Eisenstein Series and Dehn Surgery

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EISENSTEIN SERIES AND DEHN SURGERY

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1. Introduction.

1.1. In spite of its algebraic contents this paper was motivated by a purely geometric question. Let N be a hyperbolic 3-manifold of finite volume with several cusp ends E_1 , ..., E_p . Consider a "Dehn filling" (or "generalized Dehn surgery") of one of these cusps (say E_1) which is just a compactification of E_1 by adding a circle . It was proved by W.Thurston [Thu] that for all but a finite number of parameters of the Dehn filling the resulting manifolds possess complete hyperbolic structures of finite volume.

<u>Question 1.</u> What happens with the geometry of other cusps E_2 , ..., E_p after the Dehn filling ? More restrictively, do the conformal moduli , or rather the Teichmuller parameters, of these cusp tori remain the same?

A positive answer for Question 1 was given as a remark in the thesis of C.Hodgson [Ho]; however, his proof contains a fatal gap. The answer is more complicated and depends on the topology of N. In principle, the problem can be solved in terms of ideal triangulations by formulas ([Thu, Chapter 4], [NZ]); particular manifolds can be considered in this way. W.Neumann and A.Reid [NR] constructed examples of arithmetic manifolds with 2 cusps for which the answer is positive. In the general case the answer depends on the topology of the given manifold.

Basically there are two sources of these examples [NR]. First let N_1 be a hyperbolic 3-orbifold of finite volume which has two cusps $E_1 \cong E_2 \cong T^2 \times [0, \infty)$ and suppose that there exists a rigid hyperbolic 2-dimensional suborbifold $\mathcal{O} \subset N_1$ dividing E_1 from E_2. Another example starts with a hyperbolic 3-orbifold N_2 of finite volume which has two cusps $E_1 \cong \mathcal{O}_1 \times [0, \infty)$, $E_2 \cong \mathcal{O}_2 \times [0, \infty)$ where $\mathcal{O}_1$ is a nonrigid Euclidean orbifold while $\mathcal{O}_2$ is rigid. In the both cases any Dehn surgery on one cusp has no influence on the geometry of another one. The tricky question is to find manifold covers over N_1, N_2 which still have only two cusps (see [NR]).

In this paper we use the machinery of the Eisenstein series to attack the question on the "infinitesimal level". The idea of this approach is explained below.

1.2. Let $N = \mathbb{H}^3/\Gamma$, where $\Gamma \subset \mathbf{PSL}_2(\mathbb{C})$. Let $A_1, A_2, \ldots, A_p$ denote the maximal parabolic subgroups of Γ corresponding to the cusps $E_1, E_2, \ldots, E_p$. The hyperbolic Dehn filling of the cusp E_1 corresponds to the deformation of the holonomy representation $\rho_0 \colon \pi_1(N) \longrightarrow \Gamma \subset \mathbf{PSL}_2(\mathbb{C})$ in the variety

$$\mathbf{R}(\Gamma) = \mathbf{Hom}(\Gamma)/\ \mathbf{ad}(\mathbf{PSL}_2(\mathbb{C}))$$

which preserves traces of elements of the groups $A_2, \ldots, A_p$. Denote by u_j the real Lie algebra of the group U_j which is the maximal unipotent subgroup of $\mathbf{PSL}_2(\mathbb{C})$ such that $A_j \subset \mathrm{U}_j$. The tangent space to $\mathbf{R}(A_j)$ at the point [id] is isomorphic to $\mathrm{H}^1(A_j, \mathfrak{psl}_2(\mathbb{C})_{\mathrm{Ad}}) \cong \mathrm{H}^1(u_j, \mathfrak{psl}_2(\mathbb{C})_{\mathrm{Ad}}) \cong \mathbb{C}^2$.

The split torus Σ_j which normalizes the group U_j acts on $\mathrm{H}^1(u_j, \mathfrak{psl}_2(\mathbb{C})_{\mathrm{Ad}}) \cong \mathbb{C}^2$ via two characters ξ_1, ξ_2 of the weights (-4) and 0, respectively. The eigenspaces of these characters are W_1, W_2. The space W_2 is tangent to the deformations of A_j which preserve traces of elements. On the infinitesimal level we have

<u>Question</u> <u>2</u>. Let $\omega \in \mathrm{H}^1(\Gamma, \mathfrak{psl}_2(\mathbb{C})_{\mathrm{Ad}})$ be such that the projection $\omega_{1,1}$ of ω on $W_{1,1}$ doesn't vanish while $\omega_{j,1} = 0$ for

every $j \neq 1$. Is $\omega_{j,2}$ equal to zero for each $j \neq 1$?

Following [NR], if $\omega_{j,2} = 0$ then we call the cusp E_j infinitesimally isolated from E_1 .

Remark 1. The deformations of $[\rho_0]$ corresponding to Dehn fillings of the cusp E_1 (while other cusps remain to be complete) lie in the variety $R(\Gamma \mid E_2, \ldots, E_p) =$
$\{ \rho : \Gamma \longrightarrow SL_2(\mathbb{C}) : Tr(\rho(\alpha_j)) = Tr(\rho_0(\alpha_j)) , j = 2, \ldots, p \} / SL_2(\mathbb{C})$.
By transversality this variety is a smooth submanifold of $R(\Gamma)$ and the class ω above belongs to its tangent space. Thus a negative answer to Question 2 implies a negative answer to Question 1.

The Question 2 can be treated by the following analytic tools, extracted from [Ha 1].

Denote by E the space $\mathfrak{psl}_2(\mathbb{C})$ and let Ad: $Isom(X = \mathbb{H}^3) \longrightarrow GL(E)$ be the adjoint representation. Define the complex of automorphic forms
$A^q(\Gamma, X, Ad) = \{ \omega \in \Omega^q(X, E) : Ad(\gamma) \otimes R_\gamma \omega = \omega$ for every $\gamma \in \Gamma \}$,
where R_γ is the right action and d: $A^q(\Gamma, X, Ad) \longrightarrow A^{q+1}(\Gamma, X, Ad)$ is the exterior differential. Put $T_\gamma = Ad(\gamma) \otimes R_\gamma$. Suppose that the stabilizer of the point $\infty \in \partial_\infty \mathbb{H}^3$ in the group Γ is the parabolic subgroup $\Gamma_\infty = A_1$. Denote by $X(1)$ the horosphere in $\mathbb{H}^3$ such that $X(1) = \{ (x_1 , x_2 , 0) : (x_1 , x_2) \in \mathbb{R}^2 \}$. Now let
$[\varphi] \in H^1(A^*(X(1), \Gamma_\infty , Ad|_{\Gamma_\infty}) \cong H^1(\Gamma_\infty , Ad|_{\Gamma_\infty})$.

Our first aim is to extend φ to a differential form $\tilde{\varphi}$ on the whole hyperbolic space X.

(a) Consider the vertical unit vectorfield $\tau(z)$ on $X(1) \ni z$. Let $\tilde{\varphi}(\tau(z)) \equiv 0$. Suppose that $Ad(u) \otimes R_u \varphi = \varphi$ for every $u \in U_1$.

(b) Pick $s \in \mathbb{C}$. For $t \in \mathbb{C}^*$ we put

$\Lambda(t)=\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL(2, \mathbb{C})$. Then the formula

$$\mathrm{Ad}(\Lambda(t))\otimes\varphi_s(z\ \Lambda(t)) = \tilde{\varphi}(z)t^{-1-s}$$

, where $\Lambda(t)$ acts on $X \ni z \in X(1)$ in the usual way, defines the extension φ_s of the form $\tilde{\varphi}$ to the hyperbolic space. However, φ_s is not Γ-automorphic. Then the Eisenstein series

$$\text{(c)} \qquad E(\varphi_s\ ;\ x) = E(\varphi,\ s;\ x) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (T_\gamma \varphi_s)(x)$$

is the automorphic extension of φ. This form-valued function admits the meromorphic continuation $E(\varphi,\ s;\ x)$ to the whole complex plane $\mathbb{C} \ni s$. However, the form $E(\varphi,\ s;\ x)$ is closed only for two values of s.

The torus $\mathfrak{T} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}_+ \right\}$ acts on $H^1(\Gamma_\infty\ ,\ \mathrm{Ad}|_{\Gamma_\infty})$ via the adjoint representation with the weights 0 and (-4) . The eigenspace $H^1_{(-4)}(\Gamma_\infty\ ,\ \mathrm{Ad}|_{\Gamma_\infty})$ is tangent to the "loxodromic" deformations of the group Γ_∞ in $PSL(2, \mathbb{C})$, i.e. the images of Γ_∞ under these deformations are semisimple subgroups of $PSL(2, \mathbb{C})$.

It follows from [Ha 2] that for $[\varphi] \in H^1_{(-4)}(\Gamma_\infty\ ,\ \mathrm{Ad}|_{\Gamma_\infty})$ we have:

(i) the form-valued function $s \longmapsto E(\varphi,\ s;\ x)$ is holomorphic for $s = 3$;

(ii) $dE(\varphi,\ 3;\ x) = 0$;

(iii) the cohomology class $[E(\varphi,\ 3;\ x)]$ is not trivial and its restriction to U_j/A_j is equal to

$$\left[\int_{U_j/A_j} T_u E(\varphi,\ 3;\ x)\,du \right].$$

Now we are interested in the restrictions of $E(\varphi, 3; x)$ to the cusps A_j of the group Γ. They are given by the operator

$$C : \varphi \longmapsto (\ldots, \int_{U_j/A_j} T_u E(\varphi, 3; x)\,du, \ldots) \in \bigoplus_{j=1}^{p} H^1(A_j, \mathrm{Ad}|_{A_j})$$

We can consider C as the endomorphism of

$\bigoplus_{j=1}^{p} H^1(A_j, \mathrm{Ad}|_{A_j})$ if we put $\phi = (\varphi, 0, \ldots, 0) \in \bigoplus_{j=1}^{p} H^1(A_j, \mathrm{Ad}|_{A_j})$.

