orthogonal group preserving the form $\mathcal{F} = -xy + z^2 + 6w^2$. That it is reflective (generated by reflections) is essentially due to Bianchi [\[Bia92\]](#); see also [\[BM13\]](#). Applying Vinberg's algorithm to this group produces the following normal vectors

$$(0, 0, -1, 0)^t, (1, 0, 1, 0)^t, (0, 0, 0, -1)^t, \\ (6, 0, 0, 1)^t, (-1, 1, 0, 0)^t, (2, 2, 0, 1)^t.$$

Making a (choice of) change of variables from \mathcal{F} to the universal form Q in [\(48\)](#), these correspond to spheres with the following (realization of) inversive coordinates:

$$v_1 = (0, -1, 0, 0)^t, v_2 = (0, 1, 0, 1)^t, v_3 = (0, 0, -1, 0)^t, \\ (68) \quad v_4 = (0, 0, 1, \sqrt{6})^t, v_5 = (1, 0, 0, -1)^t, v_6 = (\sqrt{2}, 0, \sqrt{3}, \sqrt{2})^t.$$

See [Figure 10\(a\)](#) for the corresponding spheres (as circles in \mathbb{R}^2). Writing V for the 4×6 matrix whose columns are v_j , we can compute the 6×6 Gramian of all Q -inner products:

$$V^t Q V = \mathcal{G} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & -1 & 0 & -\sqrt{3} \\ 0 & 0 & -1 & 1 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & -\frac{1}{2} & 0 & -\sqrt{\frac{3}{2}} & 1 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\sqrt{3} & 0 & 0 & 1 \end{pmatrix}.$$

Equivalently, $\tilde{\Gamma}$ has the Coxeter diagram given in [Figure 10\(b\)](#). A different realization of v_j from [\(68\)](#) will of course have different inversive coordinates V but the Gramian and Coxeter diagram are invariants.

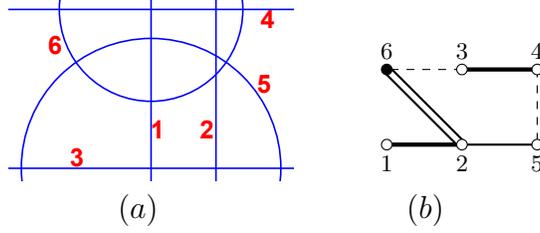


FIGURE 10. (a) The spheres with inversive coordinates (68) in Example 67. (b) Their Coxeter diagram.

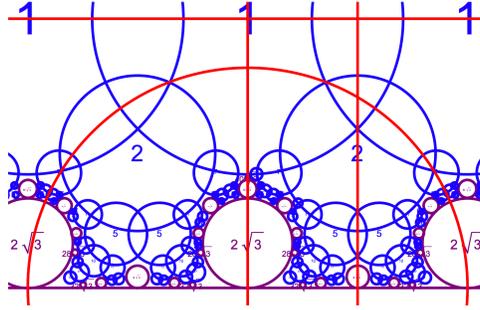


FIGURE 11. A non-integral bug; the bend of a circle is shown at its center.

Now we construct the bug. Write R_j for the Möbius action of reflection through sphere v_j , that is,

$$R_j = I_{4 \times 4} + 2v_j \cdot v_j^t \cdot Q.$$

We apply Part 2(i) of the Structure Theorem 22 with $S' = \{R_1, \dots, R_6\}$, $\langle S' \rangle = \tilde{\Gamma}$, and take $R = \{R_3, R_6\}$ so that $S = S' \setminus R$. The bug we obtain is then the orbit

$$(69) \quad \mathcal{B} = \langle S \rangle \cdot \{v_3, v_6\},$$

as shown in Figure 11. This particular realization of the bug is evidently non-integral, but we have not yet ruled out that there cannot be some other conformally equivalent realization of this bug which is integral. (Indeed, there exist realizations of the classical Apollonian packing that are non-integral.)

