Energy of harmonic functions and Gromov's proof of Stallings' theorem

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Abstract

We provide the details for Gromov's proof of Stallings' theorem on groups with infinitely many ends using harmonic functions.

1 Introduction

In his essay [7, Pages 228–230], Gromov gave a proof of the Stallings' theorem on groups with infinitely many ends using harmonic functions. The goal of this paper is to provide the details for Gromov's arguments. We note that G. Niblo [15] gave a completely different geometric proof of Stallings' theorem, based on Sageev's complex.

Let M be a complete Riemannian manifold of bounded geometry, which has infinitely many ends. Suppose, moreover, that there exists a number R such that every point in M belongs to an *R–neck*, i.e., an R-ball which separates M into at least three unbounded components. (This property is immediate if M admits a cocompact isometric group action.) Let $\bar{M} := M \cup \mathrm{End}(M)$ denote the compactification of M by its space of ends. Given a continuous function $\chi : \mathrm{End}(M) \to \{0, 1\}$, let

$$h = h_\chi : \bar{M} \to [0, 1]$$

denote the continuous extension of χ, so that $h|M$ is harmonic. Let $\mathcal{H}(M)$ denote the space of harmonic functions

$$\{h = h_\chi, \chi : \mathrm{End}(M) \to \{0, 1\} \text{ is nonconstant}\}.$$

We give $\mathcal{H}(M)$ the topology of uniform convergence on compacts in M. Let $E : \mathcal{H}(M) \to \mathbb{R}_+ = [0, \infty)$ denote the energy functional.

Definition 1.1. *The* energy gap $e(M)$ *of* M *is*

$$e(M) := \inf\{E(h) : h \in \mathcal{H}(M)\}.$$

If M admits an isometric group action $G \curvearrowright M$, then G acts on $\mathcal{H}(M)$ preserving the functional E. Therefore E projects to a lower semi-continuous (see Lemma 2.9) functional $E : \mathcal{H}(M)/G \to \mathbb{R}_+$, where we give $\mathcal{H}(M)/G$ the quotient topology. The main technical result needed for the proof of Stallings' theorem is

Theorem 1.2. *Suppose that* M *has injectivity radius bounded from below by* $\delta > 0$*, and has absolute value of the sectional curvature bounded from above by* $K < \infty$*. Then:*

1. There exists $\mu = \mu(R, dim(M), K, \delta)$ *so that* $e(M) \geq \mu > 0$*.*

2. If M *admits a cocompact isometric group action, then* $E : \mathcal{H}(M)/G \to \mathbb{R}_+$ *is proper in the sense that*

$$E^{-1}([0, T])$$

is compact for every $T \in \mathbb{R}_+$*. In particular,* $e(M)$ *is attained.*

This theorem will be proven in sections 3 and 4; the proof of part 2 in section 4 is due to Bruce Kleiner.

We now sketch our proof of the Stallings' theorem. Let $h \in \mathcal{H}(M)$ be an energy-minimizing harmonic function. We verify that the level set $\{h(x) = \frac{1}{2}\}$ is *precisely-invariant* with respect to the action of G. By choosing t sufficiently close to $\frac{1}{2}$ we obtain a smooth hypersurface $S = \{h(x) = t\}$ which is precisely-invariant under G and separates the ends of M. If this hypersurface were connected, we could use the standard construction of a dual simplicial tree T whose edges are the "walls", i.e., the images of S under the elements of G and the vertices are the components of $M \setminus G \cdot S$. In the general case, a "wall" can be adjacent to more than two connected component of $M \setminus G \cdot S$. We show however that each wall is adjacent to exactly two "indecomposable" subset of $M \setminus G \cdot S$, i.e., a subset which cannot be separated by one wall. These indecomposable sets are the vertices of T. We then verify that the graph T is actually a tree.

It was observed by W. Woess that the arguments in this paper generalize to harmonic functions on graphs. (In Appendix 3 we explain what modifications are needed to our arguments in order to extend them to graphs.) In particular, smoothness of harmonic functions (emphasized by Gromov in [7, Pages 228–230]) becomes irrelevant. One advantage of this approach is to avoid the discussion of nodal sets of harmonic functions and that the higher-order derivative bounds for harmonic functions are immediate in this setting. We observe that many of our arguments simplify if we take M to be a Riemann surface (or a graph), which suffices for the proof of Stallings' theorem. We wrote the proofs in greater generality because the compactness theorem for harmonic functions appears to be of independent interest.

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2 Preliminaries

Notations. Throughout this paper, we let M be a complete Riemannian manifold of bounded geometry, i.e., its injectivity radius is bounded from below by some $\delta > 0$ and the absolute value of the sectional curvature is bounded from above by some $K < \infty$. We say that a constant C depends only on *geometry of M* if it depends only on dimension of M, and the numbers δ and K.

Notation 2.1. *For a subset $N \subset M$ let N^c denote $M \setminus int(N)$.*

Notation 2.2. *Given a subset $N \subset M$, let $B_R(N)$ denote the collection of points in M which are within distance $\leq R$ from N. Thus, $B_R(x)$ is the closed R-ball centered at x.*

Given a function f, we set $Var(f) := \sup(f) - \inf(f)$. For an n-dimensional Riemannian manifold N (possibly with boundary), we let $|N|$ denote the n-dimensional volume of N.

We say that a metric ball $N = B_r(x) \subset M$ is an *r-neck* if N^c has at least three unbounded components.

Coarea formula. Suppose that $f : M \to \mathbb{R}$ is a smooth function on a Riemannian manifold M, and $\phi : M \to \mathbb{R}$ is a measurable function such that $\phi|\nabla f|$ is integrable. Then (by Sard's theorem) for a.e. $t \in \mathbb{R}$, the level set $f^{-1}(\{t\}) = \{f = t\}$ is a smooth hypersurface; moreover, we have the *coarea formula*

$$\int_M \phi|\nabla f| = \int_{\mathbb{R}} \left(\int_{\{f=t\}} \phi \right) dt\,, \tag{1}$$

where the integration $\int_{\{f=t\}} \phi$ is with respect to the Riemannian measure on the hypersurface. See e.g. [3, Theorem 6.3]. The same formula apples to piecewise-smooth functions f on graphs.

The application of the coarea formula that we will need is:

$$\int_{\{t_1 \le f \le t_2\}} |\nabla f|^2 = \int_{t_1}^{t_2} \left(\int_{\{f=t\}} |\nabla f| \right) dt\,, \tag{2}$$

where we take $\phi = |\nabla f|$ on $\{t_1 \le f \le t_2\}$ and zero otherwise.