Moreover, we have:

(a) j-th component $C(\phi)_j$ belongs to the eigenspace $H^1_{(0)}(A_j, \mathrm{Ad})$ for $j \neq 1$;

(b) $C(\phi)_1 = \varphi + \psi$, where $\psi \in H^1_{(0)}(A_j, \mathrm{Ad})$.

Thus, the cocycle $E(\varphi, 3; x)$ gives the desired tangent vector to the Dehn fillings of the cusp A_1 . To answer on the Question 2 we have to decide whether the elements

$$(*) \qquad \left[\int_{U_j/A_j} T_u E(\varphi, 3; x)\,du \right].$$

of the groups $H^1(A_j, \mathrm{Ad}|_{A_j})$ are nontrivial.

Unfortunately, the computation of this integral in the general case is very nontrivial problem. If we restrict ourselves to torsion free congruence subgroups Γ of Bianchi groups then the result can be obtained in terms of special values of some L-functions. In spite of the transcendental nature of the L-functions, it is easier to prove that they do not vanish than to calculate the integral (*) directly. Namely, in Theorem 1 we show that the natural restriction maps

$$res_j : H^1(\Gamma, Ad) \longrightarrow H^1(A_j, Ad)$$

are onto. This means that for every cusp E_j we can find another cusp $E_{k(j)} \neq E_j$ such that the variation of the modulus of E_j is nontrivial after the Dehn filling of the cusp $E_{k(j)}$, i.e. E_j isn't infinitesimally isolated form $E_{k(j)}$. Theorem 2 claims that Dehn fillings of every cusp E_j give nontrivial variation of another cusp $E_{k(j)} \neq E_j$ (probably for all other cusps variation is nontrivial too). Application of the transfer extends the statements of Theorems 1 and 2 to all finite-index subgroups in Γ.

1.3. There are several other interesting problems related to the main question of the present paper; they are concerned with the Kleinian subgroups of $PSL(2, \mathbb{C})$. For example:

Let N be a complete hyperbolic 3-manifold of finite volume with one cusp and suppose that ∂N is a totally geodesic surface S (which can be disconnected). Consider the manifolds $N_{p/q}$ which results from N by Dehn filling of the cusp. Then, according to Thurston, for all but finite p/q the manifolds $N_{p/q}$ admit complete hyperbolic structures with totally geodesic boundary. Denote the new hyperbolic boundary surfaces by $S_{p/q}$. The question is:

Question 3. Are the surfaces $S_{p/q}$ isometric to S ?

This question was considered by M.Fujii in [F]. He constructed an example of N for which the variation of Teichmuller parameters of S is nontrivial even on the infinitesimal level. Another example of nontrivial variation can be found in section 5 of present paper. Examples of trivial variations of S are constructed in [NR]. Question 3 can also be treated via Eisenstein series. In this context, it should be noted that the Eisenstein cohomology classes for Kleinian subgroups of $PSL_2(\mathbb{C})$ were considered by I.Kra [Kr] from the analytical point of view.

Conjecture. Let $\Gamma \subset \mathbf{PSL}_2(\mathbb{C})$ be a lattice and $A \subset \Gamma$ be an arbitrary maximal abelian subgroup. There exists a subgroup of finite index $\Gamma_0 \subset \Gamma$ such that

$$res_A : H^1(\Gamma_0 , \mathrm{Ad}) \longrightarrow H^1(A \cap \Gamma_0 , \mathrm{Ad})$$

is onto.

Theorem 1 establishes this conjecture for arithmetic groups Γ when A is unipotent.

1.4. Except for section 5 we shall closely follow the lines of the articles of G.Harder [Ha 1-4]. In this sense the current paper is just a commentary to [Ha 1-4]. Our problem was to make explicit the topological meaning of the articles of G.Harder.

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2. Notations and formulation of main theorem.

Consider the totally imaginary number field $F = \mathbb{Q}(\sqrt{-m})$, where m is a positive square-free integer. Let $\mathcal{O} = \mathcal{O}_m$ be the ring of integers of F, $\mathfrak{A} < \mathcal{O}$ be an integer ideal. Then the group $\Gamma(\mathfrak{A}) = \{ \gamma \in \mathbf{GL}_2(\mathcal{O}) : \gamma \equiv 1 \pmod{\mathfrak{A}} \}$ is a congruence subgroup of the level $\mathfrak{A}$. Let $G_0/F = GL_2/F$; B_0/F be the upper-triangular Borel subgroup of GL_2/F. The groups $G/\mathbb{Q}$, $B/\mathbb{Q}$ and so on result from G_0, B_0 etc. by the restriction of scalars. For any subgroup $H \subset G$ put $H^{(1)} = \{ h \in H : \det(h) = 1 \}$. For ring L we denote by $L^{\times}$ its group of units (invertible elements). For any subgroup $H_0 \subset G_0$ and ring $L \supset \mathbb{Q}$ we put $H(L) = H_0(L \otimes_{\mathbb{Q}} F)$. Any parabolic subgroup $B_j(\mathfrak{A})$ of $\Gamma(\mathfrak{A})$ has the type $B_j(\mathfrak{A}) = B(\mathbb{Q}) \cap b_j \Gamma(\mathfrak{A}) b_j^{-1}$ where b_j belongs to $B(\mathbb{Q})$. Let $Ad = \rho_0 : G_0 \longrightarrow GL(M_0)$ be the adjoint representation, where $M_0 = pg\ell_2(F)$. Denote by $\Delta = \Gamma(\mathfrak{A}) \cap (Z(G_0) \cong \mathcal{O}^{\times})$ the center of the group $\Gamma(\mathfrak{A})$.

THEOREM 1. Suppose that $\Gamma(\mathfrak{A})/\Delta$ is torsion-free. Then the natural restriction maps

$$res_{j,0}(\mathfrak{A}) : H^1(\Gamma(\mathfrak{A}), M_0) \longrightarrow H^1(B_j(\mathfrak{A}), M_0)$$

are epimorphisms.

3. Preliminary results for proof of theorem 1.

3.1. We shall use constructions, notations and statements from the papers of G.Harder [Ha 2-4]. In this section we introduce the following notations.

For nonarchimedian valuations $\wp$ of F put $K_{\wp}(\mathcal{O}) = GL_2(\mathcal{O}_{\wp})$ and $K_{\wp} = K_{\wp}(\mathfrak{A}) = \{ \gamma \in GL_2(\mathcal{O}_{\wp}) \mid \gamma \equiv 1 \pmod{\mathfrak{A}_{\wp}} \}$; for the infinite place we put $G_{\infty} = G(\mathbb{R}) \cong GL_2(\mathbb{C})/\mathbb{R}$; $K_{\infty} = U(2) \cdot \mathbb{C}^{*}$, where $\mathbb{C}^{*}$ is the center of G_{∞} ; then $K_f = K_f(\mathfrak{A}) = \prod_{\wp \text{ is finite}} K_{\wp}$, $K = K_{\infty} \cdot K_f$.

Remark 2. The group $K_f(\mathfrak{A})$ is normal in $K_f(\mathcal{O})$.

Consider the rings of adeles $\mathbb{A}_F$, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},\infty} \times \mathbb{A}_f$ and the groups of ideles $\mathbb{I}_F = \mathbb{I}_{F,\infty} \times \mathbb{I}_{F,f}$, $\mathbb{I} = \mathbb{I}_{\mathbb{Q}} = \mathbb{I}_{\mathbb{Q},\infty} \times \mathbb{I}_{\mathbb{Q},f}$ for the fields F, $\mathbb{Q}$. Put $\bar{G} = G \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ and extend the representation ρ_0 to the representation $\rho : \overline{G} \longrightarrow GL(M'' = M_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ which splits as

$$\rho : \overline{G} = \prod_{\sigma : F \longrightarrow \overline{\mathbb{Q}}} GL_2/\overline{\mathbb{Q}} \longrightarrow GL(M) \oplus GL(M') = 1_{\overline{\tau}} \otimes Ad_{\tau} \oplus Ad_{\overline{\tau}} \otimes 1_{\tau}$$

$=\rho \oplus \rho'$. Here 1_σ, Ad_σ are trivial and adjoint representations; σ is an embedding of F in $\overline{\mathbb{Q}}$. It suffices to prove that the natural restriction map

$$res_j(\mathfrak{A}) : H^1(\Gamma(\mathfrak{A})/\Delta, M) \longrightarrow H^1(B_j(\mathfrak{A})/\Delta, M)$$

is surjective. The modulus M decomposes as $1_{\overline{\tau}} \otimes Ad_{\tau} = M_{\overline{\tau}} \otimes M_{\tau}$, where M_σ is isomorphic to the $(d_\sigma+1)$-dimensional (over $\overline{\mathbb{Q}}$) vector space of homogeneous polynomials of the degree d_σ with the action:

$$\rho_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} : P(X_\sigma, Y_\sigma) \longrightarrow P(aX_\sigma + cY_\sigma, bX_\sigma + dY_\sigma)\cdot\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\nu(\sigma)}$$

where $d_\sigma = \begin{cases} 0, & \text{if } \sigma = \overline{\tau} \\ 2, & \text{if } \sigma = \tau \end{cases}$, $\nu_\sigma = \begin{cases} 0, & \text{if } \sigma = \overline{\tau} \\ -1, & \text{if } \sigma = \tau \end{cases}$ (sf. [Ha 4]).

3.2. Geometry of associated locally-symmetric space.

Attach to $\Gamma(\mathfrak{A})$ the locally-symmetric space

$$S_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K \cong GL_2(F) \backslash GL_2(\mathbb{A}_F) / K$$

which is not connected, its components are "counted" by the map

$$\det : S_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K \longrightarrow \mathbb{I}_{F,f} / F^{*}\cdot\det(K_f),$$

where $\det(x_f^{(j)})$ are representatives of cosets $\mathbb{I}_{F,f} / F^{*}\cdot \det(K_f)$ and $x_f^{(j)} = \begin{pmatrix} \det(x_f^{(j)}) & 0 \\ 0 & 1 \end{pmatrix}$. The connected components of the symmetric space $X_K = G(\mathbb{A})/K$ are $\mathbb{H}^3_{(j)} = G_\infty x_f^{(j)} K_f / K_\infty K_f \cong \mathbb{H}^3$.