Lemma 70. *There does not exist a conformal realization of the bug in (69) that is integral.*

Proof. We mimic the first proof of Theorem 3 by looking at a right “bends” action. This time, we will do it using an over-determined system of equations, leading to an irrational linear relation among the

bends, which is realization-independent (and hence the bug cannot ever be integral).

To begin, notice that the spheres

$$v_3, v_6, R_2 \cdot v_6, R_1 \circ R_2 \cdot v_6, \text{ and } R_5 \circ R_1 \circ R_2 \cdot v_6$$

are all in the bug \mathcal{B} . Write W for the 4×5 matrix whose columns are the inversive coordinates of these spheres. The kernel of W ,

$$\mathcal{K} := \ker(W) := \{g \in \text{Mat}_{5 \times 5}(\mathbb{R}) : Wg = 0\},$$

is independent of the realization of the bug. Indeed, if $g \in \text{Isom}(\mathbb{H}^3)$ is any isometry and we move the whole bug by left-acting by g , then the inversive coordinates matrix W changes to gW , leaving the kernel \mathcal{K} invariant. Notice that the kernel contains, e.g.,

$$K = \begin{pmatrix} -\sqrt{3} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{K}.$$

So in any realization of this bug \mathcal{B} , if the spheres in W have bends, resp., a, b, c, d , and e , say, then

$$0 = (a, b, c, d, e) \cdot K \cdot (1, 0, 0, 0, 0)^t = -\sqrt{3}a + b - 3c + e.$$

Suppose this is the case with bends a, \dots, e all integers; then we must have $a = 0$. Note that this applies to not only one particular realization of these five spheres, but to *any* such; in particular, the entire orbit of these five spheres under the symmetry group $\Gamma = \langle S \rangle$ has the bend $a = 0$. But then Γ satisfies an extra polynomial equation, and is not Zariski dense, which is a contradiction. \square

Lastly, we prove [Proposition 14](#) by exhibiting the following.

Example 71. In [\[KN19\]](#), a procedure was given for construction a packing $\mathcal{P} = \mathcal{P}(\Pi)$ modeled on a (convexly-realizable combinatorial type of a) polyhedron Π . In the case of Π being the icosahedron, the resulting packing shown in [Figure 12](#). In can be shown that the entire superpacking can be made to have bends in $\mathfrak{o} = \mathbb{Z}[\phi]$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean. But Vinberg’s criterion applied to the “super-Gramian” (in the nomenclature of [\[KN19\]](#)) shows that the supergroup is non-arithmetic, because the quadratic form it defines (over $\mathbb{Q}(\phi)$) fails to become definite under the Galois conjugate embedding. For details of this computation, see <http://math.rutgers.edu/~alexm/maths/Icosahedron.nb>.

