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DEFORMATION SPACES OF FLAT CONFORMAL STRUCTURES

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1. Introduction

The main aim of this article is to present proofs of some results concerning local and global nature of deformation spaces of flat conformal structures (FCS) on closed manifolds which have dimension $n \geq 3$. Considerable part of these results was proved by author in 1985- 86 and announced in [8, Ch. 5].

The next definitions in principle follow [11], [6].

Let M be a closed smooth n-manifold ($n \geq 3$). By *marked flat conformal structure* on M we shall mean a pair (K, φ), where $\varphi: M \to M$ is a diffeomorphism, K is an FCS on M. Two marked FCS (K'', φ') and (K'', φ'') on M are called equivalent if there exists a conformal bijection $\psi: (M, K') \to (M, K'')$ such as $\psi \circ \varphi'$ is isotopic to φ''. The corresponding space of classes of marked FCS is called *deformation space of flat conformal structures* (*Teichmuller space*) on M. This space will be denoted by $\mathcal{T}(M)$.

Topology on $\mathcal{T}(M)$ is induced by C^1- topology on space of development maps (see [11], [6]). The map

$hol: \mathcal{T}(M) \to R(\pi_1(M)) =$ Hom$(\pi_1(M), SO(n+1, 1))/ SO(n+1, 1)$

is defined in the same papers; $hol(M, K, \varphi)$ is a holonomy representation of (K, φ) (which is unique up to conjugation). Flat conformal structure K on M is called *exceptional* if

(a) (M, K) is conformally equivalent to $\mathbb{H}^{n-1} \times \mathbb{R}/ \Gamma$, where Γ is a torsion-free lattice in Isom$(\mathbb{H}^{n-1} \times \mathbb{R})$ (i.e. K is a Seifert structure);

(b) Projection of Γ to $\mathrm{Isom}(\mathbb{H}^{n-1})$ is not orientation-preserving;

(c) The maximal normal cyclic subgroup $\langle t \rangle \subset \Gamma$ is contained in kernel of the holonomy homomorphism $\rho_0 : \Gamma = \pi_1(M) \to \mathrm{Mob}(\mathbb{S}^n)$;

(d) The representation ρ_0 admits a nontrivial deformation ρ_t such that $\rho_t(\Gamma)$ does not have in $\mathbb{S}^n$ an invariant euclidean $(n-2)$-sphere.

Marked FCS (K, φ) on M is called *exceptional* if K is an exceptional structure.

The following theorem is a complement to results of [11], [6], [5] :

THEOREM 1. *The map hol is a local homeomorphism anywhere except of exceptional points of* $\mathcal{T}(M)$. *Near any exceptional point the map hol is a 2-fold branched covering.*

Consider the natural projection π: $\mathcal{T}(M) \to C(M)$, where $C(M)$ is the set of all (unmarked) flat conformal structures. The space $C(M)$ (provided with factor-topology) is called *moduli* space of FCS on M . Let $C_c(M)$ denote the subset of those FCS which have nonsurjective development maps.

THEOREM 2. *The set* $C_c(M)$ *is closed in* $C(M)$.

Let F be a normal subgroup of $G = \pi_1(M)$. Denote by $C(M, F)$ the set of all FCS whose holonomy homomorphisms have one and the same kernel F . Let $C_f(M)$ be the set of all FCS on M whose holonomy groups are discrete, with trivial center, convex cocompact subgroups of $\mathrm{Mob}(\mathbb{S}^n) \simeq \mathrm{Isom}(\mathbb{H}^{n+1})$ [12]. Put $C_f(M, F) = C_f(M) \cap C(M, F)$.

THEOREM 3. *The space* $C_f(M, F)$ *is open in* $C(M, F)$.

Denote by $S(g, e)$ total space of $\mathbb{S}^1$-fiber bundle over a closed genus g surface, which has the Euler number $e \in \mathbb{Z}$. Let

$C_s(S(g, e))$ be the space of all Seifert flat conformal structures on $S(g, e)$.

THEOREM 4. *If* $M = S(g, e)$, *then* $C_f(M) \cup C_s(M) = C_c(M)$ *is an open subset of* $C(M)$. *If* $e \neq 0$, *then number of connected components of* $C(M)$ *is not less than* $\nu(g, e) = [(\;.g-1)/|e|]$.

REMARK. Chern-Simons functional $\Phi(K)$ [4] as well as the η-invariant [2], associated with conformal class of conformally-euclidean metrics, are locally constant on $C(M)$ [4], [2]; hence they may be used to distinguish connected components of $C(M)$. On those $\nu(g, e)$ components of $C(M)$, that are found in the Theorem 4, the functional $\Phi: C(M) \to \mathbb{R}/\mathbb{Z}$ identically vanishes. On another hand, all these components have different η-invariants.