Assumption 2.3. Ubiquitous necks assumption: *There exist a number R such that R-necks cover M.*

For instance, this assumption holds if M admits a cocompact isometric action of a group with infinitely many ends.

We let $\bar{M} = M \cup \mathrm{End}(M)$ be the end compactification of M. Let $\mathcal{F} = \mathcal{F}(M)$ denote the collection of continuous functions u on $\bar{M}$, whose restriction to $\mathrm{End}(M)$ is nonconstant, and takes values in $\{0, 1\}$, while u is differentiable almost everywhere on M. We let $\mathcal{H} = \mathcal{H}(M) \subset \mathcal{F}$ denote the subset consisting of harmonic functions.

Cheeger constant. Recall that the *Cheeger constant* of a complete noncompact Riemannian manifold M is

$$\eta(M) = \inf_{U \subset M} \frac{Area(\partial U)}{Vol(U)}$$

where the infimum is taken over all bounded domains $U \subset M$ with rectifiable boundary. In other words, if $\eta(M) \ge c > 0$ (i.e., if M is non-amenable) then M satisfies the linear isoperimetric inequality with the constant c.

Quasi-isometry invariance of $\eta > 0$. One can also define Cheeger's constant for metric graphs with unit edges. It is proven by Kanai in [10] that the inequality $\eta > 0$ is a quasi-isometry invariant for quasi-isometries between Riemannian manifolds and graphs and between graphs. For the Cayley graph Γ_G of a finitely-generated group, the inequality $\eta(\Gamma_G) > 0$ is equivalent to non-amenability of G.

Theorem 2.4. *Under the assumption 2.3, $\eta(M) > 0$.*

Proof: See [16]. □

Relative version. We need a minor modification of the above setup for (metrically) complete noncompact manifolds N with boundary. For a bounded domain $\Omega \subset N$ with rectifiable boundary we define

$$\partial_1(\Omega) := \partial\Omega \cap \partial N, \quad \partial_0(\Omega) := \partial\Omega \cap int(N).$$

We then define the *relative version* of Cheeger's constant

$$\eta^{rel}(N) := \inf \frac{Area(\partial_0(\Omega))}{Vol(\Omega)}$$

where the infimum is taken over all bounded domains $\Omega \subset M$ with rectifiable boundary.

It is easy to see that for manifolds N with compact boundary

$$\eta^{rel}(N) = \eta(N),$$

since ∂N has finite area and in the definition of (absolute and relative) Cheeger's constant it suffices to consider only sequences of domains whose volumes diverge to ∞.

The first eigenvalue. Let M be a connected complete noncompact Riemannian manifold. Given a bounded domain $\Omega \subset M$ with rectifiable boundary, one defines its first eigenvalue $\lambda_1(\Omega)$ by

$$\lambda_1(\Omega) := \inf \frac{\int_\Omega |\nabla u|^2 dV}{\int_\Omega u^2 dV}$$

where the infimum is takes over all functions $u : \Omega \to \mathbb{R}$ so that $u\big|_{\partial\Omega} = 0$. Then $\lambda_1(\Omega)$ is clearly a decreasing function as Ω is increasing, so one defines the first eigenvalue $\lambda_1(M)$ of M as

$$\lambda_1(M) := \lim_{R\to\infty} \lambda_1(B_R(p))$$

where p is a fixed point in M. Cheeger's theorem [4] states that

$$\lambda_1(M) \geq \frac{\eta(M)^2}{4},$$

where $\eta(M)$ is the Cheeger's constant of M. In particular, if M satisfies a linear isoperimetric inequality, then

$$\lambda_1(M) > 0.$$

We will also need a modification of the above definition for noncompact manifolds N with boundary. For a domain $\Omega \subset N$ define

$$\lambda_1^{rel}(\Omega) := \inf \frac{\int_\Omega |\nabla u|^2 dV}{\int_\Omega u^2 dV}$$

where the infimum is takes over all functions $u : \Omega \to \mathbb{R}$ so that $u\big|_{\partial_0\Omega} = 0$ and $u\big|_{\partial_1\Omega} = 1$. As before, set

$$\lambda_1^{rel}(N) := \lim_{R\to\infty} \lambda_1^{rel}(B_R(p)).$$

(We are not sure what, if any, spectral interpretation this constant has.) Since $\eta = \eta^{rel}$ for manifolds with compact boundary, the arguments in the proof of Cheeger's inequality go through (see e.g. [17, Page 91]) and one obtains the *relative* Cheeger's inequality

$$\lambda_1^{rel}(N) \geq \frac{\eta^{rel}(N)^2}{4} = \frac{\eta(N)^2}{4}$$

for any N with compact boundary.

Facts about harmonic functions.

1. Derivative bounds. We will need the following a priori bound on the derivatives of harmonic functions u defined on a unit ball $B \subset M$ and taking values in the interval $(0, a)$:

$$|\nabla u| \leq C_1, \tag{3}$$

$$|\nabla|\nabla u|^2| \leq C_2, \tag{4}$$

where $C_i = C_i(dim(M), a, K, \delta)$, $i = 1, 2$. See e.g. [6]. (Similar bounds hold for higher derivatives as well.)

Remark 2.5. *This bound (which is a special case of a more general* elliptic regularity theory*) is the only nonelementary ingredient in the paper. The reader uncomfortable with these estimates can assume that M is a hyperbolic surface, in which case the derivative estimates follow immediately from the Cauchy integral formula for holomorphic functions. We note that an alternative proof of a bound on the gradient of u is given by S.-T. Yau, see [17, Theorem 3.1, Chapter I].*

2. An existence theorem.

Theorem 2.6. *Let M be a Riemannian manifold of bounded geometry, so that $\lambda_1(M) > 0$. Then, every continuous function χ :* End$(M) \to [0, 1]$*, admits a continuous extension to a (unique) function*

$$h = h_\chi : \bar{M} \to \{0, 1\}$$

whose restriction to M is harmonic.