Suppose that $\mathbb{H}^3_{(j)}$ is invariant under $q \in G(\mathbb{Q})$. Then

$q\, x_f^{(j)} K_f = x_f^{(j)} K_f$, hence $q \in x_f^{(j)} K_f\, (x_f^{(j)})^{-1} \cap G(\mathbb{Q})$. Thus, the individual components of S_K have the form:

$$(x_f^{(j)} K_f\, (x_f^{(j)})^{-1} \cap G(\mathbb{Q})) \backslash \mathbb{H}^3_{(j)} = \Gamma_{(j)}(\mathfrak{A}) \backslash \mathbb{H}^3_{(j)} .$$

In particular, $\Gamma(\mathfrak{A}) \equiv \Gamma_{(1)}(\mathfrak{A}) = \{ \gamma \in G(\mathbb{Q}) \mid \gamma \equiv 1 \pmod{\mathfrak{A}}\}$; the subgroup $\Delta \subset \Gamma_{(j)}(\mathfrak{A})$ acts trivially on $\mathbb{H}^3_{(j)}$ and the 1-st component of S_K is homeomorphic to $(\Gamma(\mathfrak{A})/\, \Delta) \backslash \mathbb{H}^3$.

By means of the representation ρ we define the sheaf $\tilde{M}$ of local sections of the fiber bundle over S_K (with the fiber M). Then

$$H^*((\Gamma(\mathfrak{A})/\, \Delta) \backslash \mathbb{H}^3,\, \tilde{M}) \cong H^*(\Gamma(\mathfrak{A})/\, \Delta,\, M).$$

3.3. <u>Geometry of ideal boundary of</u> S_K .

Denote by $\bar{S}_K$ the compactification of S_K such that $\partial\bar{S}_K$ is the collection of tori.

LEMMA 1. (i) The space $\partial\bar{S}_K$ is homotopy equivalent to $B(\mathbb{Q})\backslash G(\mathbb{A})/K$. (ii) The components of $B(\mathbb{Q})\backslash G(\mathbb{A})/K$ are fibers of the map $\det\circ p_f$:

$$B(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f \xrightarrow{p_f} B(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f \xrightarrow{\det} \mathbb{I}_{F,f}/\, F^*\cdot\det(K_f)$$

PROOF. See [Ha 3, Proposition 3.1] ∎

CONVENTION 1. To simplify notations we shall identify $\bar{S}_K$ and $B(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f$; below we shall drop the bar sign for $\partial\bar{S}_K$.

Consider the commutative diagram:

$$\begin{array}{ccc} B(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f & \xrightarrow{r} & B(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f \\ \downarrow q\circ r & \swarrow q & \\ B(\mathbb{A}_f)\backslash G(\mathbb{A}_f)\, /K_f & & \end{array}$$

where q, r are the natural projections. Put $\partial_1 S_K = (q\circ r)^{-1}[1]$, then $\partial_1 S_K$ is equal to $B(\mathbb{Q})\backslash\, B(\mathbb{A})/\, K_\infty^B K_f^B$ where $K_\infty^B = K_\infty \cap B_\infty$,

$K_f^B = K_f \cap B(\mathbb{A}_f)$ [Ha 3, proof of theorem 1].

Consider the (right) action of $GL_2(\mathcal{O}_f)$ on $\partial \bar{S}_K$: $[g] \in B(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f \longmapsto [g\cdot s_f]$, where $s_f \in GL_2(\mathcal{O}_f)$; this action is defined correctly since K_f is a normal subgroup of $GL_2(\mathcal{O}_f)$. Certainly, this action of $GL_2(\mathcal{O}_f)$ by right translations is eqivariant with respect to $q\circ r$:

$$\begin{array}{ccc} B(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f & \xrightarrow{s_f} & B(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f \\ \downarrow q\circ r & & \downarrow q\circ r \\ B(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f & \xrightarrow{s_f} & B(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f \end{array}$$

Hence $GL_2(\mathcal{O}_f)$ acts by right translations on fibers of the map $q\circ r$, in particular: $\partial_1 S_K \overset{s_f}{\longmapsto} B(\mathbb{Q})\backslash G_\infty B(\mathbb{A}_f)s_f K_f /K_\infty K_f \cong B(\mathbb{Q})\backslash B_\infty\cdot B(\mathbb{A}_f)s_f /K_\infty^B K_f^B$.

REMARK 3. The decomposition of ∂S_K on connected components induces the decomposition of $\partial \bar{S}_K$. The action of $SL_2(\mathcal{O}_f)=G^{(1)}(\mathcal{O}_f)$ preserves the determinant; hence the last decomposition is invariant under $SL_2(\mathcal{O}_f)$.

LEMMA 2. $G(\mathbb{A}_f) = B(\mathbb{A}_f)\cdot G(\mathcal{O}_f)$.

PROOF. $G(\mathbb{A}_f) = B(\mathbb{A}_f)\cdot SL_2(\mathbb{A}_{F,f}) = B(\mathbb{A}_f)\cdot SL_2(F)\cdot G(\mathcal{O}_f)$ (by the strong approximation for SL_2). However, $SL_2(F) \subset B(\mathbb{Q})\cdot SL_2(\mathcal{O}) \subset B(\mathbb{A}_f)\cdot G^{(1)}(\mathcal{O}_f)$ which implies the assertion of the lemma. ∎

COROLLARY 1. (i) The group $G(\mathcal{O}_f)$ acts transitively on fibers of $q\circ r$. (ii) The stabilizer of $\partial_1 S_K$ in the group $G(\mathcal{O}_f)$ is equal to $B(\mathcal{O}_f)\cdot K_f$.

PROOF. Let $B(\mathbb{A}_f)g_f K_f = B(\mathbb{A}_f)K_f$, where $g_f \in G(\mathcal{O}_f)$. Hence $g_f = b_f k_f \in B(\mathbb{A}_f)\cdot K_f \cap G(\mathcal{O}_f)$. However, $K_f \le G(\mathcal{O}_f)$; therefore $b_f \in G(\mathcal{O}_f)$ and $g_f \in B(\mathcal{O}_f)\cdot K_f$. ∎

3.4. Geometry of $\partial_1 S_K$.

3.4.1. Consider the fiber bundle

$p:\ B(\mathbb{Q})\backslash B(\mathbb{A})/K_\infty^B K_f^B \longrightarrow T(\mathbb{Q})\backslash T(\mathbb{A})/K_\infty^T K_f^T$, where $K_\infty^T = K_\infty \cap T_\infty$, $K_f^T = K_f \cap T(\mathbb{A}_f)$, T is the group of diagonal matrices. The fiber of p is the 2-torus $\mathbb{T}^2 \cong U(\mathbb{Q})\backslash U(\mathbb{A})/K_f^U = \Gamma_U \backslash U_\infty$, where $\Gamma_U = U(\mathbb{Q}) \cap K_f^U$,

$$U(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in F \right\},\ \Gamma_U = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathcal{O},\ \lambda \equiv 0 \pmod{\mathfrak{A}} \right\}$$ is the

lattice on the complex plane $U_\infty \cong \mathbb{C}$. In particular, $\pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z} = \pi_1(\text{component of } \partial_1 S_K)$, hence every component of

$\Sigma = T(\mathbb{Q})\backslash T(\mathbb{A})/K_\infty^T K_f^T$ is contractible.

3.4.2. Continue the sheaf $\tilde{M}$ to the ideal boundary $\partial_1 S_K$. Define the sheaf $\widetilde{H^1(\mathbb{T}^2, \tilde{M})}$ of local sections of bundle over Σ with the fiber $H^1(\mathbb{T}^2, \tilde{M})$, associated with the action of $T(\mathbb{Q})$ on $H^1(\mathbb{T}^2, \tilde{M})$. Then $H^1(\partial_1 S_K , \tilde{M}) \cong H^0(\Sigma , \widetilde{H^1(\mathbb{T}^2, \tilde{M})})$ (cf. [Ha 4, 2.3]).

3.4.3. $H^1(\mathbb{T}^2, \tilde{M})$ as T-modulus.

Notice that $H^1(\mathbb{T}^2, \tilde{M}) \cong H^1(u , M) \cong \bar{\mathbb{Q}} \times \bar{\mathbb{Q}}$, where u is the Lie algebra of the group $U(\mathbb{Q})$ [Ha 4]. Then decompose $H^1(u , M)$ as the direct sum of the eigenspaces for the action of the torus $T(\bar{\mathbb{Q}})$:

$$H^1(u , M) = H^1(u , M)(\chi_1) \oplus H^1(u , M)(\chi_2) = \bar{\mathbb{Q}}_{\chi_1} \oplus \bar{\mathbb{Q}}_{\chi_2} ,$$

and the direct calculation [Ha 4] of the characters $\chi_j\big|_{T(\mathbb{Q})}$ gives:

$$\chi_1 : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto t_2^{d(\tau)+\nu(\tau)+1} \cdot t_1^{\nu(\tau)-1} \cdot \left(\bar{t}_1\right)^{d(\bar{\tau})} \cdot \left(\bar{t}_1 \bar{t}_2\right)^{\nu(\bar{\tau})} = t_2^2 t_1^{-2}$$

$$\chi_2: \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto \left(t_1 t_2\right)^{\nu(\tau)} \cdot t_1^{d(\tau)} \cdot \left(\bar{t}_1\right)^{\nu(\bar{\tau})-1} \cdot \left(\bar{t}_2\right)^{d(\bar{\tau})+\nu(\bar{\tau})+1} =$$

$$\left(t_1/\bar{t}_1\right) \cdot \left(\bar{t}_2/t_2\right); \quad \chi_j = \chi_{1j}(t_1)\cdot\chi_{2j}(t_2). \text{ If } T^{(1)} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in F \right\}$$

then $\chi_1\big|_{T^{(1)}} = t^{-4}$, $\chi_2\big|_{T^{(1)}} = \bar{t}^{-2}t^{2}$, $\chi_j\big|_{Z(G)} \equiv 1$ (cf. the item 1.2).

