ON HYPERBOLIC 4-MANIFOLDS FIBERED OVER SURFACES

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Abstract. In this paper we prove that for any hyperbolic 4-manifold M , which is $\mathbb{R}^2$ bundle over surface Σ, the absolute value of the Euler number $e(\xi)$ of the fibration $\xi : M \to \Sigma$ is not greater than $\exp(\exp(10^8|\chi(\Sigma)|))$. This result partially corroborates a conjecture of Gromov, Lawson and Thurston that $|e(\xi)| \le |\chi(\Sigma)|$.

1. INTRODUCTION

1.1. The problem:
"Which 4-manifolds fibered over surfaces can have a complete hyperbolic metric ?" first appears in the paper of W.Goldman [G] in the context of flat conformal structures on 3-manifolds. It is easy to construct a hyperbolic structure on the trivial $\mathbb{R}^2$–bundle over any surface Σ of nonpositive Euler characteristic. On the other hand, the total space of any nontrivial orientable $\mathbb{R}^2$-bundle over the torus can't have any hyperbolic metric metric. The problem becomes much more difficult in the case of $\mathbb{R}^2$–bundles over hyperbolic surfaces. First examples of hyperbolic metrics on such manifolds were independently constructed by M.Gromov, H.B.Lawson and W.Thurston [GLT] and the author [Ka 1, Ka 4]. The original constructions and estimates were considerably improved in papers of N.Kuiper [Ku 1, Ku 2] and F.Luo [L]. The best (to the current date) result in this direction was obtained by N.Kuiper and P.Waterman [K W]:

THEOREM 1. For any pair of integers (e, g) such that

$$|e| \le g - 1 \tag{1.1}$$

the manifold $M^4 = M(g, e)$ admits a complete hyperbolic structure. Here and below we denote by $M(g, e)$ the total space of the oriented $\mathbb{R}^2$–bundle $\xi : M(g, e) \to \Sigma$, where $e = e(\xi)$ is the Euler number of ξ and g is the genus of Σ.

CONJECTURE 1 [GLT]. The inequality

$$|e| \le 2g - 2 \tag{1.2}$$

is necessary condition for existence of a complete hyperbolic structure on the manifold $M(g, e)$.

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There are several reasons in favor of this conjecture. All known examples satisfy the inequality (1.2). It was proven by N.Kuiper [Ku 3] and V.Marenich [Mar] that for any hyperbolic 4-manifold M and any *imbedded* minimal surface $\Sigma \subset M$ with the genus g and self-intersection number e the inequality (1.2) is satisfied (actually in such case $|e| < 2g-2$). Finally (as it was noticed in [GLT]), the inequality (1.2) appears in many cases in low-dimensional topology (we shall discuss this in the section 6).

1.2. The present paper is another corroboration of Conjecture 1. We prove

THEOREM 2. Suppose that M is a complete oriented hyperbolic 4-manifold so that $\pi_1(M) \cong \pi_1(\Sigma)$ where Σ is a closed oriented surface of genus g. Then

$$| < [\Sigma], [\Sigma] > | \leq \exp(\exp(10^8 |\chi(\Sigma)|)) \tag{1.3}$$

where $< [\Sigma], [\Sigma] >$ is the value of the intersection form of M on the generator of $H_2(M)$ represented by the homotopy– equivalence $\Sigma \to M$.

So, our result is valid for a class of 4-manifolds which is slightly bigger than the class of plane bundles over Σ. In such class we can't expect the estimate $| < [\Sigma], [\Sigma] > | \leq 2(g-1)$ because of another example constructed by N.Kuiper:

THEOREM 3 [Ku 2]. There is a sequence of complete hyperbolic 4–manifolds M_g which are homotopy equivalent to closed surfaces Σ_g of genus g such that:

$$\lim_{g\to\infty} < [\Sigma_g], [\Sigma_g] > /(2g-2) = 2/\sqrt{3} > 1 \tag{1.4}$$

In this paper we also prove the following application of our results to flat conformal structures on 3-manifolds. Denote by $\mathcal{S}(g, e)$ the orientable 3-manifold which is a circle bundle over the closed oriented surface of genus g such that the Euler number of the fibration is e.

THEOREM 4. The condition

$$|e| \leq \exp(\exp(10^8 |\chi(\Sigma)|))$$

is necessary for existence of flat conformal structures with nonsurjective development maps on the manifold $\mathcal{S}(g, e)$.

1.3. One can try to generalize the results of this paper in several directions.

THEOREM 5. There exists a function $f(B)$ of negatively curved oriented compact k-manifold B so that:

if the total space of $\mathbb{R}^k$–bundle $\xi : M^{2k} \to B$ has a complete hyperbolic metric, then

$$|e(\xi)| \leq f(B) \tag{1.5}$$

This result is an application of a deep compactness theorem due to Thurston, Morgan, Rips, Bestvina and Feighn. Unfortunately we have no any idea how the function f looks like.

In the subsequent paper [Ka 6] we will prove the following generalization of Theorem 2:

THEOREM 6. There exists a function $C(.,.)$ such that for any complete hyperbolic 4-manifold M and for any classes $[\sigma_1], [\sigma_2]$ in $H_2(M, \mathbb{Z})$ which have incompressible representatives $\sigma_j : \Sigma_j \to M$, we have:

$$| < [\sigma_1], [\sigma_2] > | \leq C(|\chi(\Sigma_1)|, |\chi(\Sigma_2)|) \tag{1.6}$$

1.4. Probably it's possible to prove Theorem 2 for certain hyperbolic 4-manifolds by comparing two η-invariants for flat conformal manifolds at infinity [Ka 2, 3]. More realistic idea was suggested to the author by M.Gromov who proposed to compactify the moduli space of all hyperbolic structures on the given fiber bundle. Formally speaking this idea doesn't work, since arbitrary large number of self-intersections of zero section can be pinched to point in the limit. However, what we are using in this paper are "pre-limit" considerations based on Mamford's compactness theorem and existence of the Margulis constant.

1.5. **Idea of the proof of Theorem 2.**

Suppose that the radius of injectivity of the manifold M is not less than ϵ. Then we can realize the class $[\sigma]$ by two transversal immersed piecewise-geodesic surfaces Σ_1, Σ_2 in M so that :

the number of simplices in both Σ_1, Σ_2 is bounded from above by $4(g-1)$ and diameter of each triangle is bounded from above by $(2g-2)/\epsilon$. Then the fact that 2 geodesic planes in $\mathbb{H}^4$ intersect transversally by not more than one point implies that the number of points of intersection between Σ_1, Σ_2 is at most

$$8(g-1)(2g-1)\exp(12(g-1)/\epsilon + \epsilon/2) \cdot \epsilon^{-3} \tag{1.7}$$

This implies the assertion of Theorem 2. One can improve this estimate by constructing Σ_j such that the diameter of each triangle is at most ϵ and the number of triangles is at most

$$10^7(g-1)/\epsilon^2 \tag{1.8}$$

(see Lemma 8). Then the number of points of intersection is at most

$$2 \cdot 10^{20}\epsilon^{-5}\exp(9\epsilon)(g-1)^2$$

Even so, the estimate is quadratic with respect to g.

