

ARITHMETIC ASPECTS OF SELF-SIMILAR GROUPS

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Let Γ be a group. A *virtual endomorphism* of Γ is a homomorphism $\varphi : \Lambda \rightarrow \Gamma$, where Λ is a finite index subgroup in Γ . An *invariant normal subgroup* of φ is a normal subgroup $N < \Gamma$ contained in Λ , so that $\varphi(N) < N$. We will be interested in the question

Question 1. When does φ have a nontrivial *invariant normal subgroup*?

We will only consider the case of lattices Γ in algebraic (linear) groups G . Therefore, without loss of generality, the subgroups Λ can be taken torsion-free. In particular, existence of a nontrivial invariant normal subgroup is equivalent to the existence of an infinite invariant normal subgroups. Our main result is

Theorem 2. *Let Γ be an irreducible lattice with trivial center in a semisimple algebraic group G . Then the following are equivalent:*

- a. Γ is virtually isomorphic to an arithmetic lattice in G , i.e., contains a finite index subgroup isomorphic to such arithmetic lattice.
- b. Γ admits a virtual endomorphism $\varphi : \Lambda \rightarrow \Gamma$ without nontrivial invariant normal subgroups.

We note that conjugacy classes of faithful self-similar actions of a group Γ on a rooted tree (of finite valency) are in 1-1 correspondence with conjugacy classes of virtual endomorphisms of Γ which contain no nontrivial invariant normal subgroups [5, 6]. Therefore,

Corollary 3. *Let Γ be an irreducible lattice with trivial center in a semisimple algebraic group G . Then the following are equivalent:*

- a. Γ is virtually isomorphic to an arithmetic lattice in G .
- b. Γ admits an irreducible faithful self-similar action on a regular rooted tree (of finite valency).

The only nontrivial ingredient in the proof of Theorem 2 is the following theorem of Margulis [4]:

Theorem 4. *A lattice Γ in an algebraic semisimple Lie group G is arithmetic if and only if its commensurator $\text{Comm}(\Gamma)$ in G is discrete.*

We start our proof of Theorem 2 with some generalities about arithmetic groups.

Let F be an (algebraic) number field. Let F_∞ denote the completion of F with respect to the valuation given by the embedding $F \rightarrow \mathbb{C}$. Let \mathcal{O} be the ring of integers

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of F . Given a prime ideal p in \mathcal{O} we let F_p denote the completion of F with respect to the nonarchimedean valuation on F determined by p .

Let \underline{G} be a semisimple algebraic group defined over \mathcal{O} . Then the group $G = \underline{G}(F_\infty)$ is a real or complex semisimple Lie group. We will regard G as the isometry group of a symmetric space X (the action has finite kernel coming from the finite center of G). Similarly, the group $G_p := \underline{G}(F_p)$ acts as a group of isometries of a Euclidean building X_p . The group of integer points $\underline{G}(\mathcal{O}_p)$ is the stabilizer in G_p of a special vertex $o_p \in X$.

Consider an irreducible arithmetic group $\Gamma \subset \underline{G}(\mathcal{O})$ in G . Recall [4] that the commensurator $Comm(\Gamma)$ of Γ is $\underline{G}(F)$, the group of F -points in \underline{G} .

Given an element $\alpha \in Comm(\Gamma)$ we let $\varphi = \varphi_\alpha$ denote the automorphism of G induced by conjugation

$$\varphi(g) = \alpha g \alpha^{-1}.$$

Thus, α induces *virtual endomorphisms* of Γ defined by taking sufficiently deep finite index subgroups $\Lambda \subset \Gamma$, so that $\varphi(\Lambda) \subset \Gamma$, and then restricting φ to Λ . (For instance, sufficiently deep congruence subgroups will work.)

Since $\underline{G}(F)$ is dense in G_p , we can choose a prime ideal p and $\alpha \in Comm(\Gamma)$ so that the corresponding isometry $\alpha : X_p \rightarrow X_p$ has unbounded orbits (i.e. is hyperbolic).

Remark 5. Moreover, if $\alpha \in \underline{G}(F)$ induces an elliptic automorphism of every Euclidean building X_p , then it belongs to a finite extension of Γ . However, we do not need this fact.

Lemma 6. *Given α as above, for every finite index subgroup $\Gamma' \subset \Gamma$, we have $\varphi(\Gamma') \neq \Gamma'$.*

Proof. If $\alpha \Gamma' \alpha^{-1} = \Gamma'$, then Γ' and α generate a discrete subgroup $\hat{\Gamma}' \subset G$, which (since Γ' is a lattice) necessarily has Γ as a finite index subgroup. Therefore, the group $\hat{\Gamma}$ generated by Γ and $\hat{\Gamma}'$ is also a finite extension of Γ . Since Γ fixes the point $o_p \in X_p$, it follows that orbit $\hat{\Gamma} \cdot o_p \subset X_p$ is bounded. This contradicts the fact that $\gamma : X_p \rightarrow X_p$ is hyperbolic. \square

Theorem 7. *Given Γ, Λ, α as above, if $N < \Lambda$ is a normal subgroup of Γ so that $\varphi(N) \subset N$, then N is finite.*

Proof. We let $M := X/N$. Then for every N as above, we obtain an isometric covering

$$q = \bar{\alpha} : M \rightarrow M$$

induced by the endomorphism φ of the fundamental group of M . Note that the group Γ acts on M isometrically, we use the notation $\bar{\gamma}$ for an isometry of M induced by $\gamma \in \Gamma$.