Proof: This theorem was proven by Kaimanovich and Woess in [9, Theorem 5] using probabilistic methods (they also proved it for functions with values in $[0, 1]$). At the same time, an analytical proof of this result was given by Li and Tam [12], see also [14, Theorem 4.1] for a detailed and more general treatment. In Section 6, we present a self-contained proof of this theorem provided by Mohan Ramachandran. □

Remark 2.7. *Note that [12] and [14] use the* analysts notion *of an end of a Riemannian manifold, which is defined as an unbounded connected component of the complement to a bounded set in a Riemannian manifold. A topologist would this* a neighborhood *of an end.*

3. Maximum Principle.

Suppose that $\chi_1, \chi_2 : \mathrm{End}(M) \to \{0, 1\}$ are such that

$$\chi_1 \leq \chi_2.$$

Then, by the maximum principle,

$$h_{\chi_1} \leq h_{\chi_2}.$$

If the equality is attained at some point of M, then $\chi_1 = \chi_2$.

We now restrict to continuous functions $\chi : \mathrm{End}(M) \to \{0, 1\}$.

Lemma 2.8. *Each function $h = h_\chi$ has finite energy*

$$E(h) = \int_M |\nabla h|^2.$$

Proof: The assertion follows immediately from Lemma 5.3 (i) [17, Page 71]. □

Lemma 2.9. *The energy function $E : \mathcal{H}(M) \to \mathbb{R}_+$ is lower semi–continuous.*

Proof: Let $h = h_\chi \in \mathcal{H}(M)$ be the limit

$$h = \lim_{n\to\infty} h_n, \quad h_n \in \mathcal{H}(M).$$

Let $\epsilon > 0$. Pick a ball $B_r(o) \subset M$. Then, since $E(h)$ is finite (Lemma 2.8), there exists $\rho \geq r$, so that for each i,

$$E(h|M_i \setminus B_\rho(o)) \leq \epsilon.$$

Let C denote the compact in M which is the union of $B_\rho(o)$ and the compact components of $B_r(o)^c$. As uniform convergence $h_n|C$ implies uniform convergence of these functions in C^1-norm (by the first derivative estimate (3)), we obtain

$$E(h\big|_C) = \lim_{n\to\infty} E(h_n\big|_C).$$

Therefore,

$$E(h) \leq E(h\big|_C) + q\epsilon \leq \lim inf_{n\to\infty} E(h_n).$$

Since q is constant and ϵ is arbitrarily small, we obtain

$$E(h) \leq \lim inf_{n\to\infty} E(h_n). \quad \square$$

3 A lower energy bound

The key to the Gromov's proof is the following

Theorem 3.1. *Suppose that M is a complete Riemannian manifold of bounded geometry with geometric bounds K and δ, so that $\eta(M) \geq \eta > 0$.*

Then there is $\mu = \mu(dim(M), \eta, \delta, K) > 0$ such that any function $f \in \mathcal{F}(M)$ has energy at least μ.

Proof: Our arguments are motivated by the proof of Theorem 3.1 in the earlier version of this paper [11] and the alternative proof given by Bruce Kleiner, who used a combination of the coarea formula and energy comparison for functions on M and on the punctured disk in $\mathbb{R}^2$.

Let $v \in \mathcal{H}(M) \subset \mathcal{F}(M)$ be a harmonic function on M and assume that $E(v) < 1$. Then, by the coarea formula (2),

$$1 > \int_M |\nabla v|^2 = \int_0^1 \int_{\{v=t\}} |\nabla v| dS\, dt.$$

Hence, for an open set of $t \in (\frac{1}{2}, 1)$, we have

$$\int_{\{v=t\}} |\nabla v| dS \; < 2. \tag{5}$$

Take a regular value $t_0 \in (\frac{1}{2}, 1)$ of the function v satisfying (5) and consider $u := \frac{v}{t_0}$. Then

$$E(v) = t_0^2 E(u) \geq E(u)/4.$$

Furthermore, we have

$$\int_{\{u=1\}} |\nabla u| dS = \frac{1}{t_0} \int_{\{v=t_0\}} |\nabla v| dS \leq \frac{2}{t_0} < 4. \tag{6}$$

We define the submanifold $N = \{u \leq 1\}$. Our next goal is to estimate how fast the function u decays on N and derive a lower energy bound on u (and, hence, on v) from that. Our argument is modelled on the proof of Lemma 5.3 in [17, Chapter II]. Actually, as in [17, Chapter II, Lemma 5.3] one also gets a uniform estimate on the exponential decay of u, but we do not need that much.

We first observe that (by the relative Cheeger's inequality)

$$\eta(N) > \eta(M) \geq \eta > 0 \Rightarrow \lambda_1^{rel}(N) > \eta^2/4.$$

Set $\rho(x) := dist(x, \partial N)$ for $x \in N$ and let N_R denote $\{\rho \leq R\}$.

1: Integral bounds. Take $R > 0$ which is a regular value of the function ρ. Let u_R denote solution of the Dirichlet problem on N_R:

$$\Delta u_R = 0, u_R\big|_{\partial N} = 1, u_R\big|_{\{\rho=R\}} = 0.$$

Integrating the equation $u_R \Delta u_R = 0$ on N_R by parts as in [17, Chapter II, Lemma 5.3] we obtain

$$\int_{N_R} |\nabla u_R|^2 dV + \int_{\partial N} u_R \frac{\partial u_R}{\partial n} dS = 0, \tag{7}$$

where $\frac{\partial}{\partial n}$ is the normal derivative to ∂N. (The integration by parts formula is Green's Second Formula, see e.g. [3, Theorem 3.20, page 144].) Since $u_R = 1$ on ∂N, in view of (6) and (7), for large R we get

$$\int_{N_R} |\nabla u_R|^2 dV \leq 4.$$

Using the definition of $\lambda_1^{rel}(N_R)$, we obtain

$$\lambda_1^{rel}(N_R) \int_{N_R} u_R^2 dV \leq \int_{N_R} |\nabla u_R|^2 dV \leq 4.$$

Hence,

$$\int_{N_R} u_R^2 dV \leq \frac{4}{\lambda_1^{rel}(N_R)}.$$

Since the gradients of the harmonic functions u_R are uniformly bounded, the family (u_R) subconverges as $R \to \infty$ to a harmonic function u_∞ on N, which equals 1 on ∂N. By applying the relative Cheeger's inequality we obtain

$$\int_N u_\infty^2 dV \le \frac{1}{\eta^2} < \infty.$$

Therefore, $u_\infty(x)$ converges to zero as $\rho(x) \to \infty$ and, hence, $u = u_\infty$. In particular,

$$\int_N u^2 dV \le \frac{1}{\eta^2}.$$

2: Decay estimates. We now estimate the decay of u as $\rho(x) \to \infty$. First of all, for $R \ge 1$,

$$Vol(N_R) \ge \frac{\Omega R}{2}$$

where $\Omega = \Omega(K, \delta)$ is the lower bound on the volumes of the unit balls in N centered at points x, $\rho(x) \ge 1$. (Actually, since N has exponential growth, one has an exponential lower bound $Vol(N_R) \ge e^{CR}$, but a linear bound will suffice.) Therefore, for

$$m = \min_{\rho(x) \ge 1} u^2(x),$$

we obtain

$$m \frac{\Omega R}{2} \le \int_N u^2 dV \le \frac{1}{\eta^2}.$$

Thus,

$$m \le \frac{2}{\eta^2 \Omega R}.$$

(One also gets a much better estimate on $\max_{\rho(x) \ge R} u(x)$ as in [17, Chapter II, Lemma 5.3] but we do not need this.)