It's impossible in general to estimate from below the radius of injectivity by a universal constant. However one can construct piecewise-geodesic surfaces Σ_j so that the "long" triangles of Σ_1, Σ_2 are contained in the $\epsilon(g)$–"thin" part of the manifold M which has very simple topological structure. Unfortunately, the function $\epsilon(g)$ has exponential decay as $g \to \infty$. Therefore, our estimate of the number of points of intersection of "small" triangles is at least an exponential function of g. The double exponent in (1.3) appear because of intersections between "short" and "long" triangles. Apriori, the "long triangles" in Σ_j can have a lot of intersections since their diameter is apriori unbounded from above and the radius of injectivity at these triangles is apriori unbounded from below. However, detailed analysis of the geometry in the "thin" part of M (Section 2) and correct choice of the surfaces Σ_j (Section 3) give the desired result.

In the section 2 we discuss the geometry of components of the "thin" parts of hyperbolic manifolds ("Margulis tubes "). The following is the reason of the difference between dimensions 3 and 4. Let $< g >$ be an infinite cyclic group of (orientation preserving) isometries of $\mathbb{H}^n$. Consider the set of points $\mathcal{K}(< g >, \nu) = \{x \in \mathbb{H}^n : d(x, g^k(x)) \leq \nu$ for some $\nu \neq 0\}$; define

$$q(x) = \text{ minimal } k > 0 \text{ such that } d(x, g^k(x)) \leq \nu$$

Then, for $n \leq 3$ the function q is constant on $\partial\mathcal{K}(< g >, \nu)$. However it's not longer true for $n \geq 4$. It's well known that the situation is the worst in the case of parabolic g, when $q(x)$ can have infinitely many different values. If g is loxodromic, then the image of q is still finite, but it depends on the element g. Something similar occurs even for $n \leq 3$ if g doesn't preserve the orientation; however, in this case q can have not more than 2 different values.

1.6. The first version of this paper and [Ka 6] was published by the author as a preprint of MSRI [Ka 5]. The proof in [Ka 5] is essentially the same as in the present paper, but in [Ka 5] we had an error in evaluation of the upper bound on the Euler number (single exponent instead of the double exponential function).

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2. GEOMETRY OF MARGULIS CONES

Many results of this section are well known, we present their proofs for the sake of completeness.

2.1. DEFINITIONS AND NOTATIONS.

We shall consider the n-dimensional hyperbolic space $\mathbb{H}^n$ which has the curvature (-1), $d(. \, , \, .)$ will denote the distance in $\mathbb{H}^n$. We shall use the upper half-space model for $\mathbb{H}^n$:

$$\mathbb{H}^n = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\} \tag{2.1}$$

$$\sinh(d(a,b)/2) = \frac{|a-b|}{2(a_n b_n)^{1/2}} \tag{2.2}$$

where $|a-b|$ is the Euclidean distance between $a, b \in \mathbb{R}^n_+$. The ideal boundary of $\mathbb{H}^n$ is $\mathbb{S}^{n-1} = \overline{\mathbb{R}}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$.

For each pair of points $a, b \in \mathbb{H}^n$ we shall denote by $[a, b]$ the geodesic segment connecting them. We denote by $[a, b, c]$ the totally geodesic 2-dimensional triangle with the vertices $a, b, c \in \mathbb{H}^n$. The union of edges of a triangle Δ is denoted by $\Delta^{(1)}$. We shall denote by $\Delta^{(0)}$ the set of vertices of a triangle Δ. If Ω is a triangulation, then $\Omega^{(j)}$ will denote the j-dimensional skeleton of Ω.

If N is a metric space then by $l_N(\gamma)$ we denote the length of the curve γ in the space N.

We shall denote by

$$dist(x, Y) = d(x, Y) = \inf\{d(x, y) : y \in Y\}$$

the hyperbolic distance between the one-point set $\{x\}$ and a nonempty set $Y \subset \mathbb{H}^n$. The open ball with the center at x and radius r will be denoted by $B(x, r)$. If N is a Riemannian manifold then $RadInj_N(z)$ denotes the injectivity radius of N at the point $z \in N$. If $h \in \text{Isom}(\mathbb{H}^n)$ then

$$\tau_h : x \mapsto d(x, h(x)) \tag{2.3}$$

is the **displacement function** of h. We shall denote by $l(h)$ the infimum of τ_h in $\mathbb{H}^n$. The set

$$Q_r = \{x : \tau_h(x) \leq r\} \tag{2.4}$$

is known to be convex for every r (see [BGS]). We recall that the element $h \in \text{Isom}(\mathbb{H}^n)$ is called **loxodromic** if it has exactly two fixed points in $\mathbb{H}^n \cup \mathbb{S}^{n-1}$. The element h is **parabolic** if it has only one fixed point in $\mathbb{S}^{n-1}$.

If $G \subset \text{Isom}(\mathbb{H}^n)$, $x \in \mathbb{H}^n$ then $Ir_G(x) = dist(x, Gx)/2$ is the injectivity radius $RadInj([x])$ at the projection $[x]$ of x in $\mathbb{H}^n/G$. We shall assume that all groups below are torsion-free.

If h is a loxodromic then A_h will denote the **axis** of h which is a geodesic where τ_h attains its minimum. Suppose that h is a parabolic element with the fixed point $p \in \mathbb{S}^{n-1}$. Then any horosphere F in $\mathbb{H}^n$ which is tangent to $\mathbb{S}^n$ at the point p is invariant under h. There exists at least one hyperbolic plane X in $\mathbb{H}^n$ which is invariant under h. We choose X and F and denote by $A_h = F \cap X$ **an invariant horocycle** of h. This horocycle (unlike the axis of a loxodromic element) is not uniquely determined by h. However, all invariant horocycles of h are parallel in the sense that for any two invariant horocycles A_h, A'_h the function $dist(x, A_h)$ is constant on A'_h.

If h is a loxodromic or parabolic transformation in $\mathbb{H}^n$ then we denote by Π the **canonical foliation** of $\mathbb{H}^n$ by totally geodesic hyperplanes orthogonal to A_h. It is easy to see that this foliation doesn't depend on the choice of A_h in the parabolic case. The projection of Π to $\mathbb{H}^n/ < h >$ will be called the **canonical foliation** of $\mathbb{H}^n/ < h >$ associated with $< h >$.