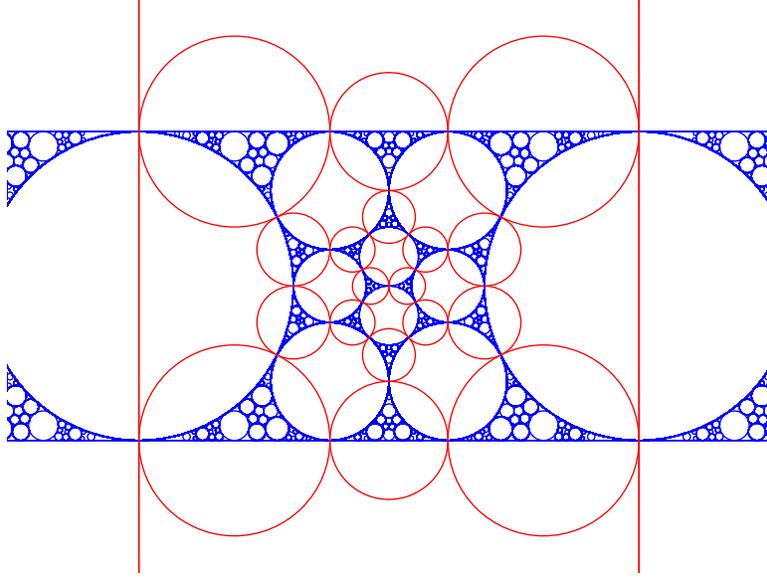


FIGURE 12. Icosahedral packing

4. HAUSDORFF DIMENSIONS OF LIMIT SETS OF PACKINGS

In [Proposition 60](#) and its application to (the forward direction of) the Classification [Theorem 9](#), we showed how to construct superintegral Kleinian packings from \mathbb{Q} -arithmetic non-uniform lattices. We now extend this construction to show that for infinitely many j 's the Hausdorff dimensions of the limit sets of the groups Γ_j 's constructed in the proof of [Proposition 60](#) are pairwise distinct and, moreover, the sequence of Hausdorff dimensions converges to the maximal Hausdorff dimension, n . This will follow from the following theorem where \dim stands for the Hausdorff dimension, which implies [Theorem 13](#). We continue with the notation introduced in the proof of [Proposition 60](#).

Theorem 72. $\lim_{j \rightarrow \infty} \dim(\Lambda(\Gamma_j)) = n$ and for all j , $\dim(\Lambda(\Gamma_j)) < n$.

Note first that the discrete groups Γ_j contain geometrically finite subgroups which are conjugates of $\pi_1(N)$. In particular,

$$\delta(\Gamma_j) \geq \dim(\Lambda(\pi_1(N))) = n - 1 \geq \frac{n}{2},$$

provided that $n \geq 2$.

We next discuss a relation between the Hausdorff dimension of the limit set, the critical exponent and the bottom of the spectrum of the Laplacian.

For a complete connected Riemannian manifold M let $\lambda(M)$ denote bottom of the L^2 -spectrum of the Laplacian of M . This number can

be computed via *Rayleigh quotients*:

$$\lambda = \inf \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}$$

where the infimum is taken over all smooth compactly supported functions (called “test” or “trial” functions) $u \in C_c^\infty(M)$ and $\Omega := \{u > 0\}$.

Theorem 73 (Semicontinuity of λ). *$\lambda(M)$ is upper semicontinuous with respect to the topology of smooth Gromov–Hausdorff convergence:*

$$M_i \rightarrow M \Rightarrow \liminf_{i \rightarrow \infty} \lambda(M_i) \leq \lambda(M).$$

Proof. This is clear using the Rayleigh quotient definition: Every test-function on M is the C^1 -limit of a sequence of test-functions on M_i 's. \square

Remark 74. 1. The same theorem (and proof) applies to Riemannian orbifolds.

2. The bottom of the spectrum is *not continuous* with respect to the topology of smooth Gromov–Hausdorff convergence. For instance, for $n \geq 3$ let M be a hyperbolic $n + 1$ -manifold of finite volume, let $M_i \rightarrow M$ be a profinite sequence of (finite) covers of M . Then the manifolds M_i converge to the hyperbolic n -space, $\lambda(M_i) = 0$, while $\lambda(\mathbb{H}^{n+1}) = n^2/4$.

Given a discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$, let $\delta(\Gamma)$ denote the critical exponent of Γ (see e.g. [Ni89]). A discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$ is called *nonelementary* if its limit set consists of more than two points. If Γ is geometrically finite and nonelementary, then $\delta(\Gamma)$ equals the Hausdorff dimension of the limit set of Γ , see [Pat76, Su84, Tu84, Ni89]. We will need the Elstrodt–Patterson–Sullivan formula (see e.g. [Ni89]), relating, for a discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$, the critical exponent $\delta = \delta(\Gamma)$ and the bottom of the spectrum $\lambda = \lambda(\mathbb{H}^{n+1}/\Gamma)$:

Theorem 75.

$$\lambda = \left(\frac{n}{2}\right)^2, \text{ if } \delta \leq \frac{n}{2},$$

$$\lambda = \delta(n - \delta), \text{ if } \delta \geq \frac{n}{2}.$$

Definition 76. *A sequence of closed subgroups Γ_i of a Lie group G is said to converge to a closed subgroup $\Gamma < G$ geometrically or in Chabauty topology if the following two conditions are met:*