REMARK 4. The characters $\chi_1 = \lambda_{01}^{-1}$, $\chi_2 = \lambda_{10}^{-1}$ are balanced and $w(\chi_1) = -4$ (in sense of G.Harder [Ha 4]).

3.4.4. Fix any isomorphisms $\bar{\mathbb{Q}}_{\chi_j} \cong \bar{\mathbb{Q}}$. Notice that $U(\mathbb{A})$ = Ker(p) is a normal subgroup in $B(\mathbb{A})$; hence $B(\mathbb{A}_f)$ acts on $T(\mathbb{A})$ via right translations and $U(\mathbb{A})$ lies in the kernel of this action. Therefore, it is possible to identify the action of $B(\mathbb{A})$ on $T(\mathbb{A})$ with the action of $T(\mathbb{A})$ on itself, which drops on $\mathfrak{T} = T(\mathbb{Q})\backslash T(\mathbb{A})/K^T$.

Now, following [Ha 4] we describe in convenient terms the cohomology $H^0(T(\mathbb{Q})\backslash T(\mathbb{A})/K_\infty^T K_f^T , \tilde{\bar{\mathbb{Q}}}_{\chi_j})$. Consider the set of Hecke characters $\Sigma_j = \{ \phi : T(\mathbb{Q})\backslash T(\mathbb{A})/K^T \longrightarrow \bar{\mathbb{Q}} : \phi = \phi_\infty \cdot \phi_f , \phi_\infty = (\chi_j)^{-1}\}$, in other words, $\phi \in \Sigma_j$ has the type χ_j at infinity. If $\phi \in \Sigma_j$ then $\phi_f\big|_{T(\mathbb{Q})} = \chi_j$. The space

$$H^0(T(\mathbb{Q})\backslash T(\mathbb{A})/K_\infty^T K_f^T , \tilde{\bar{\mathbb{Q}}}_{\chi_j})$$

is isomorphic to the space of locally constant sections $e : T(\mathbb{A})/K^T \longrightarrow \bar{\mathbb{Q}}$ such that: $e(qt) = \chi_j(q)\cdot e(t)$ for every $q \in T(\mathbb{Q})$. Put $\mathfrak{T}_f = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_f^T$.

ASSERTION 1. Every section $e(t)$ as above can be described as a linear combination of $e_\phi(t) = \phi_f(t)$, $\phi = (\chi_j^{-1}, \phi_f) \in \Sigma_j$.

PROOF. The space of sections is isomorphic to the space of $\bar{\mathbb{Q}}$-valued functions $\bar{\mathbb{Q}}[\mathfrak{T}_f]^{\#}$ on the finite abelian group $\mathfrak{T}_f$. The space $\overline{\mathbb{Q}}[\mathfrak{T}_f]^{\#}$ is spanned by the set of $\overline{\mathbb{Q}}$-valued characters of the group $\mathfrak{T}_f$. This implies the assertion. ∎

3.4.5. The items 3.4.3 and 3.4.4 imply that

$$H^1_B(\tilde{M}) \equiv H^1(\partial_1 S_K , \tilde{M}) \cong \bigoplus_{\phi\in\Sigma_1} \overline{\mathbb{Q}}e_\phi \oplus \bigoplus_{\phi\in\Sigma_2} \overline{\mathbb{Q}}e_\phi \qquad (3.4.5)$$

REMARK 5. The group $B^{(1)}(\mathbb{A}_f)$ acts transitively on $\mathfrak{T}_f$.

Now we can define the action of the group $B(\mathbb{A}_f)$ on $H^1(\partial S_K , \tilde{M})$. Namely, the action of $B(\mathbb{A}_f)$ on $\mathfrak{T}$ induces the following action on

$$H^0(\mathfrak{T} , \widetilde{H^1(\mathbb{T}^2, \tilde{M})}) : \quad R(b_f)e_\phi(t) = e_\phi(t\ b_f) = \phi_f(b_f)e_\phi(t).$$

So the decomposition (3.4.5) is invariant under the action of $B(\mathbb{A}_f)$ and the restrictions of the action to 1-dimensional subspaces $\overline{\mathbb{Q}}e_\phi$ are defined by the characters $\phi_f:B(\mathbb{A}_f) \longrightarrow \mathbb{C}^*$, where $\phi_f\big|_{U(\mathbb{A}_f)} \equiv 1$.

3.4.6. <u>The final formula for</u> $H^1(\partial S_K , \tilde{M})$ <u>as</u> $B(\mathbb{A}_f)$-<u>module</u>.

Define $\mathrm{Ind}_{B(\mathbb{A}_f),K_f}^{G(\mathbb{A}_f)} H^1_B(\tilde{M}) = \Big\{ \psi : G(\mathbb{A}_f) \longrightarrow H^1_B(\tilde{M}) : \psi(b_f g_f k_f) = R(b_f)\psi(g_f)$, for every $b_f \in B(\mathbb{A}_f)$, $g_f \in G(\mathbb{A}_f)$, $k_f \in K_f \Big\}$.

LEMMA 3. $\mathrm{Ind}_{B(\mathcal{O}_f),K_f}^{G(\mathcal{O}_f)} H^1_B(\tilde{M})$ is isomorphic to $\mathrm{Ind}_{B(\mathbb{A}_f),K_f}^{G(\mathbb{A}_f)} H^1_B(\tilde{M})$.

PROOF. Consider the linear restriction map

$$\theta: \mathrm{Ind}_{B(\mathbb{A}_f),K_f}^{G(\mathbb{A}_f)} H^1_B(\tilde{M}) \longrightarrow \mathrm{Ind}_{B(\mathcal{O}_f),K_f}^{G(\mathcal{O}_f)} H^1_B(\tilde{M}) , \ \theta(\psi) = \psi\big|_{G(\mathcal{O}_f)} .$$

(a) <u>Show that</u> θ <u>is surjective.</u> Recall that $G(\mathbb{A}_f) = B(\mathbb{A}_f)G(\mathcal{O}_f)$. Then for $\psi \in \mathrm{Ind}_{B(\mathbb{A}_f),K_f}^{G(\mathbb{A}_f)} H^1_B(\tilde{M})$ we put $\tilde{\psi}(g_f = b_f\cdot o_f) = R(b_f)\psi(o_f)$, where $o_f \in G(\mathcal{O}_f)$. Check that this definition is correct. Let $g_f = b_f o_f = a_f n_f$, where $a_f \in B_f$, $n_f \in G(\mathcal{O}_f)$. Then $c_f =$

$a_f^{-1}b_f = n_f g_f^{-1} \in G(\mathcal{O}_f) \cap B(\mathbb{A}_f) = B(\mathcal{O}_f)$; $n_f = c_f o_f$. Now $R(a_f)\psi(n_f) = R(a_f)R(c_f)\psi(o_f) = R(a_f c_f)\psi(o_f) = R(b_f)\psi(o_f)$. Hence $\tilde{\psi}$ doesn't depend on the decomposition and our definition is correct.

(b) <u>Show that</u> $\mathrm{Ker}(\theta) = 0$. Let $\psi \in \mathrm{Ind}_{B(\mathbb{A}_f),K_f}^{G(\mathbb{A}_f)} H_B^1(\tilde{M})$, $\psi|_{G(\mathcal{O}_f)} \equiv 0$. Then $\psi(g_f = b_f o_f) = R(b_f)\psi(o_f) = 0$ implies that $\psi(o_f) = 0$. ∎

Thus we obtained the decomposition:

$$(3.4.6) \quad H^1(\partial S_K , \tilde{M}) \cong \bigoplus_{\phi \in \Sigma_1 \cup \Sigma_2} \Big\{ \psi\colon G(\mathbb{A}_f) \longrightarrow \bar{\mathbb{Q}} e_\phi \cong \bar{\mathbb{Q}} : \psi(b_f g_f k_f) = \phi(b_f)\psi(g_f), \text{ for every } b_f \in B(\mathbb{A}_f),\ k_f \in K_f \Big\}.$$

A decomposition of such kind is given in [Ha 4, Theorem 1] after transition to the inverse limit by K_f .

CONVENTION 2. By the symbol e_ϕ we shall denote the element of $H^1(\partial S_K , \tilde{M})$ which is zero outside $B(\mathbb{A}_f)K_f$ and whose restriction to $B(\mathbb{A}_f)K_f$ is equal to e_ϕ .

4. Description of boundary homomorphism after Harder and proof of main theorem

4.1. Let $\phi \in \Sigma_1$; $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G(\mathbb{Q})$, $\mathrm{ad}(w_0)$ is a nontrivial element of Weil group. Define the character $|\alpha|\colon B(\mathbb{A}) \longrightarrow \mathbb{C}^*$ as $|\alpha|\colon \begin{pmatrix} t & u \\ 0 & t_2 \end{pmatrix} \longrightarrow |t_1/t_2| = \prod_{\wp} \|(t_1/t_2)_\wp\|_\wp$ (where $\wp$ runs through all valuations of F). Obviously we have

$$(4.1) \qquad |\alpha| \, \big|_{B(\mathbb{A}) \cap K} \equiv 1$$

since the matrix elements of $K_\wp \cap T(\mathbb{A})_\wp$ are units of $\mathcal{O}_\wp$.

Put $w_0 \cdot \phi = |\alpha| \cdot \phi^{w_0}$, where $\phi^{w_0}(t) = \phi(w_0 t w_0^{-1})$; therefore :
$\phi^{w_0}\Big|_{T^{(1)}(\mathbb{A})} \equiv \phi^{-1}\Big|_{T^{(1)}(\mathbb{A})}$.