For any almost Abelian discrete group $H \subset \text{Isom}(\mathbb{H}^n)$ define the **Margulis cone** as

$$\mathcal{K}(H, \nu) = \{z \in \mathbb{H}^n : Ir_H(z) \leq \nu/2\} \tag{2.5}$$

The quotient

$$\mathcal{T}(H, \nu) = \mathcal{K}(H, \nu)/H \tag{2.6}$$

is the **Margulis tube** in $\mathbb{H}^n/H$ (we assume that Margulis tubes can be noncompact; in particular cusps are also considered as Margulis tubes).

The set $\mathcal{K}(< h >, \nu)$ is the union of convex subsets

$$\{\tau_{h^n}(x) \leq \nu/2\}_{n \in Z\setminus\{0\}} \tag{2.7}$$

but in general it is not convex itself.

Suppose that the hyperbolic n-space is realized as the "upper half-space", and the loxodromic element h is a Euclidean similarity. The problem concerned with Margulis tubes in the dimension 4 (and higher) is that even for the cyclic loxodromic group $< h >= H \subset \text{Isom}(\mathbb{H}^n)$ the boundary of the Margulis cone $\mathcal{K}(H,\nu)$ is very far from been a "round" cone (like in dimensions 2 and 3), but rather looks as a cone over an ellipsoid, where the ratio of the largest and smallest axes can be arbitrary large.

We recall now a particular form of Kazdan-Margulis-Zassenhaus Theorem. Put $\mu_n = 9^{-[n/2]-2}$.

Theorem 7. For every torsion-free discrete group generated by two elements $g, h \in \text{Isom}(\mathbb{H}^n)$ which is not almost Abelian and for every point $x \in \mathbb{H}^n$ we have

$$\max\{\tau_h(x), \tau_g(x)\} > \mu_n \tag{2.8}$$

(We use here a calculation of the Margulis constant μ_n made by G.Martin in [Mart 2].)

Suppose that Γ is a torsion-free discrete subgroup of $\text{Isom}(\mathbb{H}^n)$ with the quotient $M = \mathbb{H}^n/\Gamma$. We fix the dimension n and let $\mu = \mu_n$. Denote by $M_{(0,\mu]}$ the subset of $M = \mathbb{H}^n/\Gamma$ which consists of points with injectivity radius not greater than $\mu/2$. This subset is called μ-*thin* part of the manifold M. Let $M - M_{(0,\mu]} = M_{(\mu,\infty)}$. This subset is called μ-*thick* part of M. Then Theorem 7 implies that $M_{(0,\mu]}$ is the disjoint union of embedded Margulis tubes

$$\mathcal{T}(H_j, \mu) \subset M, \qquad j = 1, 2, ... \tag{2.9}$$

where $\{H_j, j = 1, 2, ...\}$ is a family of almost Abelian subgroups of Γ.

Suppose $H =< h >\in Isom(\mathbb{H}^4)$ is a group generated by a parabolic or loxodromic element h; pick a pair of points $a, b \in \mathcal{T}(H,\nu)$ which belong to one and the same fiber Π_t of the canonical foliation on $\mathbb{H}^4/H$ associated with h. Then we define **the piecewise-geodesic annulus** F_{hab} as follows. First connect a, b by the geodesic segments I, J so that $I \subset \Pi_t$ and the closed loop $I \cup J$ is homotopic to h. Denote by γ_a, γ_b the shortest loops in $\mathbb{H}^4/H$ which contain x, y and homotopic to h. Then take the pair of geodesic triangles in M whose edges are I, J, γ_a and I, J, γ_b respectively. The union of these triangles is the desired annulus F_{hab}. (See Figure 1 for the lift of F_{hab} in $\mathbb{H}^4$).

In general F_{hab} is not entirely contained in $\mathcal{T}(H,\nu)$.

2.2. **LEMMA 1.** Let $x \in \mathbb{H}^n$ and $H \subset \text{Isom}(\mathbb{H}^n)$ be such that $Ir_H(x) \geq \nu$ for some positive ν. Then the ball $B(x,r)$ contains not more than

$$\frac{\exp((n-1)(r+\nu))}{\nu^{n-1}} \tag{2.10}$$

points from the orbit Hx.

PROOF. The balls $\{B(hx,\nu) : h \in H\}$ are disjoint. Therefore, if the points $x_j \in Hx$ $(j = 1, ..., m)$ belong to $B(x,r)$, then the volume of the union

$$\bigcup_{j=1}^{m} B(x_j) \tag{2.11}$$

is equal to $mVol(B(x,\nu)) \geq m\nu^{n-1}$. On the other hand, this volume can't be greater than $Vol(B(x,r+\nu)) \leq \exp((n-1)(r+\nu))$. This implies the inequality (2.3). ■

LEMMA 2. Under conditions of Lemma 1 the number of elements $h \in H$ such that the intersection $h(B(x,r)) \cap B(x,r)$ is not empty is not more than

$$\frac{\exp((n-1)(2r+\nu))}{\nu^{n-1}} \tag{2.13}$$

The proof of this statement is completely analogous to the proof of Lemma 1 and we omit it.

Given two numbers r, ν we define

$$p(n,r,\nu) = \frac{\exp(6(n-1)r+2\nu)}{\nu^{n-1}}\ ,\ C_1(n,r,\nu) = 2r/p(r,\nu) \tag{2.14}$$

LEMMA 3. Let H be a discrete subgroup of Isom($\mathbb{H}^4$) and $\nu/2 = Ir_H(x)$. Suppose that $d(x,y) < r$. Then $Ir_H(y) > C_1(n,r,\nu)/2$.

PROOF.
Let h be arbitrary nontrivial element of the group H. Let p_0 be such that $d(x,\ h^{p_0}(x)) \geq 3r$. Then $d(y,\ h^{p_0}(y)) \geq r$ and $d(y,\ h(y)) \geq r/p_0$. So, our aim is to estimate this p_0 from above. Notice that for $p = p(n,r,\nu)$ among the elements

$$\{1,\ h,\ ...\ , h^p\}$$

there is h^k such that $d(x,\ h^k(x)) \geq 3r$ (by Lemma 1). Then we can take $p_0 \leq p$ and $d(y,h(y)) \geq r/p$ for every $h \in H - \{1\}$. ■

REMARK 1. The function $C_1(n,r,\nu)$ has exponential decay as $r \to \infty$.

2.3. In Lemmas 4 and 5 below we shall prove a property of Margulis cones and displacement functions which will be crucial in our paper.

2.4. Let g be either parabolic or loxodromic element of Isom($\mathbb{H}^4$). We can assume that ∞ is a fixed point of g.

In the loxodromic case we shall suppose that $g = \Theta \circ \Lambda$ is a similarity in $\mathbb{E}^4$ preserving $\mathbb{H}^4$, $g(0) = 0$, $A = A_g$ is the axis of g, $d(z,A) > 2$. Here Λ is the dilation $\Lambda : x \to \lambda x$ and Θ is the rotation to the angle θ around A.