1. First, consider the case when G has rank ≥ 2 . Then, by a theorem of Margulis [4], every normal subgroup of Γ is either finite or has finite index in Γ . Assuming N has finite index in Γ , the manifold M has finite volume and, since q preserves the volume form, it has to be a diffeomorphism. Thus, $\varphi(N) = N$, which contradicts our choice of α .

2. We now consider the more interesting case when G has rank 1. We assume that N is an infinite group. Therefore, it is Zariski dense in G .

Consider the iterations q^k of the isometric endomorphism $q : M \rightarrow M$. Let M_{thick}, M_{thin} denote the thick and thin parts of M with respect to the Margulis constant of X . Pick a connected compact subset $C \subset M_{thick}$ whose fundamental group maps onto a Zariski dense subgroup H of N and which contains a fundamental domain for the action $\Gamma \curvearrowright M_{thick}$.

Then, by the Kazhdan-Margulis lemma, for every $k \geq 1$, $q^k(C)$ is never contained in M_{thin} . Therefore, for each k there exists $x_k \in C$ so that $q^k(x_k) \in M_{thick}$. Moreover, since q is isometric, there exists $\epsilon > 0$ so that for every $k \in \mathbb{N}$, the injectivity radius of M on $q^k(C)$ is bounded from below by ϵ .

Since the action $\Gamma \curvearrowright M_{thick}$ is cocompact (as X/Γ has finite volume), for every k there exists $\gamma_k \in \Gamma$ so that

$$\bar{\gamma}_k \circ q^k(x_k) \in C.$$

Therefore, the sequence of isometries $\bar{\beta}_k := \bar{\gamma}_k \circ q^k$ is precompact.

Lemma 8. *The set $\bar{I} := \{\bar{\beta}_k, k \in \mathbb{N}\}$ is finite.*

Proof. If not, then there will be arbitrarily large k, m so that $\bar{\beta}_k \neq \bar{\beta}_m$ and the restrictions $\bar{\beta}_k|_C, \bar{\beta}_m|_C$ are arbitrarily close in the sup-metric. In particular, for large k, m , they are homotopic and, hence, induce the same (up to conjugation in N) map $H \rightarrow N$ given by

$$h \mapsto \beta_k h \beta_k^{-1}, \quad h \mapsto \beta_m h \beta_m^{-1}.$$

Since H is Zariski dense in G , its centralizer in $G/Z(G)$ is trivial and, thus we have the equality of the cosets

$$\beta_k \cdot N \cdot Z(G) = \beta_m \cdot N \cdot Z(G).$$

Since the center of G acts trivially on X , we obtain that $\bar{\beta}_k = \bar{\beta}_m$. Contradiction. \square

We continue with the proof of Theorem 7. Let $I \subset \underline{G}(F)$ denote the finite set of representatives of lifts of the isometries $\bar{\beta}_k \in \bar{I}$. Hence, for each $k \in \mathbb{N}$,

$$\gamma_k \cdot \alpha^k \cdot N \subset I \cdot N,$$

and, thus,

$$\alpha^k \in \gamma_k^{-1} \cdot I \cdot N.$$

We now consider the action of the isometries in the above equation on the building X_p : the group Γ fixes a point $o \in X_p$, the images $I \cdot o$ form a finite set. Therefore, the set

$$\gamma_k^{-1} \cdot I \cdot N \cdot o = \gamma_k^{-1} \cdot I \cdot o$$

is bounded in X . However, by our assumption, the orbit $\alpha^k \cdot o, k \in \mathbb{N}$ is unbounded. Contradiction. \square

Corollary 9. *Suppose that Γ has trivial center. Then, if $N < \Lambda$ is a normal subgroup of Γ so that $\varphi(N) \subset N$, then N is trivial.*

Proof. Since Γ has trivial center, its only finite normal subgroups are trivial. \square

We now consider virtual endomorphisms of non-arithmetic lattices.

Proposition 10. *Let G be a rank 1 semisimple Lie group (with finitely many components), which is not locally isomorphic to $SL(2, \mathbb{R})$ and $\Gamma < G$ be a non-arithmetic lattice with trivial center. Then for every virtual endomorphism φ of Γ , there exists an infinite normal subgroup $N < \Gamma$ which is φ -invariant.*

Proof. 1. Suppose that $\varphi : \Lambda \rightarrow \Gamma$ is not injective. Let K denote the kernel of φ . Note that this subgroup is not necessarily normal in Γ . Our assumption that Γ has trivial center implies that Γ acts faithfully on the symmetric space X associated with G and, hence, the group K is necessarily infinite.