Notice that: (a) $w_0 \cdot \phi$ is a Hecke character. Really, $w_0 \cdot \phi\big|_{T(\mathbb{Q})} \equiv 1$ follows from $|\alpha|\big|_{T(\mathbb{Q})} \equiv 1$ (see [Lg, p.85-86]). Next, (4.1) implies that $|\alpha|\big|_{{}_K T} \equiv 1$. (b) $w_0 \cdot \phi$ has the type χ_2 at infinity because:

1. $\phi_\infty = \phi\big|_{T_\infty} = \chi_1^{-1}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto (t_1/t_2)^2$;

2. $\mathrm{Ad}(w_0)\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}$;

3. $|\alpha|_\infty : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longrightarrow t_1\bar{t}_1/(t_2\bar{t}_2)$.

$$\text{So} \quad |\alpha|_\infty \cdot (\chi_1^{-1})^{w_0}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = (t_1/t_2)^2 t_1\bar{t}_1/(t_2\bar{t}_2) = \chi_2^{-1}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

and $w_0 : \phi \in \Sigma_1 \longmapsto w_0 \cdot \phi \in \Sigma_2$. The same arguments show that:
$w_0 : \phi \in \Sigma_2 \longmapsto w_0 \cdot \phi \in \Sigma_1$.

4.2. The results of (4.1) and (3.4.6) give us the isomorphism:

$$H^1(\partial S_K, \tilde{M}) \cong \bigoplus_{\substack{\{\phi, w_0\phi\} \\ \phi \in \Sigma_1}} \left(\mathrm{Ind}_{B(\mathbb{A}_f), K_f}^{G(\mathbb{A}_f)} \bar{\mathbb{Q}} e_\phi \oplus \mathrm{Ind}_{B(\mathbb{A}_f), K_f}^{G(\mathbb{A}_f)} \bar{\mathbb{Q}} e_{w_0 \cdot \phi} \right) =$$

$$= \bigoplus_{\phi \in \Sigma_1} \left(V_{\phi_f} \oplus V_{w_0 \cdot \phi_f} \right).$$

Now decompose V_{ϕ_f}, $V_{w_0 \cdot \phi_f}$ in local factors. We have $\phi_f(b_f) = \prod_{\wp \text{ is finite}} \phi_\wp(b_\wp)$, $b_\wp \in B(F_\wp)$. Put $B_\wp = B(F_\wp)$, $G_\wp = GL_2(F_\wp)$, $K_\wp$ is the projection of K_f in $G_\wp$, i.e.

$K_{\mathcal{Y}} = \left\{ \gamma \in GL_2(F_{\mathcal{Y}}) : \gamma \equiv 1 \pmod{\mathfrak{a}_{\mathcal{Y}}} \right\}$. Furthermore: $V_{\phi_{\mathcal{Y}}} =$
$\left\{ \psi : G_{\mathcal{Y}} \longrightarrow \overline{\mathbb{Q}} : \psi(b_{\mathcal{Y}} g_{\mathcal{Y}} k_{\mathcal{Y}}) = \phi_{\mathcal{Y}}(b_{\mathcal{Y}}) \psi(g_{\mathcal{Y}}), \text{ for every } b_{\mathcal{Y}} \in B_{\mathcal{Y}}, \quad k_{\mathcal{Y}} \in K_{\mathcal{Y}} \right\}$

ASSERTION 2. The space V_{ϕ_f} is isomorphic to $\bigotimes_{\mathcal{Y} \text{ is finite}} V_{\phi_{\mathcal{Y}}}$.

PROOF. For $\psi \in V_{\phi_f}$ we put $\xi_{\mathcal{Y}}(\psi) = \psi_{\mathcal{Y}} = \psi|_{G_{\mathcal{Y}}}$,

$\xi(\psi) = \prod_{\mathcal{Y} \text{ is finite}} \psi_{\mathcal{Y}} \in \bigotimes_{\mathcal{Y} \text{ is finite}} V_{\phi_{\mathcal{Y}}}$. We have: $G_{\mathcal{Y}} = B_{\mathcal{Y}} K_{\mathcal{Y}}$ for all

but finite places; hence the product $\xi(\psi)$ is convergent.
The map ξ is surjective and has zero kernel. ∎

4.3. <u>Local intertwining operators</u> $T_{\mathcal{Y}} : V_{\phi_{\mathcal{Y}}} \longrightarrow V_{w_0 \cdot \phi_{\mathcal{Y}}}$.

4.3.1. Suppose that the quasicharacter $\phi^{(1)} = \mu_1 / \mu_2$ is ramified over $\mathcal{Y}$, i.e. $\phi|_{T^{(1)}(\mathcal{O}_{\mathcal{Y}}^{\times})}$ is not trivial. Then put:

$$(4.3.1) \qquad T_{\mathcal{Y}}(\psi_{\mathcal{Y}})(g_{\mathcal{Y}}) = \int_{U(F_{\mathcal{Y}})} \psi_{\mathcal{Y}}(w_0 u_{\mathcal{Y}} g_{\mathcal{Y}}) du_{\mathcal{Y}},$$

where $\psi_{\mathcal{Y}} \in V_{\phi_{\mathcal{Y}}}$; $U(F_{\mathcal{Y}})$ is isomorphic to $F_{\mathcal{Y}}$ and $du_{\mathcal{Y}}$ is the natural measure on $U(F_{\mathcal{Y}})$ induced by this isomorphism i.e.:

(1) $du_{\mathcal{Y}}$ is invariant under translations;

(2) $\int_{U(\mathcal{O}_{\mathcal{Y}})} du_{\mathcal{Y}} = 1$; (3) $\mathrm{mes}(\|z\|_{\mathcal{Y}} \cdot E_{\mathcal{Y}}) = \|z\|_{\mathcal{Y}} \cdot \mathrm{mes}(E_{\mathcal{Y}})$ for every $z \in F_{\mathcal{Y}}$ and measurable $E_{\mathcal{Y}} \subset F_{\mathcal{Y}}$.

To show that the integral in (4.3.1) is finite we reproduce some calculations from [Ha 2]. Let $g_{\mathcal{Y}} \in G(\mathcal{O}_{\mathcal{Y}})$.

$$T_{\mathcal{Y}}(\psi_{\mathcal{Y}})(g_{\mathcal{Y}}) = \int_{U(F_{\mathcal{Y}})} \psi_{\mathcal{Y}}(w_0 u_{\mathcal{Y}} g_{\mathcal{Y}}) du_{\mathcal{Y}} = \int_{U(\mathcal{O}_{\mathcal{Y}})} \psi_{\mathcal{Y}}(w_0 u_{\mathcal{Y}} g_{\mathcal{Y}}) du_{\mathcal{Y}} +$$

$$\sum_{N=1}^{\infty} \|\pi_{\mathcal{Y}}^{-N}\|_{\mathcal{Y}} \cdot \int_{\mathcal{O}_{\mathcal{Y}}^{\times}} \psi_{\mathcal{Y}}\left(w_0 \begin{pmatrix} 1 & x_{\mathcal{Y}} \cdot \pi_{\mathcal{Y}}^{-N} \\ 0 & 1 \end{pmatrix} g_{\mathcal{Y}}\right) dx_{\mathcal{Y}}, \text{ where } \mathrm{ord}_{\mathcal{Y}}(\pi_{\mathcal{Y}}) = 1.$$

The first summand is finite since $\mathcal{O}_{\wp}$ is compact and $\psi_{\wp}$ is continuous. Then consider the infinite sum. Notice that:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{\wp}\cdot\pi_{\wp}^{-N} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -x_{\wp}\cdot\pi_{\wp}^{-N} \end{pmatrix} =$$

$$\begin{pmatrix} -x_{\wp}^{-1}\cdot\pi_{\wp}^{N} & 1 \\ 0 & -x_{\wp}\cdot\pi_{\wp}^{-N} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_{\wp}^{-1}\cdot\pi_{\wp}^{N} & 1 \end{pmatrix}$$

hence $x_{\wp}^{-1}\cdot\pi_{\wp}^{N} \equiv 0 \pmod{\mathfrak{a}_{\wp}}$ for all $N \geq N_0$ (since $\mathfrak{a}_{\wp}$ is a power of $(\pi_{\wp})$ in $\mathcal{O}_{\wp}$). Therefore, for $N \geq N_0$ the matrix $\begin{pmatrix} 1 & 0 \\ x_{\wp}^{-1}\cdot\pi_{\wp}^{N} & 1 \end{pmatrix} = k_{\wp}$ belongs to $K_{\wp}$. Hence

$$\sum_{N=N_0}^{\infty} \|\pi_{\wp}^{-N}\|_{\wp} \cdot \int_{\mathcal{O}_{\wp}^{\times}} \psi_{\wp}(w_0 \begin{pmatrix} 1 & x_{\wp}\cdot\pi_{\wp}^{-N} \\ 0 & 1 \end{pmatrix} g_{\wp})\, dx_{\wp} =$$

$$\sum_{N=N_0}^{\infty} \phi_{\wp}^{(1)}(-\pi_{\wp}^{N}) \|\pi_{\wp}^{-N}\|_{\wp}\, \psi_{\wp}(g_{\wp})\cdot\left(\int_{\mathcal{O}_{\wp}^{\times}} \phi_{\wp}^{(1)}(x_{\wp})\, dx_{\wp} = 0\right) = 0 \quad ([Lg]).$$

4.3.2. Suppose that the quasicharacter $\phi^{(1)} = \mu_1 / \mu_2 \in \Sigma_j$ is unramified over $\wp$, i.e. $\phi\big|_{T^{(1)}(\mathcal{O}_{\wp}^{\times})}$ is trivial. Decompose $\phi_{\wp}$ as

$$\phi_{\wp}: \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto \mu_{1\wp}(t_1)\cdot\mu_{2\wp}(t_2).$$

For $G_{\wp} \ni g_{\wp} = b_{\wp}k_{\wp} \in B(F_{\wp})\cdot GL_2(\mathcal{O}_{\wp})$ we put:

$$\Psi_{0,\wp}^{(j)}(g_{\wp}) = \Psi_{0,\wp}^{(j)}(b_{\wp}k_{\wp}) = \phi_{\wp}(b_{\wp})\mu_{1\wp}(\det(k_{\wp}))$$

$\Psi_{0,\wp}^{(j)}$ is a "spherical" function. Show that this definition is correct. Suppose that $b_{\wp}k_{\wp} = \tilde{b}_{\wp}\tilde{k}_{\wp}$. Then (1) $\mu_{1\wp}(\varepsilon) = \mu_{2\wp}(\varepsilon)$

for every $\varepsilon \in O_{\psi}^{\times}$; (2) $k_{\psi} = k'_{\psi}\tilde{k}_{\psi}$, where $k'_{\psi} = \begin{pmatrix} k_1 k_3 \\ 0\ k_2 \end{pmatrix}$, $\tilde{b}_{\psi} = b_{\psi}k'_{\psi}$.