3: Lower energy bound. We now take $R = R(\eta, \Omega)$ so that

$$\left(\frac{2}{\eta^2 \Omega R}\right)^{1/2} \le \frac{1}{4}.$$

Pick a point $x \in N$ so that

$$u(x) = m^{1/2} \le \left(\frac{2}{\eta^2 \Omega R}\right)^{1/2} \le \frac{1}{4}.$$

Let $x' \in \partial N$ be the point nearest to x and let $\ell = d(x, x') \le R$. Then, by the mean value theorem, there exists a point y in the geodesic segment $\overline{xx'} \subset N$ so that

$$|\nabla u(y)| \ge \frac{|u(x) - u(x')|}{\ell} \ge \frac{1}{4\ell} \ge \frac{1}{4R}.$$

Hence, we obtain a lower energy-density bound at one point:

$$|\nabla u(y)|^2 \geq c := \frac{1}{16R^2}.$$

We next promote this to an energy bound for u. Let $L = C_2(dim(M), 2, K, \delta)$ be an upper 2-nd derivatives bound as in (3). Then for

$$r := \frac{c}{2L} = \frac{1}{32LR^2}$$

we have

$$d(y,z) \leq r \Rightarrow |\nabla u(z)|^2 \geq |\nabla u(y)|^2/2 \geq \frac{1}{32R^2}. \tag{8}$$

We also have

$$Vol(B_r(y)) \geq \omega = \omega(K, \delta, r) > 0 \tag{9}$$

for the r-ball in M.

By combining the inequalities (8) and (9), we obtain

$$E(u) \geq E(u\big|_{B_r(y)}) \geq \frac{\omega|\nabla u(y)|^2}{2} \geq \frac{\omega}{32R^2}. \quad \square$$

Hence,

$$E(v) = E(u)/4 \geq \epsilon := \frac{\omega}{128R^2}.$$

Since harmonic functions in $\mathcal{H}(M) \subset \mathcal{F}$ are energy-minimizers, for every $f \in \mathcal{F}$ we obtain

$$E(f) \geq E(v) \geq \mu := \min(\epsilon, 1),$$

where $v \in \mathcal{H}(M)$ is the harmonic function with the same values at infinity as f. Theorem follows. $\square$

Let $U \subset M$ be a smooth codimension 0 submanifold with compact boundary C. Recall that the *capacitance* of the pair (U, C) is the infimal energy of compactly supported functions $u : U \to [0,1]$ which are equal to 1 on C.

Corollary 3.2. *For each U and C as above, the capacitance is at least μ.*

Proof: Given a function $u : U \to [0,1]$ which equals 1 on C, we extend u by 1 to the rest of M. Then, clearly, the extension $\tilde{u}$ has the same energy as u and $u \in \mathcal{F}$. Therefore, $E(u) = E(\tilde{u}) \geq \mu$. $\square$

The main application of this result is a compactness theorem for harmonic functions in $\mathcal{H}$ which have bounded energy, provided that M has ubiquitous necks. A proof of this theorem occupies the next section.

4 A compactness theorem for harmonic functions

The arguments below were explained to me by Bruce Kleiner; they replace the earlier much more complicated proof which appeared in the preprint [11].

We first need a corollary of Theorem 3.1:

Corollary 4.1. *Assume that every point in M belongs to an R-neck. Then for all $0 < a < b < 1$, $E \in [0, \infty)$, there is an $r \in (0, \infty)$ with the following property. If $u : M \to (0, 1)$ is a proper map, and $p \in M$, then either*

1. *$u(B_r(p))$ is not contained in the interval $[a, b]$, or*
2. *The energy of u is at least E.*

Proof: Given $0 < R < \infty$ and $p \in M$, we let C denote the collection of unbounded components of $M \setminus B_R(p)$. Let $u : M \to (0, 1)$ be a proper map so that $u(B_r(p)) \subset [a, b]$. For each $U \in C$, the function u takes the values in $[a, b]$ on ∂U. Consider the two functions $u^+ = \max\{b, u\}$ and $u^- = \min\{a, u\}$ on U. Then

$$E(u^\pm) \le E(u\big|_U)$$

and $u^+\big|_{\partial U} = b, u^-\big|_{\partial U} = a$. Let $\tilde{u}^\pm$ denote the extension of $u^\pm$ to the rest of M so that

$$\tilde{u}^\pm\big|_{U^c} \equiv u^\pm\big|_{\partial U}.$$

Then

$$E(\tilde{u}^\pm) = E(u^\pm) \le E(u\big|_U).$$

Consider the function $\tilde{u}^-$: Its values on End(M) belong to $\{0, a\}$. If it does not take zero values on End(M) then $u\big|_{End(U)}$ takes only the value 1. Assuming that this does not happen, we see that $\frac{1}{a}\tilde{u}^-$ belongs to $\mathcal{F}(M)$ and, hence,

$$E(u\big|_U) \ge E(\tilde{u}^-) \ge a^2\mu$$

by Theorem 3.1. If $u\big|_{End(U)}$ takes only the value 1, then $\tilde{u}^-$ is constant (equal to a) on End(M) and we obtain no contradiction. In this case, we use the function $\tilde{u}^+$: It takes the values b and 1 on End(M). We then consider the function

$$\tilde{v} := 1 - \tilde{u}^+$$

and, similarly, obtain

$$E(u\big|_U) \ge E(\tilde{v}) \ge b^2\mu.$$

In either case, we conclude that

$$E(u|_U) \geq a^2\mu > 0.$$

Since the number of elements of C grows exponentially with R, the statement follows.