In the parabolic case we assume that $g = \Theta \circ \Lambda$, where Λ is a Euclidean translation to the distance λ along A_g and Θ is the rotation to the angle θ around $A_g = A$.

Let L be the geodesic containing the points ∞, z; let $w \in L$ be a point such that z lies between w and ∞. (See Figure 2).

LEMMA 4. Suppose that under conditions above:

$$\nu \le d(g(z), z) \le R;\ d(g(w), w) \le R \tag{2.15}$$

Then $d(z, w) \le R + \frac{1}{\nu}$.

PROOF. First we consider the case of parabolic transformation. The points $z, z' = g(z)$ belong to a horosphere F, the points $w, w' = g(w)$ belong to a horosphere F'. Therefore, the formula (2.2) for the distance d implies that

$$R \ge d(w, w') \ge d(z, z') + d(z, w) \tag{2.16}$$

Hence,

$$d(z, w) \le R - \nu < R + \nu^{-1} \tag{2.17}$$

This concludes the proof in the parabolic case.

Now we consider the more difficult case of loxodromic transformations.

Step 1. Put $g(z) = z'$ $, g(w) = w'$. Denote by $\alpha(u)$ the angle between the horosphere P with center at ∞ containing the point z and the Euclidean line passing through the points z, u ; $\alpha(u) \le \pi/2$. Then the condition $d(z, A) > 2$ guarantees that $\alpha(\lambda z) \le \pi/3$. However $|z - \Theta\lambda z| \ge |z - \lambda z|$, thus

$$\beta = \alpha(gz) \le \alpha(\lambda z) \le \pi/3 \tag{2.18}$$

Step 2. Due to the Step 1 it suffice to consider the case: $z,\ z' = \lambda z,\ w,\ w' = \lambda w \in \mathbb{H}^2 \subset \mathbb{C}$, $\arg(z) = \beta \le \pi/3$; $d(z, w) = d(z', w'),\ Re(z) = Re(w)$. (Figure 3). Without loss of generality we can suppose that $Im(z) = 1, y = Im(z')$.

Then we have: $\rho = |z - z'|, y = \rho \sin(\beta) + 1$,

$q = 2 \sinh \frac{d(z,z')}{2} = \frac{\rho}{\sqrt{y}} = \frac{\rho}{\sqrt{1+\rho\sin(\beta)}}$.

So $q^2 + \rho q^2 \sin(\beta) - \rho^2 = 0$,

$$\rho = \left(q^2 \sin(\beta) + \sqrt{q^4 \sin^2(\beta) + 4q^2} \right) /2 \ge q \ge 2 \sinh \frac{\nu}{2} \tag{2.19}$$

On the other hand we have: $\sin\beta \le \sqrt{3}/2$, so $\rho \le 2q^2 + 2 \le 8 \sinh^2(R/2) + 2$,

$$\sinh \nu/2 \le \sinh d(z, z'')/2 = |z - z''|/2 = \rho(\cos\beta)/2 \tag{2.20}$$

$$\le (8 \sinh^2(R/2) + 2) \cos\beta \le 8 \sinh^2(R/2) + 2 \tag{2.21}$$

and $d(w, w') \ge d(w, w'') - d(w', w'') \ge d(w', w'') - R$.

Let $s = Im(w)$, then $d(z, w) = \log(1/s)$ and

$$\sinh(d(w, w'')/2) = |z - z''|/(2s) \ge \rho/(4s) \ge \sinh(\nu/2)/(2s) \tag{2.22}$$

$$s \geq \frac{\sinh(\nu/2)}{2\sinh(d(w,w'')/2)} \geq \frac{\sinh(\nu/2)}{2\sinh(R)} \tag{2.23}$$

since $d(w,w'') \leq R + d(w,w')$ and $d(w,w') \leq R$. Now

$$d(z,w) = \log(1/s) \leq \log 2 + \log\sinh(R) - \log(\sinh(\nu/2)) \tag{2.24}$$

However $\log\sinh a = (a^2-1)/2a$ and $2\sinh b \leq e^b$. Therefore: $d(z,w) \leq R + \frac{1}{\nu}$. Lemma 4 is proved. ∎

Now suppose that $a, b, z \in \mathbb{H}^4$ be points such that: $d(z,[a,b]) \leq R$.
Denote by L_a, L_b the geodesic rays connecting the points a, b and the point $\infty \in \mathbb{S}^3$. For a point $w \in \mathbb{H}^4$ we denote by $m(w)$ the number

$$m(w) = \min\{d(w,L_a)\ , d(w,L_b)\} \tag{2.25}$$

PROPOSITION 4. (Cf. [B]) Under the conditions above we have:

$$m(z) \leq 2 + R \tag{2.26}$$

PROOF. Denote by c the point of $[a,b]$ such that $d(c,z) \leq R$. Let L'_x be the geodesic containing L_x. Then we have:

$$\cosh dist(c,L'_a) = \frac{1}{\sin\alpha}\ ,\ \cosh dist(c,L'_b) = \frac{1}{\sin\beta} \tag{2.27}$$

(see Figure 4) and $\alpha + \beta \geq \pi/2$ so

$$\sin^2\alpha + \sin^2\beta \geq 1 \tag{2.28}$$

Now there are two possibilities (up to the change of notations: $\beta \leftrightarrow \alpha$):
(i) $b_4 \geq w_4$ for every $w \in [a,b]$
(ii) otherwise.
Consider (ii). Then $\phi < \pi/2\ , \psi < \pi/2$ where ϕ, ψ are nonzero angles of the triangle formed by $L_a,\ L_b,\ [a,b]$. Therefore:

$$dist(c,L'_a) = dist(c,L_a)\ ,\ dist(c,L'_b) = dist(c,L_b) \tag{2.29}$$

Now if $\sin^2\alpha \leq 1/4$ then $\sin^{-1}\beta \leq 2$.
This means that

$$\min\{\cosh dist(c,L_a), \cosh dist(c,L_b)\} \leq 2 \tag{2.30}$$

so $m(z) \leq 2 + R$ in the case (ii).
Consider the case (i). Then $\phi < \pi/2\ , \psi > \pi/2$, however $\alpha \geq \pi/4$ (since the arc of the geodesic passing through a, b is greater then the quarter of circle).
Then $1/\sin\alpha \leq \sqrt{2}$ and $e^x/2 \leq \cosh x = \cosh(d(c,L'_a)) \leq \sqrt{2}$; $x \leq \log(3) < 2$.
However $\phi < \pi/2$, then $d(c,L'_a) = d(c,L_a)$ that means $d(c,L_a) < 2$.