Hence $\mu_{1\psi}(\det(k'_{\psi})) = \mu_{1\psi}(k_1)\cdot \mu_{2\psi}(k_2)$; $\phi_{\psi}(b_{\psi})\mu_{1\psi}(\det(k_{\psi})) = \phi_{\psi}(b_{\psi})\mu_{1\psi}(k_1)\cdot \mu_{2\psi}(k_2)\mu_{1\psi}(\det\tilde{k}_{\psi}) = \phi_{\psi}(\tilde{b}_{\psi})\mu_{1\psi}(\det(\tilde{k}_{\psi}))$. Therefore the value of $\Psi_{0,\psi}^{(j)}$ does not depend on the decomposition of g_{ψ} . Notice that $|\alpha|_{\psi}$ is unramified, hence $w_0\cdot\phi_{\psi}^{(1)}$ and $\phi_{\psi}^{(1)}$ are unramified simultaneously. Now put:

$$T_{\psi} : \Psi_{0,\psi}^{(1)} \in V_{\phi_{\psi}} \longmapsto \Psi_{0,\psi}^{(2)} \in V_{w_0\cdot\phi_{\psi}}$$

This operator extends linearly to an isomorphism between the B_{ψ}-modules $V_{\phi_{\psi}}$ and $V_{w_0\cdot\phi_{\psi}}$ [Ha 3, 4]. Put

$$T_{\psi}(\psi_{\psi})(g_{\psi}) = (L_{\psi}(0)/L_{\psi}(-1)) \cdot \int_{U(F_{\psi})} \psi_{\psi}(w_0 u_{\psi} g_{\psi})du_{\psi} ,$$

where $L_{\psi}(s) = L_{\psi}(s, \phi_{\psi}^{(1)}) = \left(1 - \phi_{\psi}^{(1)}(\pi_{\psi})/\mathbf{N}((\pi_{\psi}))^{-s}\right)^{-1}$ and for $s \in \mathbb{Z}_+$ we have $\mathbf{N}((\pi_{\psi}))^{s} =$ (number of elements of $O/(\pi_{\psi})^{s}) = \|\pi_{\psi}\|_{\psi}^{-s}$, valuation ψ is determined by the prime ideal (π_{ψ}) . The function $L_{\psi}(s, \phi^{(1)})$ admits a meromorphic continuation to the whole complex plane. Calculate the value $T_{\psi}(\psi_{\psi} = \Psi_{0,\psi}^{(1)})(g_{\psi})$ for $g_{\psi} \in G^{(1)}(O_{\psi}) = SL_2(O_{\psi})$.

Consider the function

$$\psi_{\psi} = \Psi_{0,\psi,s}^{(1)}(b_{\psi}k_{\psi}) = (|\alpha|_{\psi}^{s/2}\cdot\phi_{\psi}(b_{\psi}))\cdot|\det(k_{\psi})|_{\psi}^{s/2}\mu_{1\psi}(\det(k_{\psi}))$$

which is a spherical function with respect to the character ${}^{s}\phi_{\psi} = |\alpha|_{\psi}^{s/2}\cdot\phi_{\psi}(b_{\psi})$.

According to (4.3.1) we have:

$$\int_{U(F_\wp)} \Psi^{(1)}_{0,\wp,s}(w_0 u_\wp g_\wp)\,du_\wp = \left(\int_{\pi(O_\wp)} \psi_\wp(w_0 u_\wp g_\wp)\,du_\wp = 1 \right) +$$

$$\sum_{N=1}^{\infty} {}^s\phi^{(1)}_\wp(\pi^N_\wp)\,\|\pi^{-N}_\wp\|_\wp \cdot \psi_\wp(g_\wp) \cdot \left(\left(\int_{O^\times_\wp} {}^s\phi^{(1)}_\wp(x_\wp)\,dx_\wp \right) = 1 - \|\pi_\wp\|_\wp \right) =$$

$$\left(1 - \phi^{(1)}_\wp(\pi_\wp)\,\|\pi^s_\wp\|_\wp\right)\left(1 - \phi^{(1)}_\wp(\pi_\wp)\,\|\pi^{s-1}_\wp\|_\wp\right)^{-1} =$$

$$L_\wp(s-1, \phi^{(1)}_\wp)/L_\wp(s, \phi^{(1)}_\wp).$$

Remark 6. Apriori this infinite sum is divergent for $s=0$ however it is convergent for sufficiently large $\mathrm{Re}(s)$; we can use a regularization by a meromorphic continuation to the whole complex plane to obtain the desired value $L_\wp(s-1, \phi^{(1)}_\wp)/L_\wp(s, \phi^{(1)}_\wp)\big|_{s=0}$. In fact, as we shall see later, this series is convergent.

So $T_\wp(\psi_\wp = \Psi^{(1)}_{0,\wp})(g_\wp \in SL_2(O_\wp)) = 1$. On other hand, $\Psi^{(2)}_{0,\wp}(g_\wp) = 1$ for $g_\wp \in G^{(1)}(O_\wp) = SL_2(O_\wp)$. Direct calculations show that $T_\wp(\psi_\wp)(b_\wp g_\wp) = w_0 \cdot \phi_\wp(b_\wp) T_\wp(\psi_\wp)(g_\wp)$ and $T_\wp \circ R(g_\wp) = R(g_\wp) \circ T_\wp$. This implies that $T_\wp(\Psi^{(1)}_{0,\wp}) = \Psi^{(2)}_{0,\wp}$ and we really obtained the desired extension.

4.4. The formula for a global intertwining operator is

$$T_{\phi_f} = \bigotimes_{\wp \text{ is finite}} (T_{\phi_\wp} : V_{\phi_\wp} \longrightarrow V_{w_0 \cdot \phi_\wp})$$

Let $L_F(s, \phi^{(1)}) = \prod_{\substack{\wp \text{ is finite} \\ \phi^{(1)}_\wp \text{ is unramified}}} L_\wp(s, \phi^{(1)}_\wp)$,

where the local factors

$$L_\wp(s, \phi^{(1)}_\wp) = \left(1 - \phi^{(1)}_\wp(\pi_\wp)/N((\pi_\wp))^{-s}\right)^{-1}$$

were introduced in 4.3.2.

Then the function $L_F(s, \phi^{(1)})$ admits a meromorphic continuation to the whole complex plane. Introduce the "constant"

$$c_\phi = L(-1, \phi^{(1)}_\psi)/L(0, \phi^{(1)}_\psi) \in \mathbb{C} \cup \{\infty\}.$$

THEOREM H (Harder [Ha 4]). Let

$$res : H^1(S_K, \tilde{M}) \longrightarrow H^1(\partial S_K, \tilde{M})$$

be the restriction homomorphism. Then:

(1) $\mathrm{Im}(res) = \bigoplus_{\phi\in\Sigma_1} \mathrm{Im}(res_\phi) \subset V_{\phi_f} \oplus V_{w_0\cdot\phi_f}$;

(2) $\mathrm{Im}(res_\phi) = \left\{ (\psi, \alpha_F\cdot c_\phi\cdot T_{\phi_f}(\psi)) \mid \psi \in V_{\phi_f}\right\}$,

where $\alpha_F \in \mathbb{C}^*$ is a constant.

PROOF. The field F is a totally imaginary quadratic extension of $\mathbb{Q}$ and $\phi_\infty = \lambda_{01}$ (see the items 3.4.2, 3.4.3). Then the character ϕ is "balanced" (in the sense of Harder). Therefore we can apply the theorem 2 from [Ha 4].

QED.

For the completeness we repeat here briefly the arguments of [Ha 4, proof of theorem 2; Ha 2, Ha 3]. Put $\omega(g, \phi, \psi) = \omega\Big((b_\infty k_\infty, g_f), \phi, \psi\Big) = \phi_\infty(b_\infty)\psi(g_f)\Big(ad^*_{p_\infty}(k^{-1}_\infty)\otimes\rho_\infty(k^{-1}_\infty)\varepsilon_{\phi_\infty}\Big)$, where p_∞ is the orthogonal complement to the Lie algebra of K_∞ in $\mathfrak{gl}_2(\mathbb{C})$, $\varepsilon_{\phi_\infty} \in \mathrm{Hom}(p_\infty, M)$ has the weight ϕ^{-1}_∞ with respect to the adjoint representation of $\mathbf{T}(\mathbb{R})$. Let

$$E(g, {}^s\phi, \psi) = \sum_{a\in \mathbf{B}(\mathbb{Q})\backslash \mathbf{G}(\mathbb{Q})} \omega(a\cdot g)$$

Consider the "constant term"

$$(*) \qquad E^0(g, {}^s\phi, \psi) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(ug, {}^s\phi, \psi)\,du$$

where du is the Tamagawa measure on $U(\mathbb{A})$. Then the restriction of $E(g, \phi, \psi)$ to $U(\mathbb{Q})\backslash U(\mathbb{A})$ is a closed 1-form which is cohomologious to the restriction of $E^0(g, \phi, \psi)$. The computation of the integral (*) is relatively simple in our case since we have of the Bruhat decomposition $G(\mathbb{Q}) = B(\mathbb{Q}) + B(\mathbb{Q})w_0U(\mathbb{Q})$. Then

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(ug, {}^s\phi, \psi)du=$$

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \sum_{a\in B(\mathbb{Q})\backslash G(\mathbb{Q})} \omega(a\cdot ug)du = \omega(g) + \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \sum_{u\in U(\mathbb{Q})} \omega(w_0u\cdot g)$$

$$= \omega(g)+ \int_{U(\mathbb{A})} \omega(w_0u\cdot g)du \quad \text{since} \quad Vol(U(\mathbb{Q})\backslash U(\mathbb{A}))= 1.$$

The form ω decomposes as $\omega_\infty \otimes \bigotimes_{\wp \text{ is finite}} \psi_\wp$ and the integral

$\int_{U(\mathbb{A})} \omega(w_0u\cdot g)du$ is equal to

$$d_F\cdot\left(\prod_{\wp \text{ is finite}} \int_{U(F_\wp)} \psi_\wp(w_0u_\wp\cdot g_\wp)du_\wp\right)\times \int_{U(F_\infty)} \omega_\infty(w_0u_\infty\cdot g)du_\infty ,$$

where the nonzero number d_F is coming from the Tamagawa measure. Direct calculations [Ha 4] show that the last multiple (for $g=1$) is equal to $\beta_\phi\varepsilon_{w_0\cdot\phi_\infty}$ where $\beta_\phi \in \mathbb{C}^*$ is a scalar depending only on the character ϕ_∞. Then $\alpha_F= d_F\cdot\beta_\phi$.