Introduce the notations: universal covering of manifold M will be denoted by $p: \tilde{M} \to M$, it's deck-transformation group is $G \cong \pi_1(M)$. Development map of FCS K is denoted by d, corresponding holonomy representation is $d_*: G \to H$, where $H \subset \mathrm{Mob}(\mathbb{S}^n)$ is the holonomy group.

2. Proof of Theorem 2.

Suppose that there exists a sequence of FCS K_n and development maps $d_n: \tilde{M} \to \mathbb{S}^n$ of these structures such that $\lim d_k = d$ is development map of a structure $K \notin C_c(M)$, i.e. $d(\tilde{M}) = \mathbb{S}^n$. Let $x = d(y) \in \mathbb{S}^n$, $y \in \tilde{M}$. Since the index of map is constant under small C^1-perturbations, then there are a neighborhood $U(x)$ of the point x and a number $N(x) \in \mathbb{N}$ such that $U(x) \subset d_k(\tilde{M})$ for every $k \geq N(x)$.

By compactness of $\mathbb{S}^n$ we obtain that $d_k(\tilde{M}) = \mathbb{S}^n$ for any sufficiently large $k \in \mathbb{N}$. This contradiction shows that $C_c(M)$ is closed in $C(M)$. ∎

3. Proof of theorem 3.

Consider a compact flat conformal manifold (M, K) whose development map isn't surjective and the holonomy group $H = d_*(G)$ is discrete, convex cocompact group with trivial center. Then $d(\tilde{M}) = \Omega$ is an invariant component of the discontinuity set $R(H)$. Suppose that $d_m \to d$ is a sequence of development maps of (M_m, K_m) such that $Ker(d_{m*}) \supseteq Ker(d_*)$. Then we have a sequence of representations $\rho_m : H \to Mob(S^n)$ such that $\rho_m \circ d_* = d_{m*}$, $\lim \rho_m = id$. The group H is convex cocompact, hence the sequence of limit sets $L(H_m = d_{m*}(H))$ converges to $L(H)$ [12]. It means that any neighborhood U of $L(H)$ contains $L(H_m)$ for any large $m \in \mathbb{N}$. Let Φ be a compact fundamental domain for action of G on $\tilde{M}$. Then $\lim(d_m \Phi) = d(\Phi)$ and hence $d_m(\Phi) \cap L(H_m) = \emptyset$ for large m. So, $d_m(\tilde{M}) \neq S^n$ and the groups H_m are convex cocompact according to [12].

REMARK. The condition of H to be convex cocompact may be weakened. Namely we can suppose that the group H is geometrically finite and rank of any maximal abelian parabolic subgroup of H is equal to $\max(3, n-1)$.

4. Proof of theorem 4.

Step 1. Let $K \in C_c(M)$ be a FCS with the holonomy group H. If H isn't discrete, then K is a Seifert structure [7, 10] and $M = S(g, 0)$, $g \geq 2$. Suppose K is not Seifert and H is a discrete group. Then, arguing analogously to [9, Theorem 3] (see also [8]) we obtain the group H to be torsion-free and pseudofuchsian. Thus we are to show that H is convex cocompact. The group H is isomorphic to a surface group $\pi_1(S_g)$, H acts as a convergence group on the topological circle $L(H)$. Hence, due to [13], there exists a homeomorphism

$f: L(H) \to S^1$ conjugating H and a fuchsian group $F \simeq \pi_1(S_g)$, S_g is a closed surface. Clearly the group F does not contain parabolics and each point of $L(F)$ is a point of approximation [3]. The same is true for the group H and H is convex cocompact. So $C_c(M) = C_s(M) \cup C_f(M)$.

Step 2. Let $\langle t \rangle$ be the center of $G = \pi_1(M)$. Then for all $K \in C_c(M) \setminus C_s(M)$ the kernel of holonomy contains $\langle t \rangle$ (according to step 1). We have that $C_c(M) \setminus C_s(M) = C_f(M, \langle t \rangle)$ is open in $C(M, \langle t \rangle)$. Remark that if d_* is a holonomy homomorphism of K such that $Ker(d_*) \not\supset \langle t \rangle$, then the group $d_*(G)$ has an invariant euclidean circle. Hence in this case $K \in C_s(M)$ [9].