Therefore

$$m(z) \leq 2 + d(c, z) \leq 2 + R \tag{2.31}$$

■

2.5. **LEMMA 5.** Suppose that $g \in \mathrm{Isom}(\mathbb{H}^4)$ is a loxodromic or parabolic element. If g is loxodromic we put $A = A_g$ as in Lemma 4, otherwise let $A = \emptyset$. Let $a, b, z \in \mathbb{H}^4$ be points such that:

$$dist(z, [a, b]) \leq R, dist(z, A) \geq 2 + R \tag{2.32}$$

$$\nu < d(g(z), z) < R, d(a, g(a)) < R, d(b, g(b)) < R \tag{2.33}$$

Then:

$$\min\{d(z, a), d(z, b)\} \leq 4R + 6 + 1/k \tag{2.34}$$

where

$$k = k(R, \nu) = 2(2 + R)\nu^3 / \exp(18(2 + R) + 2\nu) \tag{2.35}$$

PROOF. According to Proposition 4 we can assume that $d(z, L_a) \leq 2 + R$. Let $u \in L_a$ be a point such that $d(z, u) < 2 + R$. Then we have: $dist(u, A) > 2$ (in the loxodromic case), $k < d(g(u), u) < 3R + 4$ (the last follows from Lemma 3). Now we can apply Lemma 4 to the points a, u to obtain: $d(a, u) < 3R + 4 = 1/k$ and $d(z, a) < 4R + 6 + 1/k$.

■.

COROLLARY 5. Let h be a parabolic or loxodromic isometry of $\mathbb{H}^4$, $x \in \mathbb{H}^4$ is such that: $Ir_{<h>}(x) > \nu$ and there is a geodesic segment $L = [a, b]$ such that $d(x, L) < R$ for some R and

$$max\{\tau_h(x),\ \tau_h(a),\ \tau_h(b)\} < R \tag{2.36}$$

Then either

$$\min\{d(x, a), d(x, b)\} < C_+(R, \nu) = 4R + 6 + 1/k \text{ (the parabolic alternative)} \tag{2.37}$$

$$\text{or } l(h) > C_-(R, \nu) = C_1(R + 2, \nu) > 0 \quad \text{(the hyperbolic alternative)} \tag{2.38}$$

Here $k = 2(2 + R)\nu^3 / \exp(18(2 + R) + 2\nu)$ (as in (2.35)) and the function C_1 is defined by Lemma 3.

PROOF. Combine Lemma 5 and Lemma 3. ■

Denote by $H(t, A) = \{w \in \mathbb{H}^2 : dist(w, A) = t\}$ the "hypercycle" whose axis is the geodesic A.

2.6. **LEMMA 6.** Let z_1, z belong to a connected component of $H(t, A)$. Then

$$d_{H(t,A)}(z_1, z) \leq 2\sinh(d(z_1, z)/2) \tag{2.39}$$

where d_H is the metric on $H = H(t, A)$ induced from the hyperbolic plane.

PROOF. We can suppose that $|z_1| = 1, |z| = r$, $\log r$ is the distance between orthogonal projections of the points z_1, z on the geodesic A. Let $\pi - 2\theta$ be the Euclidean angle at the vertex of $H(t, A)$. Then

$$2\sinh(d(z_1, z)/2) = \frac{r-1}{\sin(\theta)\sqrt{r}} \tag{2.40}$$

for $\cosh(t)\sin\theta = 1$. Moreover,

$$a = d_H(z_1, z) = \log(r)/\sin\theta, a\sin\theta = \log r \tag{2.41}$$

Our aim is to show that:

$$\log(r) \leq \frac{r-1}{\sqrt{r}} = \sqrt{r} - \frac{1}{\sqrt{r}} \tag{2.42}$$

if $r \geq 1$. Let $x = \sqrt{r}$, then $2\log x \leq x - 1/x$ since for $x = 1$ we have the equality and derivative of the left side is not greater than derivative of the right side. ■

REMARK 2. In this situation we have also:

$$\cosh t \leq 2\frac{1}{\log r}\ \sinh\frac{d(z_1, z)}{2} \tag{2.43}$$

2.7. We will need some facts about triangulations of hyperbolic surfaces. Let S be a closed hyperbolic surface, $\nu > 0$ be a number such that the ν-thin part $S_{(0,\nu]}$ of the surface S is a disjoint union of tubes. We denote by S' the closure of the complement $S_{(\nu,\infty)} = S - S_{(0,\nu]}$. According to Lemma 6, the length of each boundary component of S' is at most $2\sinh(\nu/2)$. We denote by g the genus of S and by m the number of boundary curves of S' ($m \leq 2(g+1)$). Our aim is to find a triangulation of S' so that the edges of triangulation are either geodesic segments or arcs of $\partial S'$, the number of triangles $s(g, \nu)$ is a linear function of g and diameter of triangles is bounded from above by ν.

Consider the Riemannian metric on S' induced from S and the corresponding distance function on connected components.

Cover S' by a maximal set of disjoint discs $D(z_j, \nu/4)$. Put points z_i on each boundary curve such that distance between any two consecutive points is $\nu/2$ (with exception of two points on each boundary component which have distance at most $\nu/2$). Let n be the number of these points. The number of the discs is at most

$$4Area(S')/\nu^2 \leq 8(g-1)/\nu^2 \tag{2.44}$$

The number of points on $\partial S'$ is at most

$$2m\sinh\nu/(2\nu) \leq 4(g+1) \tag{2.45}$$

The set of points $Z = \{z_j\}_j$ has the following properties:

(a) for each point $x \in S'$ there is at least 4 points from Z on the distance $\leq \nu$;

(b) for each point $x \in S'$ there is at most 25 points from Z at the distance $\leq \nu$;
(c) the number of points in Z is

$$\#(Z) \leq 8(g-1)/\nu^2 + 4(g+1) \tag{2.46}$$

This implies

LEMMA 7. (Cf. [Bo], [Ab]) The diameter of each component of S' is at most

$$C_2(g,\nu) = 16(g-1)/\nu$$

Now we connect any two points in Z by geodesic segment (which can be a boundary arc) iff the distance between these points is at most ν. This gives us a cell decomposition of S'. Each segment that we construct this way can intersect not more than 300 other segments. Therefore, the total number of vertices in this cell decomposition is at most

$$\#(Z) \cdot 45 \cdot 10^3 \tag{2.47}$$

Each cell in the decomposition has at most 25 vertices. Therefore, if we complete this cell decomposition to a triangulation of S', then the number of triangles in the triangulation is at most

$$\#(Z) \cdot 45 \cdot 25 \cdot 10^3 \leq 9 \cdot 10^6 (g-1)/\nu^2 \tag{2.48}$$

which is a linear function of g and the diameter of each triangle is at most ν. Thus, we proved

LEMMA 8. The surface $S' = S_{(\nu,\infty)}$ admits a geodesic triangulation such that the diameter of each triangle is at most ν and the number of triangles is at most

$$9 \cdot 10^6 (g-1)/\nu^2 \tag{2.49}$$

2.8. Now we extend the triangulation to the whole surface S. On each component γ^* of $\partial S'$ we single out a point $z_\gamma \in Z$. In our original triangulation we substitute each arc on every boundary curve γ^* by the shortest geodesic segment in S with the same vertices. Let curve γ^* be adjacent to a certain tube W in $S - S'$. Denote second boundary component of W by α^*. Let z_α be another distinguished point of Z on α. Connect z_γ by the shortest geodesic segments with each point $z_i \in Z \cap \gamma^*$ and repeat the same for z_α. All these segments are contained in W. Finally, we construct two "long triangles" which intersect both curves α^*, γ^* as follows.