According to 4.3.1, 4.3.2 for any finite $\wp$ we have:

$$\int_{U(F_\wp)} \psi_\wp(w_0u_\wp\cdot g_\wp)du_\wp = \begin{cases} T_\wp(\psi_\wp)(g_\wp), & \text{if } \phi^{(1)}_\wp \text{ is ramified} \\ (L_\wp(-1, \phi^{(1)}_\wp)/L_\wp(0, \phi^{(1)}_\wp))\cdot T_\wp(\psi_\wp)(g_\wp), & \text{else} \end{cases}$$

This calculation proves the theorem.

■

4.5. **Lemma 4.** $c_\phi \notin \{0, \infty\}$.

Proof. First we notice that $c_\phi \neq \infty$, $L(0, \phi^{(1)}) \notin \{0, \infty\}$ which follows from [Ha 4, proof of theorem 2, p.82]. So we only have to show that $L(-1, \phi^{(1)}) \neq 0$. Denote $\phi^{(1)}$ by η. For $L(s, \eta)$ we put [Lg]:

$$\Lambda(s, \eta) = ((2\pi)^{-2} \cdot d_\eta)^{s/2} \cdot \Gamma(s_\infty/2) \cdot L(s, \eta)$$

where $d_\eta \in \mathbb{Q}^*$, the complex number s_∞ will be defined later and the function $\Lambda(s, \eta)$ satisfies the functional equation:

$$|\Lambda(s, \eta)| = |\Lambda(\overline{1-s}, \eta)|.$$

Hence $|\Lambda(-1, \eta)| \in \mathbb{C}^* \iff |\Lambda(2, \eta)| \in \mathbb{C}^*$.

Assertion 3. $L(2, \eta) \in \mathbb{C}^*$.

Proof. The function $\Lambda(s, \eta)$ is holomorphic and $L'(s)/L(s)$ is bounded for $\mathrm{Re}(s) > 1+1/2$ [Lg]. Hence $L(2, \eta) \neq 0, \infty$. ■

Therefore we have $|L(-1, \eta)| = |\Gamma(2_\infty/2)\Gamma((-1)_\infty/2)|^{-1} \cdot \mathrm{const}$, where const $\notin \{0, \infty\}$. Consider $\Gamma(s_\infty/2)$, $s \in \{2, -1\}$. The map $s \longmapsto s_\infty$ is constructed via the character $\eta_\infty(z) = z^{-4}$, $z \in \mathbb{C}$ in the following way.

We decompose $\eta_\infty(z) = z^{-4}$ as $z^{-4} = (z/|z|)^m \cdot |z|^{2c} = z^m \cdot |z|^{2c-m} = z^{m/2+c}(\bar{z})^{c-m/2}$ (see [Lg]) . Then $m = m_\infty = -4$, $s_\infty/2 = s + |m|/2$ (since $|F: \mathbb{Q}| = 2$), $s_\infty = s + 2$; $2_\infty/2 = 4$, $(-1)_\infty/2 = 1$. However $\Gamma(4) \cdot \Gamma(1)^{-1} \in \mathbb{Q}^*$. Therefore $|L(-1, \eta)| \neq 0$. Lemma is proved. ■

4.6. Recall that $H^1(\partial S_K, \tilde{M}) \cong \mathrm{Ind}_{B(\mathcal{O}_f)K_f}^{G(\mathcal{O}_f)} H^1(\partial_1 S_K, \tilde{M})$. We have to single out the different components of ∂S_K in this induced modulus. The group $G(\mathcal{O}_f)$ acts transitively on the decomposition $\partial S_K = \partial_1 S_K \cup \partial_2 S_K \cup \dots$. This decomposition corresponds to $G(\mathcal{O}_f) = B(\mathcal{O}_f)K_f + B(\mathcal{O}_f)K_f g_f^{(2)} + \dots$ (where $g_f^{(1)} = 1$). The restriction of $H^1(\partial S_K, \tilde{M})$ on $\partial_j S_K$ is equivalent to the restriction of the function $\psi: G(\mathbb{A}_f) \longrightarrow H^1_B(\tilde{M})$ on $B_f g_f^{(j)} K_f$. Then the image of $H^1(\partial_1 S_K , \tilde{M})$ in $H^1(\partial S_K , \tilde{M})$ corresponds to

$$\left\{\psi : G(\mathcal{O}_f) \longrightarrow H^1_B(\tilde{M}) : \operatorname{Supp}(\psi)= B_f K_f , \psi(b_f k_f)= R(b_f)\psi(1) \right\}.$$

Respectively, $H^1(\partial_j S_K , \tilde{M}) \subset H^1(\partial S_K , \tilde{M})$ corresponds to

$$\left\{\psi : G(\mathcal{O}_f) \longrightarrow H^1_B(\tilde{M}): \operatorname{Supp}(\psi)= B_f K_f g_f^{(j)}, \psi(b_f g_f^{(j)} k_f)= R(b_f)\psi(g_f^{(j)}) \right\}.$$

In the same way, on the level of $\operatorname{Ind}^{G(\mathbb{A}_f)}_{B(\mathbb{A}_f),K_f} H^1_B(\tilde{M})$ we can single out the space $H^1_j(\tilde{M})$ as the space of functions with support at $B(\mathbb{A}_f) g_f^{(j)} K_f$.

4.7. **Assertion 4.** If for every j the map

$$res : H^1(S_K , \tilde{M}) \longrightarrow H^1_j(\partial S_K , \tilde{M})$$

is <u>onto</u> then $res : H^1(S_K , \tilde{M}) \longrightarrow H^1(\mathbb{T}^2_{ij} , \tilde{M})$ is <u>onto</u> for every boundary torus $\mathbb{T}^2_{ij} \subset \partial S_K$.

Proof. $H^1_j(\partial S_K , \tilde{M})= \bigoplus_i H^1(\mathbb{T}^2_{ij} , \tilde{M})$ ∎

The space $H^1_B(\tilde{M})$ and hence $\tilde{H}^1_j(\tilde{M})$ is the direct sum of two subspaces where $B(\mathbb{A})$ acts by two (quasi)characters which are different at the infinite place. Denote these subspaces by V^1_j , V^2_j (where 1, 2 correspond to χ_1 , χ_2 - values at infinity), $V^i= \bigoplus_j V^i_j$.

4.8. <u>Principal lemma.</u>

Lemma 5. $V^2_j \subset \operatorname{Im}(res_j: H^1(S_K , \tilde{M}) \longrightarrow H^1_j(\partial S_K , \tilde{M}))$ for every $j\neq 1$.

Proof. Let $\phi \in \Sigma_1$, $\phi_f= \mu_{1f}\cdot\mu_{2f}$, where μ_{kf} are defined on $\mathbb{I}_f$; $\phi_f^{(1)}= \mu_{1f}(t_f)/\mu_{2f}(t_f)$, $t_f\in \mathbb{I}_f$.

4.8.1. Let $\psi \in \tilde{H}^1_1(\tilde{M}) \setminus \{0\}$ be an eigenfunction of $B(\mathbb{A}_f)$ corresponding to the quasicharacter ϕ. Then

$$T_{\phi_f}:\psi \longmapsto \bar{\mathbb{Q}}\cdot e_{w_0\cdot\phi_f} \subset V_{w_0\cdot\phi_f} .$$

Our aim is to show that $T_{\phi_f}(\psi)\Big|_{B_f g_f^{(j)} K_f} \neq 0$. If it is done

then $\underset{\phi\in\Sigma_1}{\mathrm{Span}}(T_{\phi_f}(\psi)\Big|_{B_f g_f^{(j)} K_f}) = \underset{\phi\in\Sigma_1}{\mathrm{Span}}(e_{w_0}\cdot\phi_f\Big|_{B_f g_f^{(j)} K_f}) = V_j^2$.

4.8.2. Thus, take $\psi\in V_{\phi_f}$, $\mathrm{Supp}(\psi)= B_f K_f$. The space M_ϕ of such functions is 1-dimensional (over $\overline{\mathbb{Q}}$) then we can choose ψ in the following way:

(a) If there exists a place $\mathfrak{p}$ where $\phi_f^{(1)}$ is ramified then we take $\psi_{\mathfrak{p}} = e_{\phi_{\mathfrak{p}}}$ for all such places; for other places $\mathfrak{z}$ (where $\phi_f^{(1)}$ is unramified) we take $\psi_{\mathfrak{z}} = \Psi^{(1)}_{0,z}$ (see the Convention from 3.4.6).

(b) If $\phi_f^{(1)}$ is unramified at every place, then we pick a prime $\pi \in \mathcal{O}_F$ and put: $\psi_{\mathfrak{p}} = e_{\phi_{\mathfrak{p}}}$ (for $\mathfrak{p} = (\pi)$) and $\psi_{\mathfrak{z}} = \Psi^{(1)}_{0,z}$ (for every $\mathfrak{z} \neq \mathfrak{p}$).

In any case, the function $\psi = \underset{\mathfrak{p}\text{ is finite}}{\otimes} \psi_{\mathfrak{p}}$ belongs to M_ϕ .