So for any $K \in C_c(M)$ there exists a neighborhood $U(K) \subset C(M) \setminus C_s(M)$ such that: holonomy d'_* of any structure $K' \in U(K)$ drops to a representation $\rho: H \to d'_*(G)$, $\rho \circ d_* = d'_*$. Furthermore, without a loss of generality we can assume that $Ker(\rho) = 1$ [12], since H is convex cocompact. It follows that $C_c(M) \setminus C_s(M)$ is open not only in $C(M, \langle t \rangle)$ but also in $C(M)$ itself. Evidently $C_s(M) \setminus C_f(M)$ is open in $C(M)$; so $C_c(M)$ is open in $C(M)$.

Step 3. Thus we obtained the following description of $C_c(M)$: $C_c(M)$ is the union of two closed sets $C_s(M)$ and $C_f(M)$. If $K \in C_f(M)$, $d: \tilde{M} \to R(H)$ - its development map, which is a covering with deck-transformation group Deck(d), then we put $\nu(d) = |\mathrm{Deck}(d) : \langle t \rangle|$; $\nu: C_f(M) \to \mathbb{N}$ is a continuous function. Denote $\nu^{-1}(n)$ by $C_f^n(M)$. Flat conformal manifolds $(M, K^1) \in C_f^1(M)$ are uniformizable by their holonomy groups. Manifolds (M, K^n) are not uniformizable, if $n > 1$, but finitely cover uniformizable flat conformal manifolds. Consider the case

(a) $M = S(g, 0)$. Then $C_f^n(M) \neq \emptyset$, $hol(C_f^n(M)) = hol(C_f^1(M))$

for any $n \in \mathbb{N}$. If $K^n \in C_f^n(M)$ then there exists an n- fold conformal covering $(M, K^n) \to (M, K^1)$, where $K^1 \in C_f^1(M)$. Consider the case

(b) $M = S(g, e)$, $e \neq 0$. Then for all $K^n \in C_f^n(M)$ the manifold (M, K^n) is an n-fold conformal covering of the manifold $(S(g, ne), K^1)$, where $K^1 \in C_f^1(M)$ is uniformizated by a pseudofuchsian group $H = H(g, ne)$ (see [8]). It is easy to deduce from [11] that in the space of all pseudofuchsian groups the groups $H(g, ne)$ and $H(g, me)$ lie in different connected components (if $n \neq m$).

Step 4. Now we are ready to present $[(g - 1)/11e]$ connected components of $C(M)$. Let $M = S(g, e)$, $e \neq 0$. Then for any $n = 1, \dots, [(g - 1)/11e]$ the manifold $S(g, en)$ admits an uniformizable flat conformal structures L_n [8]. Consider the cyclic covering $p_n : S(g, e) \to S(g, ne)$ and lift L_n via p_n to a FCS $\tilde{L}_n \in C_f^n(M)$. Then $\tilde{L}_n$ lie in different connected components $\mathfrak{U}(n)$ of $C(M)$, $n = 1, \dots, [(g-1)/11e]$. This fact follows from the above considerations, however we shall deduce it from computations of η-invariant, presented below.

Introduce conformally-euclidean metrics g_n compatible with the structures L_n on the manifold $Y_n = S(g, en)$. Consider the extension $H_n^+ \subset \mathrm{Isom}(\mathbb{H}^4)$ of the holonomy group H_n of the structure L_n. Then the compact manifold $X_n = (\mathbb{H}^4 \cup R(H_n))/H_n$ has the boundary Y_n. Clearly the metric g_n may be extended from ∂X_n to a conformally-euclidean metric g_n^+ on the manifold X_n. So the η-invariant $\eta(Y_n, g_n^+)$ is equal to $1/3 \cdot \int_{X_n} p_1(g_n^+) - \mathrm{Sign}(X_n)$ [2].

The pontrjagin form $p_1(g_n^+)$ identically vanishes since g_n^+ is conformally-flat. The manifold X_n is diffeomorphic to a disc bundle over closed surface S_g (of genus g) with euler number equal to e. Hence $\mathrm{Sign}(X_n) = 1$ and $\eta(Y_n, g_n) = -1$. For Chern-Simons invariant we have the formula: $\Phi(Y_n, g_n) \equiv 1/2 \cdot \mathrm{Sign}(X_n) - 3/2 \cdot \eta(Y_n, g_n) \pmod{\mathbb{Z}}$ [2]; thus $\Phi(Y_n, g_n) \equiv 0 \pmod{\mathbb{Z}}$.

Next lift the metric g_n to a metric $\tilde{g}_n$ on $\tilde{Y}_n = S(g, e)$, compatible with the structure $\tilde{L}_n$. Then $\Phi(\tilde{Y}_n, \tilde{g}_n) \equiv n \cdot \Phi(Y_n, g_n) \equiv 0 \pmod{\mathbb{Z}}$. So, the different connected components $\mathcal{U}(n)$ of $C(M)$ can not be distinguished via Chern-Simons functional.