Take the shortest curve in S which contains z_γ and the shortest curve which contains z_α. Connect z_γ and z_α by two geodesic segments in W. The choice of these segments is not canonical, we shall discuss this in the section 3.7. So, for the pair (z_γ, z_α) we have constructed 4 segments in S which bound "long" two triangles. (See Figure 1.)

Thus, we have a triangulation of S such that the number of triangles is not greater than

$$9 \cdot 10^6 (g-1)/\nu^2 + 10(g-1) \tag{2.50}$$

This triangulation has 2 "long" triangles for each component of $S - S'$. All other triangles will be called "short". Their diameters do not exceed ν.

3. PROOF of THEOREM 2

3.1. **Step 1.** Suppose that $M = \mathbb{H}^4/G$ is a hyperbolic manifold where G is isomorphic to the fundamental group of a closed orientable surface Σ. Denote by $\psi : \pi_1(\Sigma) \to G$ the corresponding isomorphism and let

$$\pi : \mathbb{H}^4 \to M$$

be the universal covering with the group G of the deck transformations.

Let $[\sigma] \in H_2(M, \mathbb{Z})$ be a homology class represented by the homotopy-equivalence

$$\sigma : \Sigma \to M$$

We recall that $\mu = 1/9^4$ is the Margulis constant for $\mathbb{H}^4$.

3.2. **Step 2.** The group $G = \psi(\pi_1(\Sigma))$ contains at least one loxodromic element and is not almost cyclic; hence there is a hyperbolic structure on Σ and a pleated map

$$f^0 : \Sigma \to M \text{ inducing } \psi : \pi_1(\Sigma) \to G \tag{3.1}$$

(see [Th 1 , Th 2], [Bo]).

Pick a maximal union L_0 of simple closed disjoint geodesics γ on Σ such that

$$0 < l_\Sigma(\gamma) < \mu \tag{3.2}$$

3.3. **Step 3.** For every component $P_j \subset \Sigma - L_0$ define the set

$$W_\mu(P_j) = \{z \in P_j : RadInj_\Sigma(z) \leq \mu/2\} \tag{3.3}$$

Each ideal boundary component $\alpha \subset \partial P_j$ has orientation induced from P_j so we shall distinguish curves $\alpha \subset L_0$ with different orientations but equal underlying sets. Put:

$$W_\mu(\alpha, P_j) = \{z \in P_j : \text{ there exists a loop } \beta_z \text{ on } P_j$$
$$\text{which is homotopic to } \alpha \text{ and passes through } z, \text{ so that } l_\Sigma(\beta_z) \leq \mu\} \tag{3.4}$$

Then

$$W_\mu(P_j) = \bigcup_{\alpha \subset \partial P_j} W_\mu(\alpha, P_j) \tag{3.5}$$

The properties of the Margulis constant imply that for different boundary components α, β of P_j we have :

$$W_\mu(\alpha, P_j) \cap W_\mu(\beta, P_j) = \emptyset \tag{3.6}$$

Put

$$\Sigma_\mu = \{z \in \Sigma : RadInj(z) \geq \mu/2\} \tag{3.7}$$

Let $P_j^0 = P_j - W_\mu(P)$ for every j.

We recall that diameter of each component P_j^0 is at most $C_2(g,\mu) = 16(g-1)/\mu$ (Lemma 7).

REMARK 3. Unfortunately it isn't true that Σ_μ has a decomposition in a union of pairs of pants such that the diameter of each component is bounded from above by a constant which is independent of g. For example, let $\Gamma \subset PSL(2,\mathbb{R})$ be a cocompact arithmetic group, $\Gamma \supset \Gamma(2) \supset ... \supset \Gamma(p)...$ be a decreasing sequence of congruence subgroups. Put $\Sigma(p) = \mathbb{H}^2/\Gamma(p)$. Then for sufficiently large p the surface $\Sigma_\mu(p)$ coincides with $\Sigma(p)$ and the length of the smallest closed geodesic on $\Sigma(p)$ grows linearly as $p \to \infty$. As we shall see later this is the reason of the exponential estimate in Theorem 2.

3.5. **Step 4.**

For each $\gamma \subset L_0$ we pick the Margulis tube $T_\gamma(\mu) \subset M_{(0,\mu]}$ whose fundamental group is generated by an element in the free homotopy class of $f^0(\gamma)$.

For every such geodesic γ we have two (possibly equal) components $P_i, P_j \subset \Sigma - L_0$ adjacent to γ. Then

$$f^0(W_\mu(\gamma, P_j)) \subset T_\gamma(\mu)\ (k = i, j)$$

Choose points $x_k = x_{\gamma,k} \in \partial W_\mu(\gamma, P_k)$ $(k = i, j)$ such that:

$$f^0(x_i), f^0(x_j) \in \Pi_t$$

for some fiber of the canonical foliation of $T_\gamma(\mu)$ associated with $\psi(\gamma)$. Let

$$\nu = RadInj_M(f^0(\Sigma_\mu)) \tag{3.8}$$

Lemma 9. Put

$$C_3(g) = C_1(4, C_2(g,\mu), \mu) \tag{3.9}$$

Then $\nu \geq C_3(g)$.

PROOF. For every i we have a point $o_i \in P_i - (f^0)^{-1}(M_{(0,\mu]})$ since $\psi(\pi_1(P_i))$ is not almost Abelian (because ψ is a monomorphism). Then for each $x \in P_i^0$

$$d_M(f^0 o_i, f^0 x) \leq C_2(\mu, g) \tag{3.10}$$

(by Lemma 7). Therefore (by Lemma 3)

$$RadInj_M(f^0 x) \geq C_1(4, C_2(g,\mu), \mu) = C_3(g) \tag{3.11}$$

■

REMARK 4. The function $C_3(g)$ has exponential decay as $g \to \infty$ and

$$C_3(g) > 7 \cdot 10^{-6}(g-1)\exp(-2 \cdot 10^6(g-1)) \tag{3.12}$$

Let $\alpha_k^*, \beta_k^*, ..., \omega_k^*$ be the boundary components of P_k^0.