Corollary 4.2. *Suppose M is as above, and $E \in (0, \infty)$. If $u \in \mathcal{H}(M)$ has energy at most E, and u is nearly constant on a large ball B, then it is nearly equal to* 0 *or* 1 *on B. (I.e., the supremum-norm of $u|_B$ or of $(u-1)|_B$ converges to zero as $Var(u|_B) \to 0$.)*

We can now prove part 2 of the Compactness Theorem 1.2. Recall that $\mathcal{H}(M)$ is the space of $f \in \mathcal{F}(M)$ which are harmonic on M. Consider a sequence $f_n \in \mathcal{H}(M)$ and $x_n \in f_n^{-1}(1/2)$. By applying elements of the isometry group G, we can assume that the points x_n belong to a fixed compact $A \subset M$. By passing to a subsequence, we may assume that $\lim x_n = x \in A$. The gradient bound (3) implies that the derivatives of the functions f_n are uniformly bounded. Therefore, the sequence (f_n) subconverges uniformly on compacts to a harmonic function f which attains the values $1/2$ at $x \in A$. We have to show that $f \in \mathcal{H}(M)$. By lower semicontinuity of energy, f has finite energy. Suppose first that f is constant on M. Then for each $\epsilon > 0$ and $r > 0$ there exists n so that

$$Var(f_n|_{B_r(x)}) < \epsilon.$$

By taking r sufficiently large, we conclude that f_n is approximately equal to 0 or 1 on $B_r(x)$, which contradicts the assumption that $f_n(x_n) = 1/2$. Therefore, f cannot be constant.

Suppose now that f either does not extend continuously to a point $\xi \in \mathrm{End}(M)$ or that the extension $f(\xi)$ exists but is different from 0 or 1. Then there exist $a, b \in (0, 1)$ and a sequence $p_i \in M$ converging to ξ in the topology of $\bar{M}$ so that

$$0 < a \leq f(p_i) \leq b < 1, \quad \forall i.$$

Take $r = r(a, b, E+1)$ as in Corollary 4.1. Since $E(f) < \infty$, $Var(f|_{B_r(p_i)})$ converges to 0 as $i \to \infty$. (Compare Step 3 of the proof of Theorem 3.1.) Since for each fixed i

$$\lim f_n|_{B_r(p_i)} = f|_{B_r(p_i)},$$

we see that (for large n and i) the function f_n contradicts Corollary 4.1. Thus $f \in \mathcal{H}(M)$. □

Remark 4.3. *One could remove the cocompactness assumption by saying that any sequence $u_i \in \mathcal{H}(M)$ has a pointed limit $u \in \mathcal{H}(M')$ defined on a pointed Gromov–Hausdorff limit (M', x) of a sequence (M, x_n) (where M' is another bounded geometry manifold with a linear isoperimetric inequality and ubiquitous R-necks).*

5 Proof of Stallings' theorem

The goal of this section is to present the rest of Gromov's proof of the Stallings' theorem on groups with infinitely many ends. The following was proven by Stallings [18] for torsion-free groups, his proof was extended by Bergman [2] to groups with torsion:

Theorem 5.1 (Stallings, Bergman)**.** *Let G be a finitely–generated group with infinitely many ends. Then G splits nontrivially as a graph of groups with finite edge groups.*

Proof: Our argument is a slightly expanded version of Gromov's proof in [7, Pages 228–230]. Since G is finitely–generated, it admits a cocompact isometric properly discontinuous action $G \curvearrowright M$ on a connected Riemannian manifold M. For instance, if G is k–generated, and F is a Riemann surface of genus k, we have an epimorphism

$$\phi : \pi_1(F) \to G.$$

Then G acts isometrically and cocompactly on the covering space M of F so that $\pi_1(M) = \ker(\phi)$. Thus, M has infinitely many ends. The manifold M has bounded geometry since it covers a compact Riemannian manifold.

Let $\mathcal{H}(M)$ denote the space of harmonic functions $h : M \to (0, 1)$ as in the Introduction. According to Theorem 1.2, there exists a function $h \in \mathcal{H}(M)$ with minimal energy $E(h) = e(M) > 0$. Then, for every $g \in G$, the function

$$g^* h := h \circ g$$

has the same energy as h and equals

$$h_{g^*(\chi)}.$$

For $g \in G$, define

$$g_+(h) := \max(h, g^*(h)), \quad g_-(h) := \min(h, g^*(h)).$$

Define the *nodal set*

$$\Lambda = \Lambda_g := \{x : h(x) = g^* h(x)\} = \{x : h(x) = h(g(x))\} \subset M.$$

We note that if $h \neq g^*(h)$ then, by [8] or [1], the set Λ has measure zero. In particular,

$$E(g_\pm(h)|_{M\setminus\Lambda}) = E(g_\pm(h)).$$

Lemma 5.2.

$$E(g_+(h)) + E(g_-(h)) = 2E(h).$$

Proof: Without loss of generality, we may assume that $h \neq g^*(h)$. Set

$$M_- := \{x \in M : h(x) > g^* h(x)\}, M_+ := \{x \in M : h(x) < g^* h(x)\}.$$

We obtain:

$$E(g_+(h)) + E(g_-(h)) =$$

$$\int_{M_-} |\nabla h(x)|^2 + \int_{M_+} |\nabla g^* h(x)|^2 + \int_{M_-} |\nabla g^* h(x)|^2 + \int_{M_+} |\nabla h(x)|^2 =$$

$$= E(h) + E(g^*(h)) = 2E(h). \quad \square$$

Remark 5.3. *A direct proof of the above lemma (which does not use the geometry of nodal sets) due to Mohan Ramachandran is presented in the Appendix 2, Lemma 7.2.*

Note that the functions $g_+(h), g_-(h)$ have continuous extension to $\bar{M}$ (since h does and G acts on $\bar{M}$ by homeomorphisms). By construction, the restrictions

$$\chi_+ := g_+(h)\big|_{\mathrm{End}(M)}, \quad \chi_- := g_-(h)\big|_{\mathrm{End}(M)}$$

take the values 0 and 1 on $\mathrm{End}(M)$. Let

$$h_\pm := h_{\chi_\pm}$$

denote the corresponding harmonic functions on M. Then

$$E(h_\pm) \leq E(g_\pm(h)),$$

$$E(h_+) + E(h_-) \leq 2E(h) = 2e(M).$$

Note that it is, a priori, possible that χ_- or χ_+ is constant. Set

$$G_c := \{g \in G : \chi_- \text{ or } \chi_+ \text{ is constant}\}.$$

We first analyze the set $G \setminus G_c$. For $g \notin G_c$, both h_- and h_+ belong to $\mathcal{H}(M)$ and, hence,

$$E(h_+) = E(h_-) = E(h) = e(M).$$

Therefore,

$$E(g_+(h)) = E(h_+), \quad E(g_-(h)) = E(h_-).$$

It follows that $g_{\pm}(h)$ are both harmonic. Since

$$g_{-}(h) \leq g_{+}(h),$$

the maximum principle implies that either $g_{-}(h) = g_{+}(h)$ or $g_{-}(h) < g_{+}(h)$. Hence, the set Λ_g is either empty or equals the entire M, in which case $g^*(h) = h$. Therefore, for every $g \in G \setminus G_c$ one of the following holds:

1. $g^*h = h$.
2. $g^*h(x) < h(x),\ \forall x \in M$.
3. $g^*h(x) > h(x),\ \forall x \in M$.