Now we are to compute $\eta(\tilde{Y}_n, \tilde{g}_n)$. Let $\tilde{X}_n$ be total space of disc bundle over S_g with the Euler number e, $\partial\tilde{X}_n = \tilde{Y}_n$. Then the covering p_n may be extended to a branched cyclic covering $p_n^+ : \tilde{X}_n \to X_n$, whose branch set is the zero section of the bundle $\tilde{X}_n$. Therefore formulas of [1, 2] imply that:

$$\eta(M, \tilde{g}_n) = -1 - \sum_{m=1}^{n-1} e/\sin(\pi \cdot m/n)$$

It is easy to verify that $\eta(\tilde{M}, \tilde{g}_n) > \eta(\tilde{M}, \tilde{g}_k)$ if $n < k$. So the η-invariant has distinct values on different components $\mathcal{U}(n)$ of $C(M)$. Theorem 4 is proved. ■

5. Proof of Theorem 1.

Due to [6] the map $hol: \mathcal{T}(M) \to \mathrm{Hom}(G, \mathrm{Mob}(S^n))/\mathrm{ad}(\mathrm{Mob}(S^n))$ may be lifted to a local homeomorphism $hol': \mathcal{T}'(M) \to \mathrm{Hom}(G, \mathrm{Mob}(S^n))$, which is $\mathrm{Mob}(S^n)$- equivariant (here $\mathcal{T}'(M)$ is a space of *pointed* marked flat conformal structures).

REMARK. It follows that the map *hol* is locally injective near any structure (K, φ) such that:

(1) the orbit $ad(Mob(\mathbb{S}^n))(d_*(G))$ is closed in $Mob(\mathbb{S}^n)$, and
(2) the orbit $ad(Mob(\mathbb{S}^n))(d_*)$ has principle type with respect to action $ad(Mob(\mathbb{S}^n))$ near (d_*) in $Hom(G, Mob(\mathbb{S}^n))$.

Remark that if the limit set $L(H)$ of the holonomy group $H= d_*(G)$ has more than one point, then the condition (1) is satisfied. The exceptional cases are reduced to euclidean structure K , and claim of Theorem 1 easy follows.

So we are to consider only those structures K , such that $L(H)$ is contained in some euclidean sphere $\mathbb{S}^\nu \subset \mathbb{S}^n$ $(1 \le \nu \le n-2)$. Then K has a nonsurjective development map $d: \tilde{M} \to \mathbb{S}^n \setminus L(H)$ [9]

(a) Suppose $\mathbb{S}^n \setminus L(H)$ to be simply connected. Then d is a homeomorphism and the group H is a convex cocompact Kleinian group. Hence some neighborhood U of K is contained in $C_c(M) \cap C_f(M)$ and any $K' \in U$ is uniformizable by it's holonomy group isomorphic to G . Thus, nearly (K,φ) the space $\mathcal{T}(M)$ is isomorphic to a neighborhood of $[d_*] \in Hom(G, Mob(\mathbb{S}^n))/ad(Mob(\mathbb{S}^n))$ and theorem's proof is finished here.

(b) It remains the case : $L(H)= \mathbb{S}^{n-2}$, M is a Seifert fibered space and $K \in C_s(M)$ is a Seifert structure. Denote by $\langle t \rangle$ the maximal normal cyclic subgroup of $\pi_1(M)$. Due to the Remark above, we are to consider only $K \in C_s(M) \cap C(M, \langle t \rangle)$ such as:

the representation $d_*: G \to Mob(\mathbb{S}^n)$ is not isolated in $Hom(G, Mob(\mathbb{S}^n)) \setminus hol'(C_s(M))$, else hol is injective near (K,φ) . Without a loss of generality we can suppose that $L=L(H)$ is an extended euclidean $(n-2)$-plane in $\overline{\mathbb{R}}^n$. There are two opportunities:

1. Action of H on L is not sense-preserving (i.e. K is an exceptional structure);

2. Action of H on L is sense-preserving.

Consider the most difficult first case. Let τ be euclidean rotation to the angle π around the plane L. Then $ad(\tau)\circ d_* = d_*$. Let $r^+_m: H \to Mob(S^n)$ be a sequence of representations such as $\lim(r^+_m) = id_H$ and limit sets of $H^+_m = r^+_m(H)$ are not euclidean $(n-2)$-spheres in S^n. The group H is convex cocompact and hence stable [11]. So for m is sufficiently large, there are diffeomorphisms $f_m: R(H) \to R(H^+_m)$ such that $f_m h = r^+_m(h) f_m$ (for any $h \in H$) and the sequence (f_m) converges in C^1 on compacts to identity.