We first consider the case when Λ is normal in Γ . Let $\gamma_1, \dots, \gamma_n$ be the generators of Γ . Consider the conjugates

$$K_i := \gamma_i K \gamma_i^{-1} \subset \Gamma, \quad K_0 := K.$$

Then

$$K' := \bigcap_{i=0}^n K_i$$

is a normal subgroup in Γ . This subgroup is the kernel of the homomorphism

$$\begin{aligned} \Phi : \Lambda &\rightarrow \prod_{i=0}^n \Gamma, \\ \Phi &= (\varphi_0, \dots, \varphi_n), \end{aligned}$$

where

$$\varphi_i : \Lambda \rightarrow \Gamma$$

is given by

$$\varphi_i(g) = \varphi(\gamma_i g \gamma_i^{-1}), \quad i = 1, \dots, n; \quad \varphi_0 := \varphi.$$

We set let Γ_i denote the i -th factor of the product group $\prod_{i=0}^n \Gamma$.

If K is infinite, it contains a free nonabelian subgroup H . Thus, each group K_i contains a free nonabelian subgroup H_i , $i = 0, \dots, n$. Assume, for a moment, that K' is finite.

The group Λ/K' embeds in the product group $\prod_{i=0}^n \Gamma$. Therefore, the intersections

$$\Phi(\Lambda) \cap \Gamma_i$$

contain free nonabelian subgroups (isomorphic to H_i), $i = 0, \dots, n$. Hence, the group Λ/K' contains a direct product of free nonabelian subgroups. This is impossible since the group Λ/K' is isomorphic to a discrete subgroup of isometries of the negatively curved symmetric space X . Contradiction. Thus, K' is infinite and we obtain an infinite normal subgroup $N = K' < \Lambda$ of the group Γ , so that $\varphi(N) = 1 \subset N$.

We now consider the case when Λ is not necessarily normal in Γ . If $\varphi : \Lambda \rightarrow \Gamma$ is a virtual endomorphism with infinite kernel, we find a finite index subgroup $\Lambda' \subset \Lambda$ which is normal in Γ . Then the restriction $\varphi' = \varphi|_{\Lambda'}$ still has infinite kernel and we obtain a contradiction as above.

2. Suppose now that φ is injective. Then, by Mostow rigidity theorem, the homomorphism φ is induced by conjugation via some $\alpha \in \text{Comm}(\Gamma)$. Recall that, by the Margulis theorem, $\text{Comm}(\Gamma)$ is discrete. Therefore, since Γ , is a lattice,

$\hat{\Gamma} := \text{Comm}(\Gamma)$ is a finite extension of Γ . Therefore, Λ has finite index in $\hat{\Gamma}$ and, hence, contains a finite index subgroup $N < \Lambda$ which is normal in $\hat{\Gamma}$. In particular,

$$\alpha N \alpha^{-1} = N$$

and $N < \Gamma$ is a normal subgroup. Clearly, N is an infinite group as required by the proposition. \square

If G is locally isomorphic to $SL(2, \mathbb{R})$ then the abstract commensurator of a lattice $\Gamma < G$ is not isomorphic to the commensurator of Γ in G . However, Γ will contain a finite index torsion-free subgroup Λ which is isomorphic to an arithmetic subgroup of G . Therefore, Γ will have a virtual endomorphism φ without nontrivial invariant normal subgroups.

By combining the above results we obtain Theorem 2. \square

We observe that our results should, in principle, generalize to Gromov-hyperbolic groups which are not lattices. The problem, however, is that:

1. Among hyperbolic groups Γ , (if we ignore torsion) only Poincaré duality groups (fundamental groups of closed aspherical manifolds) are known to be *weakly cohopfian*, in the sense that if $\Lambda < \Gamma$ is a finite index subgroup and $\varphi : \Lambda \rightarrow \Gamma$ is an injective homomorphism, then the image of φ is a finite index subgroup of Γ . Hyperbolic groups which act geometrically on rank 2 hyperbolic buildings provide good candidates for weakly cohopfian groups [1]. On the other hand, there are no known examples of 1-ended hyperbolic groups which are not weakly cohopfian.

2. Among hyperbolic groups Γ which are Poincaré duality groups, the only known examples where the abstract commensurator $\text{Comm}(\Gamma)$ is a finite extension of Γ , are the non-arithmetic lattices. There are few more classes of hyperbolic groups with small abstract commensurators: Fundamental groups of compact hyperbolic n -manifolds with totally-geodesic boundary, $n \geq 3$, as well as some rigid examples constructed in [3]. (In all these examples, the whole quasi-isometry group of Γ is a finite extension of Γ .) Conjecturally, the fundamental groups of Gromov-Thurston manifolds [2] should also have small abstract commensurators.

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