Thus, the set

$$L := h^{-1}\left(\frac{1}{2}\right)$$

is *precisely–invariant* under the elements of $G \setminus G_c$: for every $g \in G \setminus G_c$, either

$$g(L) = L$$

or

$$g(L) \cap L = \emptyset.$$

We now consider the elements of G_c. Suppose that g is such that $\chi_- = 0$. Then

$$g^*(\chi) \leq 1 - \chi$$

and, hence,

$$g^*(h) \leq 1 - h.$$

Since these functions are harmonic, in the case of the equality at some $x \in M$, by the maximum principle we obtain $g^*(h) = 1 - h$. The latter implies that

$$g(L) = L.$$

If

$$g^*(h) < 1 - h$$

then $g(L) \cap L = \emptyset$. The same argument applies in the case when χ_+ is constant.

To summarize, for every $g \in G$ one of the following holds:

$$g^*h = h,\ \ g^*h < h,\ \ g^*h > h,\ \ g^*h = 1-h,\ \ g^*h < 1-h,\ \ g^*h > 1-h. \qquad (10)$$

We conclude that L is precisely–invariant under the action of the entire group G. Moreover, if $g(L) = L$ then either $g^*h = h$ or $g^*h = 1 - h$. Since L is compact, its stabilizer G_L in G is finite.

By construction, the hypersurface L separates M in at least two unbounded components.

Since L is compact, there exists $t \in (0,1) \setminus \frac{1}{2}$ sufficiently close to $\frac{1}{2}$, which is a regular value of h, so that the hypersurface $S := h^{-1}(t)$ is still precisely–invariant under G. Let $G_S \subset G_L$ denote the stabilizer of S in G.

We now show that G splits nontrivially over a subgroup of G_S. (The proof is straightforward under the assumption that S is connected, but requires extra work in general.) We proceed by constructing a simplicial G–tree T on which T acts without inversions, with finite edge–stabilizers and without a global fixed vertex.

Construction of T. Consider the family of functions $\mathcal{M} = \{f = g^* h : g \in G\}$. Each function $f \in \mathcal{M}$ defines the *wall* $W_f = \{x : f(x) = t\}$ and the *half-spaces* $W_f^+ := \{x : f(x) > t\}$, $W_f^- := \{x : f(x) < t\}$ (these spaces are not necessarily connected).

Let $\mathcal{E}$ denote the set of walls. We say that a wall W_f *separates* $x, y \in M$ if

$$x \in W_f^+, \quad y \in W_f^-.$$

Maximal subsets V of

$$M^o := M \setminus \bigcup_{f \in \mathcal{H}} W_f$$

consisting of points which cannot be separated from each other by a wall, are called *indecomposable* subsets of M^o. Note that such sets need not be connected. Set

$$\mathcal{V} := \{\text{indecomposable subsets of } \ M^o\}.$$

We say that a wall W is *adjacent* to $V \in \mathcal{V}$ if $W \cap cl(V) \neq \emptyset$.

The next lemma follows immediately from the inequalities (10), provided that t is sufficiently close to $\frac{1}{2}$:

Lemma 5.4. *No wall W_{f_1} separates points of another wall W_{f_2}.*

Lemma 5.5. *1. Let $V \in \mathcal{V}$ and $W \in \mathcal{E}$ be adjacent to V. Then, for each component C of V, we have $C \cap W \neq \emptyset$.*

2. $W \in \mathcal{E}$ is adjacent to $V \in \mathcal{V}$ iff $W \subset cl(V)$.

Proof: 1. Suppose that $V \subset W^+$. A generic point $x \in C$ is connected to $W = W_f$ by a gradient curve $p : [0,1] \to M$ of the function f. The curve p crosses each wall at most once. Since V is indecomposable and for sufficiently small $\epsilon > 0$, $p(1-\epsilon) \in V$, it follows that p does not cross any walls. Therefore the image of p is contained in the closure of C and $p(1) \in W \cap cl(C)$

2. Lemma 5.4 implies that for $x, y \in W^+$ (resp. $x, y \in W^-$) which are sufficiently close to W, there is no wall which separates x from y. Therefore, such points x, y belong to the same indecomposable set V^+ (resp. V^-) which is adjacent to W and $W \subset cl(V^\pm)$. Clearly, V^+, V^- are the only indecomposable sets which are adjacent to W. □

Hence, each wall W is adjacent to exactly two elements of $\mathcal{V}$ (contained in W^+, W^- respectively). We obtain a graph T with the vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, where a vertex V is incident to an edge W iff the wall W is adjacent to the indecomposable set V.

From now on, we abbreviate W_{f_i} to W_i.

Lemma 5.6. *T is a tree.*

Proof: By construction, every point of M belongs to a wall or to an indecomposable set. Hence, connectedness of T follows from connectedness of M.

Let

$$W_1 - V_1 - W_2 - .... - W_k - V_k - W_1$$

be an embedded cycle in T. This cycle corresponds to a collection of paths $p_j : [0, 1] \to cl(V_j)$, so that

$$p_j(0) \in W_j, \quad p_j(1) \in W_{j+1}, j = 1, ..., k$$

and points of $p_j([0, 1])$ are not separated by any wall, $j = 1, ..., k$. By Lemma 5.4, the points $p_j(1), p_{j+1}(0)$ are not separated by any wall either. Therefore, the points of

$$\bigcup_{j=1}^{k} p_j([0, 1])$$

are not separated by W_1. However,

$$p_1((0, 1]) \subset W_1^+, \quad p_k([0, 1)) \subset W_1^-$$

or vice–versa. Contradiction. □

We next note that G acts naturally on T since the sets $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{V}$ are G–invariant and G preserves adjacency. If $g(W_f) = W_f$, then $g^* f = f$, which implies that g preserves W_f^+, W_f^-. Hence, g fixes the end-points of the edge corresponding to W, which means that G acts on T without inversions. The stabilizer of an edge in T corresponding to a wall W is finite, since W is compact and G acts on M properly discontinuously.