1. *For every $\gamma \in \Gamma$ there exists a sequence $\gamma_i \in \Gamma_i$ which converges to γ .*
2. *If a sequence $\gamma_i \in \Gamma_i$ subconverges to $\gamma \in G$, then $\gamma \in \Gamma$.*

Suppose now that X is a complete connected Riemannian manifold, G is the isometry group of X . Then G (equipped with the compact-open topology) is a Lie group. Fix a base-point $x \in X$. Consider a sequence of subgroups $\Gamma_i < G$ and a subgroup $\Gamma < G$ and quotient manifolds/orbifolds $M_i = X/\Gamma_i, M = X/\Gamma$. Let \bar{x}_i, \bar{x} denote the projections of x to M_i, M respectively.

The following theorem was proven in [BP92] in the case when X is the hyperbolic space (which will suffice for us), but the same proof works for any complete connected Riemannian manifold.

Theorem 77 (See [BP92]). *A sequence of discrete subgroups $\Gamma_i < G$ converges geometrically to a discrete subgroup $\Gamma < G$ if and only if the sequence of pointed Riemannian manifolds/orbifolds (M_i, \bar{x}_i) converges to (M, \bar{x}) in the smooth Gromov–Hausdorff topology.*

Corollary 78 (Semicontinuity of λ and δ). *(a) Suppose that Γ_i is a sequence of discrete (nonelementary) subgroups of $\text{Isom}(\mathbb{H}^{n+1})$ converging to a discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$. Then*

$$\liminf_{i \rightarrow \infty} \lambda(\mathbb{H}^{n+1}/\Gamma_i) \leq \lambda(\mathbb{H}^{n+1}/\Gamma).$$

(b) Suppose, in addition, that $\inf\{\delta(\Gamma), \delta(\Gamma_i), i \in \mathbb{N}\} \geq n/2$. Then

$$(79) \quad \liminf_{i \rightarrow \infty} \delta(\Gamma_i) \geq \delta(\Gamma).$$

In particular, if $\delta(\Gamma_i) \leq \delta(\Gamma)$ for all i (e.g. if $\Gamma_i < \Gamma$) then

$$\lim_{i \rightarrow \infty} \delta(\Gamma_i) = \delta(\Gamma).$$

Proof. The first inequality is a direct corollary of the two theorems above. The second inequality follows from the relation of δ and λ :

$$\lambda = \delta(n - \delta) \text{ if } \delta \geq \frac{n}{2},$$

see [Theorem 75](#). □

Lastly, we need the following theorem due to T. Roblin [[Ro05](#)] (see also R. Brooks [[Br85](#)]):

Theorem 80. *Let $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$ be a geometrically finite¹⁴ subgroup and $\hat{\Gamma} < \Gamma$ a normal subgroup with amenable quotient $\Gamma/\hat{\Gamma}$. Then $\delta(\hat{\Gamma}) = \delta(\Gamma)$.*

¹⁴More generally, a group of divergence type, i.e. a group whose Poincaré series diverges at the critical exponent.

Thus, if Γ is a lattice and $\hat{\Gamma} < \Gamma$ is a normal subgroup with cyclic quotient, then $\delta(\hat{\Gamma}) = n$.

We can now finish the proof of Theorem 72. Let Γ_j be the discrete groups as in the theorem.

Lemma 81. *The sequence Γ_j geometrically converges to $\hat{\Gamma}$.*

Proof. This follows from the fact that the sequence of domains D_j exhausts \hat{M} (see Remark 62), which ensures Gromov–Hausdorff convergence of the corresponding hyperbolic manifolds, hence, geometric convergence of discrete subgroups. \square

Thus, we conclude that

$$\lim_{j \rightarrow \infty} \dim(\Lambda(\Gamma_j)) = \dim(\Lambda(\Gamma)) = n.$$

On the other hand, $\delta(\Gamma_j) < n$ since each Γ_j is geometrically finite and its limit set is a proper subset of \mathbb{S}^n , (Sullivan [Su84] and Tukia [Tu84], independently). This concludes the proof of Theorem 72 and, hence, of Theorem 13. \square

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