Then we can triangulate Σ as in Sections 2.8, 2.7 so that:

(a) for each k the points $x_{\alpha,k}, x_{\beta,k}, ..., x_{\omega,k}$ belong to the set of vertices of this triangulation Ω_f (they are "distinguished points" as in Section 2.8);

(b) all vertices of the triangulation are contained in Σ_μ,

(c) diameters of all "short" triangles are bounded from above by μ,

(d) the number of triangles is not greater than

$$2 \cdot 10^{10}(g-1)/\nu^2 + 10(g-1) \tag{3.13}$$

3.6. **Step 5.** Now, for each k we map the triangulated surface P_k^0 to a piecewise-geodesic surface in M by the new map $f : P_k^0 \to M$ which is a (local) isometry on each triangle so that:

for every edge e of the triangulation we have: $f(e) \sim f^0(e)(rel\ \partial e)$.

Hence $l_M(f(e)) \leq l_\Sigma(e) \leq \mu$.

Now consider the thin part of the surface Σ.

3.7. **Step 6.** Fix $x_{\gamma,i}$, $x_{\gamma,j}$ lying on the components P_i , P_j adjacent to a curve $\gamma \subset L_0$. Then connect the images $f(x_{\gamma,i})$, $f(x_{\gamma,j})$ by the piecewise-geodesic annulus

$$F = F_{\psi(\gamma) f(x_{\gamma,i}) f(x_{\gamma,j})}$$

(see Definition 2). The boundary of F is equal to $f\gamma_i^* \cup f\gamma_j^*$. The annulus F consists of two geodesic triangles. These triangles will be called "long" triangles corresponding to $T_\gamma(\mu)$. The annulus F itself will be called "long piecewise-geodesic annulus corresponding to $T_\gamma(\mu)$".

Finally, we can resolve the ambiguity in the choice of triangulation in Section 2.8. Namely, we connect the points $x_{\gamma,i}$, $x_{\gamma,j} \in \Sigma$ by geodesic segments which are homotopic to the pull-back of the 1-dimensional skeleton of F. So we extended our map from Σ_μ to the piecewise-geodesic map $f : \Sigma \to M$ which is homotopic to σ. We also constructed a triangulation Ω_f of the surface Σ. The map f is geodesic on each triangle of Ω_f.

We shall use the notation Σ^μ for the union of "short" triangles in Σ, then $\Sigma_\mu \subset \Sigma^\mu$.

We recall that the upper bound for the diameter of each component of Σ^μ is

$$R = C_2(g, \mu) = 16(g-1)/\mu = 16 \cdot 9^4(g-1) \tag{3.14}$$

LEMMA 10. Suppose that $\psi(\gamma)$ is loxodromic and $d(\pi(A_{\psi\gamma}), z) \leq \mu + 2$ for some $z \in f(\Sigma_\mu)$. Then

$$diam(f\Delta) \leq \frac{4\sinh(\mu/2)}{C_1(4, R+\mu+2, \mu)} \tag{3.15}$$

for every long triangle $f(\Delta)$ corresponding in $T_\gamma(\mu)$.

PROOF. We have $d(z, f\ o_i) \leq R, f(o_i) \in M_{(\mu,\infty]} \cap f(P_i^0), d(f(o_i), \pi(A_{\psi\gamma})) \leq R+2+\mu$ and thus $l(\gamma) \geq C_1(4, R+\mu+2, \mu)$ by Lemma 3. On the other hand, for every $z \in f(\Delta)$ we have the inequality:

$$\tau_{\psi(\gamma)}(z) \leq \mu \tag{3.16}$$

and therefore

$$d(p(A_{\psi\gamma}), z) \leq \frac{2\sinh(\mu/2)}{C_1(4, R+\mu+2, \mu)} \tag{3.17}$$

for every $z \in f(\Delta)$. Hence for the triangle $f(\Delta)$ corresponding in $T_\gamma(\mu)$ we have:

$$f(\Delta) \subset T_\gamma(\mu)$$

and

$$diam(\Delta) \leq \frac{4\sinh(\mu/2)}{C_1(4, R+\mu+2, \mu)}$$

■

Our construction of the map f is sufficiently flexible; thus, varying points $x_{\gamma,j}$, we can construct (as above) two transversal piecewise-geodesic maps $f_i : \Sigma \to M$ which are homotopic to f^0.

3.8. **Step 7**. Below we summarize the properties of the maps f_i.

(1) f_i are piecewise-geodesic with respect to triangulations Ω_i of the surfaces Σ. The number of triangles in Ω_i is not greater than $2 \cdot 10^{10}(g-1)/\nu^2 + 10(g-1)$.

(2) In the triangulation Ω_i there are "short" and "long" triangles. Namely, internal diameter of each "short" triangle is not greater than μ and the union of short triangles is a surface Σ_i^μ so that $\Sigma - \Sigma_i^\mu = W_{\mu,i}$ is the union of pairwise disjoint nonhomotopic tubes. All vertices of Ω_i are contained in Σ_i^μ. Each point $z \in \partial W_{\mu,i}$ and curve $\gamma^* \subset \partial\Sigma_i^\mu$ passing through z have the property:

$$\tau_{\psi(\gamma^*)}(z) \leq 2\mu$$

(3) Put $\nu = \nu(g) = C_3(g)$ as in Lemma 9. Then $f_i(0-$ skeleton of $\Omega_i) \subset M_{[\nu,\infty)}$.

(4) Every component $Q_{ji} \subset W_{\mu,i}$ consists of two "long" triangles. The tube Q_{ji} contains a geodesic γ_j so that the maps

$$f_i : Q_{ji} - \Omega_i^{(1)} \to M \tag{3.18}$$

can be lifted to the fundamental domain $\Phi_j \subset \mathbb{H}^4$ of the group $< \psi(\gamma_j) >$. The fundamental domain Φ_j is bounded by a pair of fibers of the canonical foliation of $\mathbb{H}^4$ corresponding to $< \psi(\gamma_j) >$. This fundamental domain depends only on the component Q_{ji} and doesn't depend on $i = 1, 2$.

(5) Suppose that $d(\pi(A_{\psi\gamma_j}), z) \leq \mu+2$ for some $z \in f_i(\Sigma_i^\mu)$, where γ_j is generator of $\pi_1(Q_{ji})$ which has loxodromic image under ψ. Then

$$diam(f_i\Delta) \leq 2\sinh(\mu/2)/C_1(R+2+\mu, \mu) \tag{3.19}$$

for every long triangle $\Delta \subset Q_{ji}$, for both $i = 1, 2$ (see Lemma 10).