Suppose that $G \curvearrowright T$ has a fixed vertex. This means that the corresponding indecomposable subset $V \subset M$ is G–invariant. Since G acts cocompactly on M, it follows that $M = B_r(V)$ for some $r \in \mathbb{R}_+$. The indecomposable subset V is contained in the half-space W^+ for some wall W. Since W is compact and W^- is not, the subset W^- is not contained in $B_r(W)$. Thus $W^- \setminus B_r(V) \neq \emptyset$. Contradiction.

Therefore T is a nontrivial G–tree and we obtain a nontrivial graph of groups decomposition of G where the edge groups are conjugate to subgroups of the finite group G_S. □

6 Appendix 1: "An existence theorem for harmonic functions" by Mohan Ramachandran

Theorem 6.1. *Let* $\chi : \mathrm{End}(M) \to \{0, 1\}$ *be a continuous function. Then* χ *admits a harmonic extension to* M.

Proof: (M. Ramachandran.) Let φ denote a smooth extension of χ to M so that $d\varphi$ is compactly supported.

We let $W_o^{1,2}(M)$ denote the closure of $C_c^\infty(M)$ with respect to the norm

$$||u|| := ||u||_{L_2} + \sqrt{E(u)}.$$

Consider the affine subspace of functions

$$\mathcal{G} := \varphi + W_o^{1,2}(M) \subset L^2_{loc}(M).$$

Then the energy is well-defined on $\mathcal{G}$ and we set $E := \inf_{f \in \mathcal{G}} E(f)$.

Note that, since $\mathcal{G}$ is affine, for $u, v \in \mathcal{G}$ we also have

$$\frac{u+v}{2} \in \mathcal{G},$$

in particular,

$$E(\frac{u+v}{2}) \geq E$$

and we set

$$E(u, v) := 2E(\frac{u+v}{2}) - \frac{E(u) + E(v)}{2}.$$

The latter equals

$$E(u, v) := \int_M \langle \nabla u, \nabla v \rangle$$

in the case when u, v are smooth. We thus obtain

$$E(u, v) \geq 2E - \frac{E(u) + E(v)}{2}$$

for all $u, v \in \mathcal{G}$. Hence,

$$E(u - v) = E(u) + E(v) - 2E(u, v) \leq 2E(u) + 2E(v) - 4E. \tag{11}$$

Pick a sequence $u_n \in \mathcal{G}$ such that

$$\lim_{n\to\infty} E(u_n) = E.$$

Then, according to (11),

$$E(u_m - u_m) \leq 2E(u_n) + 2E(u_m) - 4E = 2(E(u_n) - E) + 2(E(u_m) - E).$$

Since $\lambda := \lambda_1(M) > 0$, we obtain

$$\lambda \int_M f^2 \leq E(f) \tag{12}$$

for all $f \in W_o^{1,2}(M)$. Therefore, the functions $v_n := u_n - \varphi \in W_o^{1,2}(M)$ satisfy

$$\|v_n - v_m\| \leq (2 + \lambda^{-1})(E(u_n) - E + E(u_m) - E).$$

Hence, the sequence (v_n) is Cauchy in $W_o^{1,2}(M)$. Set

$$v := \lim_n v_n, u := \varphi + v \in \mathcal{F}.$$

By semicontinuity of energy, $E(u) = E$. Therefore, u is harmonic and, hence, smooth. Since $d\varphi$ is compactly supported, the function v is also harmonic away from a compact subset $K \subset M$. By the inequality (12), we have

$$\int_M v^2 \leq \lambda^{-1} E(v) < \infty. \tag{13}$$

Let $r > 0$ denote the injectivity radius of M. Pick a base-point $o \in M$. Then (13) implies that there exists a function $\rho : M \to \mathbb{R}_+$ which converges to 0 as $d(x, o) \to \infty$, so that

$$\int_{B_r(x)} v^2(x) \leq \rho(x)$$

for all $x \in M$. By the gradient estimate, there exists $C_1 < \infty$ so that

$$\sup_{B_r(x)} v^2 \leq C_1 \inf_{B_r(x)} v^2$$

provided that $d(x, K) \geq r$. Therefore,

$$v^2(x) \leq \frac{C_1}{Vol(B_r(x))} \int_{B_r(x)} v^2 \leq C_2 \rho(x).$$

Thus

$$\lim_{d(x,o) \to \infty} v(x) = 0.$$

Therefore the harmonic function u extends to the function χ on End(M). □

7 Appendix 2: "Energy of minimum and maximum of two smooth functions" by Mohan Ramachandran

Let M be a smooth manifold and f be a C^1-smooth function on M. Define the function $f^+ := max(f, 0)$ and the closed set

$$\Gamma := \{x \in M : f(x) = 0, df(x) = 0\}.$$

Set

$$\Omega := \{x \in M : f(x) = 0, df(x) \neq 0\} = f^{-1}(0) \setminus \Gamma.$$

By the implicit function theorem, Ω is a smooth submanifold in M and, hence, has measure zero. Clearly, f^+ is smooth on $M \setminus \Omega$.

Lemma 7.1. *Under the above conditions, a.e. on M we have:*
$df^+(x) = df(x)$ *if* $f(x) > 0$ *and* $df^+(x) = 0$ *if* $f(x) \leq 0$.

Proof: Since Ω has measure zero, it suffices to prove the assertion for points $x_0 \in \Gamma$. Choose local coordinates on M at a point $x_0 \in \Gamma$, so that $x_0 = 0$. Since f has zero derivative at 0, we have:

$$\lim_{v \to 0} \frac{|f(v)|}{||v||} = 0.$$

Since $0 \leq |f^+| \leq |f|$, it follows that

$$\lim_{v \to 0} \frac{|f^+(v)|}{||v||} = 0.$$

Therefore, f^+ is differentiable at x_0 and $df^+(x_0) = 0$. □

Consider now two C^1-smooth functions f_1, f_2 on M. Define

$$f_{max} := \max(f_1, f_2), \quad f_{min} := \min(f_1, f_2), \quad f := f_1 - f_2.$$

Lemma 7.2. $E(f_1) + E(f_2) = E(f_{max}) + E(f_{min})$.