Consider the sequence of development maps $d^+_m = f_m \circ d$ and structures $\tilde{K}^+_m$ lifted to $\tilde{M} = \mathbb{H}^{n-1} \times \mathbb{R}$ by d^+_m. Without a loss of generality we can suppose that the structure (K, φ) is marked by identity diffeomorphism, which is lifted to $id: \tilde{M} \to \tilde{M}$. Then we have: (K^+_m, id) converges in $\mathcal{T}(M)$ to (K, id), where $K^+_m = p(\tilde{K}^+_m)$.

REMARK. Considerations analogous to presented in the proof of Theorem 3 show that any sequence convergent to (K, φ) arises in this way.

Put $r^-_m = ad(\tau) \circ r^+_m$; evidently $\lim(r^-_m) = id_H$. The same considerations as above imply that r^+_m are induced by diffeomorphisms f^-_m convergent to id_H and the sequence (K^-_m, id) converges to (K, id) in $\mathcal{T}(M)$. Also we have $hol(K^-_m, id) = hol(K^+_m, id)$ in $Hom(G, Mob(S^n))/ad(Mob(S^n))$. Finally we are to show that the marked structures (K^-_m, id) and (K^+_m, id) are not equivalent. Let $\tilde{\lambda}_m: (\tilde{M}, \tilde{K}^+_{\text{II}}) \to (\tilde{M}, \tilde{K}^-_{\text{II}})$ be a conformal diffeomorphism such that $\tilde{\lambda}_m \gamma = \gamma \tilde{\lambda}_m$ for any $\gamma \in \pi_1(M) = G$. Then $\tilde{\lambda}_m$ projects to a mobius transformation λ_m such that $ad(\lambda_m)(h) = ad(\tau)(h)$ for any $h \in (H^+_m = d^+_{m*}(H))$. Hence

$\lambda_m=\tau$. However any lift of $ad(\tau):H_m^+\to H_m^-$ to $\tilde{\tau}_*: G\to G$ is outer automorphism of G. This contradiction implies that the structures (K_m^-, id) and (K_m^+, id) are not equivalent. So the map hol is $2q$-fold branched covering near the point (K, φ). It remains to notice that the centralizer of the group H in $\mathrm{Mob}(S^n)$ is equal to $\langle\tau\rangle\simeq \mathbb{Z}_2$; hence $q=1$ and hol is 2-fold branched covering near (K,φ).

Consider the second case, when action of H on L is sense-preserving. Let C_m^+ , C_m^- be structures near K such that their holonomy representations δ_{m*}^+ , δ_{m*}^- are related by a mobius transformation τ_m , $ad(\tau_m)\circ\delta_{m*}^+ = \delta_{m*}^-$. We have $(\lim \tau_m)\in H\times U(1)$, where $U(1)$ is the group of euclidean rotations around L . The group H preserve orientation on L , hence any lift $\tilde{\tau}_0$ of $\tau_0 \in H \times U(1)$ to $\tilde{M}$ induces internal automorphism of the group G. Therefore marked structures (C_m^+, id) , (C_m^-, id) are equivalent and the map hol is injective near the structure (K, id).

Theorem 1 is proved. ∎

REMARK. Theorem 1 shows that the conclusion of [6, Corollary of Deformation Theorem] about local injectivity of hol was too optimistic. It would be interesting to understand local nature of this map in cases of geometric structures other than flat conformal.

6. Some conjectures and speculations.

Let M be a closed 3-manifold. Consider the map $\eta: C(M)\to \mathbb{R}$ given by η-invariant of Atiyah-Patodi-Singer.

CONJECTURE 1. The set $\eta^{-1}(x)$ is connected for any $x\in \mathbb{R}$.

Consider a flat conformal manifold (M , K) with holonomy representation $r_K: \pi_1(M)\to SO(4, 1)$. Associate with r_K a flat $so(4, 1)$ -connection A_r . Then we consider the η-invariant

$\eta(A_r) \equiv \eta(r_K)$.

CONJECTURE 2. There exists a finite-to-one function $\varphi: \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ \eta(K) \equiv \eta(r_K)$.

REMARK. If Conjecture 2 is true then $\eta(C(M))$ consists only of finitely many points.

CONJECTURE 3. The space $C(M)$ consists only of finitely many connected components.

REMARK. (Conj. 1) & (Conj. 2) $\Rightarrow$ (Conj. 3).

REMARK. May be more easy way to handle out the Conjecture 3 is to find a "natural" compactification for the space $C(M)$.

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