4.8.3. If $\psi_{\mathfrak{z}} = \Psi^{(1)}_{0,z}$ then $T_{\phi_{\mathfrak{z}}}(\psi_{\mathfrak{z}})(g_{\mathfrak{p}})\neq 0$ for every $g_{\mathfrak{p}}$. So, our problem is reduced to the calculation of the term

$$\mathrm{const}\cdot T_{\mathfrak{p}}\psi_{\mathfrak{p}}(g_{\mathfrak{p}}^{(j)})= \int_{U(F_{\mathfrak{p}})} \psi_{\mathfrak{p}}(w_0 u_{\mathfrak{p}} g_{\mathfrak{p}}^{(j)})\,du_{\mathfrak{p}} \quad \text{in the case } \psi_{\mathfrak{p}} = e_{\phi_{\mathfrak{p}}}.$$

Remark 7. If necessary we use here the regularization (4.3.2) of the integral above.

Fix the Bruhat decomposition: $G(F_{\mathfrak{p}})= B(F_{\mathfrak{p}}) + U(F_{\mathfrak{p}})\cdot w_0 B(F_{\mathfrak{p}})$. The element $g_{\mathfrak{p}}^{(j)}$ belongs to the second summand since $j\neq 1$. Then:

$$g_{\mathfrak{p}} = g_{\mathfrak{p}}^{(j)} = \begin{pmatrix}1 & u'\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}b_1 & b_3\\ 0 & b_2\end{pmatrix}$$ and we have:

$$\int_{U(F_\psi)} \psi_\psi(w_0 u_\psi g_\psi^{(j)})\, du_\psi =$$

$$\int_{U(F_\psi)} \psi_\psi\left(w_0 \begin{pmatrix} 1 & u_\psi + u' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix}\right) du_\psi =$$

$$\int_{U(F_\psi)} \psi_\psi\left(\begin{pmatrix} 1 & 0 \\ u_\psi + u' & 1 \end{pmatrix} \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix}\right) du_\psi =$$

$$\int_{U(F_\psi)} \psi_\psi\left(\begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1-b_3 z & -b_3^2 z \\ z & b_3 z + 1 \end{pmatrix}\right) du_\psi \quad , \text{ where } \quad z = (u_\psi + u')\, b_1 b_2^{-1}.$$

Recall that $\mathrm{Supp}(\psi_\psi) = B_\psi K_\psi$, then $\psi_\psi\left(\begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1-b_3 z & -b_3^2 z \\ z & b_3 z + 1 \end{pmatrix}\right)$

vanishes unless $\begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1-b_3 z & -b_3^2 z \\ z & b_3 z + 1 \end{pmatrix} \equiv 1 \pmod{\pi_\psi^{N_0}}$ for some

$\begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix} \in B(F_\psi)$. However this is equivalent to

$a_2 \equiv a_1 \equiv 1 \pmod{\pi_\psi^{N_0}}$, $a_3 \equiv (u_\psi + u')\, b_1 b_2^{-1} \equiv 0 \pmod{\pi_\psi^{N_0}}$ and

$(u_\psi + u')\, b_1 b_2^{-1} \in \pi_\psi^{N_0} \mathcal{O}_\psi$. Hence, u_ψ runs through the set $b_2 b_1^{-1} \pi_\psi^{N_0+1} \mathcal{O}_\psi + u' = \Xi$. Notice that $\mathrm{mes}(\Xi) = \|b_1 b_2^{-1} \pi_\psi^{N_0}\|_\psi \cdot (1 - \|\pi_\psi\|_\psi) \neq 0$ and

$$\psi_\psi\left(\begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1-b_3 z & -b_3^2 z \\ z & b_3 z + 1 \end{pmatrix}\right) = \phi_\psi\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Therefore, $T_\psi \psi_\psi(g_\psi^{(j)}) = \mathrm{const}\cdot \phi_\psi \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \mathrm{mes}(\Xi) \neq 0$. So, Lemma is proved. ∎

4.9. **COROLLARY 2.** $V_j^2 \subset \mathrm{Im}(res_j)$ for every $j = 1, 2, \ldots$.

PROOF. The group $G(\mathcal{O}_f)$ acts on S_K, ∂S_K (and on $H^1(S_K, \tilde{M})$, $H^1(\partial S_K, \tilde{M})$) and this action commutates with the restriction maps. However, this action is transitive on $\{\partial_j S_K, j = 1, 2, \ldots\}$.

Therefore, the corollary follows from Lemma 5. ∎

4.10. **COROLLARY 3.** $V_1^1 \subset \mathrm{Im}(res_1)$.

PROOF. Let $\psi \in H_1^1(\tilde{M}) \cap V_{\phi_f}$, then $\mathrm{Supp}(\psi) = B(\mathbb{A}_f)K_f$ and

$$(\psi, c_\phi T_\phi \psi \big|_{B(\mathbb{A}_f)K_f}) = (\psi, \mathrm{const}\cdot \prod_{\mathfrak{q} \text{ is finite}} \int_{U(F_\mathfrak{q})} \psi_\mathfrak{q}(w_0 u_\mathfrak{q} b_\mathfrak{q} k_\mathfrak{q})\, du_\mathfrak{q})$$

$$= (\psi, \mathrm{const}\cdot\mathrm{const}'\cdot e_{w_0\cdot\phi_f}\big|_{B_f K_f}) \in \mathrm{Im}(res_\phi)\big|_{B_f K_f} .$$

Remark. The scalar const′ vanishes iff $\phi_f^{(1)}$ is ramified over some place as it follows from (4.3.1).

However, according to Corollary 1, every vector $(0, \varphi)$ belongs to $\mathrm{Im}(res_\phi)\big|_{B_f K_f}$; therefore

$$\left\{(\psi, 0) : \psi \in H_1^1(\tilde{M}) \cap V_{\phi_f}\right\} \subset \mathrm{Im}(res_\phi)\big|_{B_f K_f}$$ for every Hecke character $\phi \in \Sigma_1$. This proves the corollary. ∎

4.11. Now we apply the action of the group $G(\mathcal{O}_f)$ in the same manner as in the section 4.9 to obtain:

COROLLARY 4. $V_j^1 \subset \mathrm{Im}(res_j)$ for every $j = 1, 2, \ldots$.

<u>This corollary finishes the proof of Theorem 1.</u> ∎

4.12. <u>Proof of theorem 2.</u>

Below we use notations introduced in 1.2.

Theorem 2. Let $\mathbb{T}^2_{nj} \subset \partial S_K$. Then for every $\omega_{nj,1} \in H^1(\pi_1(\mathbb{T}^2_{nj}), M_0 = pg\ell_2(\mathbb{C}))$ there exists a class $\omega \in H^1(\Gamma = \Gamma(\mathfrak{A}), M_0)$ such that:

(i) projection of ω to $W_{nj,1}$ is equal to $\omega_{nj,1}$;

(ii) $\omega_{kl,1} = 0$ for every torus $\mathbb{T}^2_{kl} \neq \mathbb{T}^2_{nj}$;

(iii) for every $k \neq j$ there exists $l = l(k)$ such that the projection $\omega_{kl,2}$ doesn't vanish.

Proof. Using the isomorphism $M_0 \cong M$ and preserving notations for sub (upper)scripts we identify classes the $\omega_{kl,p}$ with elements of $H^1(\Gamma, M)$; the eigenvectors of the character ξ_1 correspond to elements of V^1 under the isomorphism $H^1(\partial S_K, \tilde{M}) \cong V^1 \oplus V^2$. Applying the action of $SL_2(\mathcal{O}_{F,f})$ we can suppose that $j = 1$.

Then $V^1_1 = \bigoplus_{s=1}^{|\Sigma_f|} W_{s1,1}$ (under our convention on notations above).

Put $T = \bigoplus_{\phi_f} T_{\phi_f}$. For $\omega_{n1,1} \equiv \psi \in V^1_1$ let $\omega \in H^1(\Gamma, pg\ell_2(\mathbb{C}))$ be a class corresponding to $(\psi, T\psi) \in \mathrm{Im}(res)$. Notice that the operator $\varphi \in V^1_1 \longmapsto (T\varphi)_k \in V^2_k$ is injective (4.8), therefore the class ω has the desired properties. ∎

Thus, after infinitesimal deformation of the group $\Gamma = \Gamma(\mathfrak{A})$ corresponding to a Dehn surgery on $\mathbb{T}^2_{nj}$, we have nontrivial infinitesimal variation of Euclidean structure on some other cusp tori of a connected component of S_K containing $\mathbb{T}^2_{nj}$.

5. Example.

5.1. Example. Let M be the complete hyperbolic manifold homeomorphic to the complement of the "Figure 8" knot. Then (according to C.Maclachlan [M]) M contains an immersed closed totally geodesic surface Σ. Let

$$R(\pi_1 M) = \mathrm{Hom}(\pi_1 M, SL_2(C))/SL_2(C) .$$

ASSERTION 5.1. Under deformations of holonomy representation of $\pi_1 M$ the group $\pi_1\Sigma$ varies nontrivially. So, under small deformations of id= $\rho_o : \pi_1 M \longrightarrow SL_2(C)$ the image of $\rho_o(\pi_1\Sigma)$ becomes quasifuchsian.

PROOF. As it was shown by R.Riley [Ri] (see also [KK]) in the space $R(\pi_1 M)$ there is an irreducible complex curve σ, which contains $[\rho_o]$ and some solvable representation $[\rho_s]$. Hence there exists an element $\gamma \in \pi_1\Sigma$ such that $\rho_s(\gamma)= 1$ while $\rho_o(\gamma) \neq 1$. This means that $\rho_o(\gamma)$ varies nontrivially under the deformation of ρ_o along σ. ■

Let $\tilde{M} \longrightarrow M$ be a finite -sheeted covering such that Σ lifts in the embedded closed totally geodesic surface $\tilde{\Sigma}$ in $\tilde{M}$. Then the assertion above means that after a Dehn filling of $\tilde{M}$ with sufficiently large parameter the surface $\tilde{\Sigma}$ deforms to non-totally geodesic incompressible surface. In general case the surface $\tilde{\Sigma}$ doesn't split $\tilde{M}$. Let N be the hyperbolic manifold with totally geodesic boundary which is the compactification of $\tilde{M} \setminus \tilde{S}$. Then the the Mostow's rigidity theorem implies that the answer to Question 3 (see Introduction) is negative for the manifold N.

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