Proof: Set

$$M_1 := \{f_1 > f_2\}, M_2 := \{f_2 > f_1\}, M_0 := \{f_1 = f_2\}.$$

Since

$$f_{max} = f_2 - f^+, \quad f_{min} = f_1 - f^+,$$

by the above lemma we have:

$$\nabla f_{max} = \nabla f_2, \quad \nabla f_{min} = \nabla f_1 \quad \text{a.e. on} \quad M_0.$$

Clearly,

$$\nabla f_{max} = \nabla f_i|_{M_i}, \nabla f_{min} = \nabla f_{i+1}|_{M_{i+1}}, i = 1, 2.$$

Hence,

$$\int_{M_i} |\nabla f_{max}|^2 + |\nabla f_{min}|^2 = \int_{M_i} |\nabla f_1|^2 + |\nabla f_2|^2, i = 0, 1, 2.$$

Therefore,

$$E(f_1) + E(f_2) = E(f_{max}) + E(f_{min}). \quad \square$$

8 Appendix 3: Harmonic functions on graphs

The goal of this appendix is to explain how to modify the arguments of this paper in order to replace harmonic functions on manifolds with harmonic functions on graphs.

Let Γ be a connected locally-finite metric graph with the vertex set V and the edge set E. We will be assuming that every pair of vertices in Γ is connected by at most one edge and that no edge connects a vertex to itself. We will use the notation $|e|$ to denote the length of an edge $e = [x, y]$ in Γ. For a subgraph $\Lambda \subset B$ we let $\partial\Lambda$ denote the topological frontier of Λ in Γ. Clearly, $\partial\Lambda \subset V$. We will abuse the terminology and, given a graph Γ (which is not regarded as a subgraph of another graph), we let $\partial\Gamma$ denote the set of leaves of Γ, i.e., the set of vertices of valence 1. This is the usual conflation between the notation for the topological frontier and the boundary of a manifold.

We will consider the space $C(\Gamma)$ of continuous functions on Γ which are linear on the edges. Every such function is determined by its restriction to V. A point $p \in \Gamma$ is said to be *regular* for a function $f \in C(\Gamma)$ if p belongs to the interior of an edge e of Γ and p is a regular point for $f\big|_e$. Accordingly, we define regular values of f. Then, for a function $f \in C(\Gamma)$, almost every $t \in \mathbb{R}$ is a regular value of f.

Given $f \in C(\Gamma)$ and an oriented edge $e = [x, y]$ we have the gradient

$$\nabla_e f = \frac{f(y) - f(x)}{|e|}.$$

It will be convenient to think of ∇f as a vector field defined on $\Gamma \setminus V$ where f is differentiable in the usual sense. Accordingly, the energy of f is

$$E(f) = \int_\Gamma (\nabla f)^2 = \sum_{e \in E} (\nabla_e f)^2 |e|.$$

Here and in what follows we integrate with respect to the natural Lebesgue measure on Γ, in particular, V has measure zero.

With this definition it is clear that if $0 \le f \le C$ on Γ then $|\nabla_e f| \le C$ for every $e \in E$ and, hence, for all edges $e_1, e_2 \in E$ we have

$$|\nabla_{e_1} f - \nabla_{e_2} f| \le C.$$

In this sense, the second derivative of a bounded function f is also a priopri bounded. (This observation replaces the a priori bounds on the 1-st and 2-nd derivatives of harmonic functions on manifolds.)

The coarea formula for the functions $f \in C(\Gamma)$ and measurable functions ϕ on Γ is immediate.

For $f \in C(\Gamma)$ define the *Laplacian* Δf as a function $\Delta f : V \to \mathbb{R}$,

$$\Delta f(x) = \sum_{[x,y]} \nabla_{[x,y]} f$$

where the sum is taken over all edges $[x, y]$ containing the vertex x. It is convenient to set $\Delta f(z) = 0$ for every $z \in \Gamma \setminus V$ since f is linear on the edges of Γ. A function $f \in C(\Gamma)$ is *harmonic* if $\Delta f = 0$ on V.

One verifies that f is harmonic iff it is locally energy-minimizing, i.e., for every finite subgraph $\Lambda \subset \Gamma$ the function $f\big|_\Lambda$ is energy-minimizing among all functions $g \in C(\Lambda)$ so that $g\big|_{\partial\Lambda} = f\big|_{\partial\Lambda}$.

The integration by parts formula for functions on graphs is

$$\sum_{x \in V} \Delta f(x) g(x) = -\frac{1}{2} \sum_{x,y \in V} (\nabla_{[x,y]} f)(\nabla_{[x,y]} g)|xy| = -\sum_{e \in E} (\nabla_e f)(\nabla_e g)|e|$$

where $f, g \in C(\Gamma)$ and either f or g has finite support on V, see e.g. [5]. The reader can easily verify this formula using induction on the number of vertices in the support set. The integration by parts formula can be also rewritten as

$$\int_\Gamma \Delta f(z) g(z) = \int_\Gamma \nabla f(z) \nabla g(z).$$

Suppose now that $\Gamma = \Lambda$ is a finite graph with the set of leaves $\partial\Lambda$. Assume that $f = g$ and f is harmonic on $int(\Lambda) = \Lambda \setminus \partial\Lambda$. Then the integration by parts formula becomes

$$\int_{\partial\Lambda} f(x)\frac{\partial f(x)}{\partial n} = \sum_{x\in\partial\Lambda} f(x)\nabla_{[x,y]}f(x) = -\int_{\Gamma}|\nabla f|^2$$

where n is the tangent vector to Λ at x directed inward. Here we equip $\partial\Lambda$ with the obvious counting measure. With this notation, the integration by parts formula is the exact analogue of the one used in Step 1 of the proof of Theorem 3.1 with Λ playing the role of the submanifold with boundary N_R and f replacing the function u_R. With these observations, the arguments of the proof of the compactness theorem for harmonic functions go through with a Cayley graph Γ of the group G replacing the manifold Riemannian M. Here we assume that the edges of the Cayley graph have unit length. Sublevel sets $\{f \leq t\}$ of functions $f \in C(\Gamma)$ with t being a regular value of f, replace submanifolds with smooth boundary $\{f \leq t\}$ in M. The sublevel sets $\{f \leq t\} \subset \Gamma$ have obvious structure of metric graphs with the metric induced from Γ, this is why we set up the formalism of harmonic functions on metric graphs in the first place. (The more traditional viewpoint is to consider only graphs with unit edges.) The arguments in Section 5 remain virtually unchanged, we only note that gradient curves of functions in $C(\Gamma)$ may have branch points at the vertices of the graph but this does not affect the arguments.

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