3.9. **Step 8.** Now we can count the number of points of intersection

$$\#(f_1(\Sigma_1) \cap f_2(\Sigma_2)) \tag{3.20}$$

(i) Consider intersections of "short" triangles. Pick a pair of such triangles

$$f_1(\Delta_1) \subset f_1(\Sigma), f_2(\Delta_2) \subset f_2(\Sigma)$$

let $\tilde{\Delta}_j \subset \mathbb{H}^4$ be the geodesic triangles covering them ($j = 1, 2$). Then we have to estimate the number of elements $h \in G$ such that $h\tilde{\Delta}_1 \cap \tilde{\Delta}_2$ isn't empty. Recall that diam $\tilde{\Delta}_j \leq \mu$ and for any point $y_j \in \tilde{\Delta}_j$ we have

$$Ir_G(y_j) \geq \nu \geq C_3(g)$$

(Lemma 9). Therefore we can apply Lemma 2 to obtain:

$$\#(f_1(\Delta_1) \cap f_2(\Delta_2)) \leq \exp(36\mu + 2C_3(g))C_3(g)^{-3} \tag{3.19}$$

Therefore, the number of points of intersection $f_1(\Sigma_1^\mu) \cap f_2(\Sigma_2^\mu)$ is not greater than

$$\exp(36\mu + 2C_3(g))C_3(g)^{-3} \frac{4 \cdot 10^{20}(g-1)^2}{C_3(g)^4} =$$

$$4C_3(g)^{-7} \exp(36\mu + 2C_3(g)) \cdot 10^{20}(g-1)^2$$
$$\leq \exp(15 \cdot 10^6(g-1)) \tag{3.20}$$

(ii) Consider the case when Δ_1 is short while Δ_2 is long.

Suppose that $h(z) \in h(\tilde{\Delta}_1) \cap \tilde{\Delta}_2$. Then we can apply Lemma 5 and the property 6 of the maps maps f_i to obtain

$$d(z, \tilde{\Delta}_2^{(0)}) \leq \max\{C_+(\mu, \nu), \mu + \frac{4\sinh(\mu/2)}{C_1(\mu + 2 + R, \mu)}\} = C_4 \tag{3.21}$$

Let $\{w_1, w_2, w_3\} = \tilde{\Delta}_2^{(0)}$.

Hence we obtain estimate in the same manner as in the case (i):

$$\#(f_1(\Delta_1) \cap f_2(\Delta_2)) \leq \#\{h \in G : h(B(w_i, C_4)) \cap \tilde{\Delta}_1 \neq \emptyset, i = 1, 2, 3\}$$
$$\leq 3\exp^3(C_4 + \mu/2))/\nu^3 \tag{3.22}$$

since diam $\Delta_1 \leq \mu$.

Direct calculation shows that

$$C_+(\mu, \nu) > \mu + \frac{4\sinh(\mu/2)}{C_1(\mu + 2 + R, \mu)} \tag{3.23}$$

and

$$C_4 \leq 10^{30}(g-1)^{-3}\exp(6\cdot 10^6(g-1)) \tag{3.24}$$

Therefore, the number of points of intersection between short and long triangles is not greater than

$$12\cdot 10^7(g-1)^2\exp[3\cdot 10^{30}(g-1)^{-3}\exp(6\cdot 10^6(g-1))]/\nu^5 \tag{3.25}$$

(iii) Assume now that both Δ_1, Δ_2 are long. Denote by $T_{\gamma_j}(\mu) \subset M_{(0,\mu]}$ the corresponding Margulis tubes so that $f_j(\Delta_j) \subset T_{\gamma_j}(\mu)$.

If these tubes are different then the intersection between $f_1(\Delta_1), f_2(\Delta_2)$ is empty. This, we can assume that

$f_1\Delta_1, f_2\Delta_2$ are in the same tube $T_\gamma(\mu) \subset M_{(0,\mu]}$. The lifts $\tilde{\Delta}_j$ to $\mathbb{H}^4$ belong to one and the same fundamental domain of the cyclic group $< \psi(\gamma) >$ (property 4 of the maps f_j). Thus we have not more than 1 point of intersection between the "long" triangles $f_1(\Delta_1), f_2(\Delta_2)$.

Hence the total estimate of the number of points of intersection between long triangles is

$$6(g-1) \tag{3.26}$$

Therefore, the number $| < [\sigma], [\sigma] > |$ can be estimated as:

$$3\cdot 10^8(g-1)^2\exp[3\cdot 10^{30}(g-1)^{-3}\exp(6\cdot 10^6(g-1))]/\nu^5 \leq$$

$$3\cdot 10^8(g-1)^{-2}\exp[3\cdot 10^{30}(g-1)^{-3}\exp(6\cdot 10^6(g-1))]10^{30}7^{-5}\exp(10^7(g-1)) \leq$$

$$2\cdot 10^{34}(g-1)^{-2}\exp(10^7(g-1))\exp[3\cdot 10^{30}(g-1)^{-3}\exp(6\cdot 10^6(g-1))] \leq$$

$$\exp[\exp(10^8(g-1))] \tag{3.27}$$

This finishes the proof of Theorem 2. ■

REMARK 5. Actually, we proved somesing stronger. Namely, denote by σ the homotopy class of continuous maps $\sigma : \Sigma \to M$ which induce the isomorphism $\psi : \pi_1(\Sigma) \to \pi_1(M)$. Then define the **geometric intersection number** $j(\sigma, \sigma)$ in M to be the

$$\min\{\#(h_1(\Sigma) \cap h_2(\Sigma)) : \ h_j \in \sigma\}$$

where $\#(h_1(\Sigma) \cap h_2(\Sigma))$ is the number of points of intersection between $h_1(\Sigma)$ and $h_2(\Sigma)$. It's clear that $j(\sigma, \sigma) \geq | < [\sigma], [\sigma] > |$. So, our result is:

$$j(\sigma,\sigma) \leq \exp[\exp(10^8(g-1))] \tag{3.28}$$

4. FLAT CONFORMAL STRUCTURES on SEIFERT MANIFOLDS

We recall that a flat conformal structure on a manifold M of dimension $n \geq 3$ is a maximal atlas C on M with Moebius transition maps. For any conformally flat manifold (M, C) there is a conformal developing map

$$d : (\tilde{M}, \tilde{C}) \to \mathbb{S}^n$$

where $(\tilde{M}, \tilde{C})$ is the universal covering of (M, C). This map is unique up to composition with conformal transformations of $\mathbb{S}^n$. The holonomy representation of (M, C) is a homomorphims

$$d_* : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^{n+1})$$

such that

$$d \circ \gamma = d_*(\gamma) \circ d \quad \text{for every } \gamma \in \pi_1(M)$$

where we consider $\pi_1(M)$ as the groups of deck transformation for the covering $\tilde{M} \to M$. If M is compact and has infinite fundamental group then it is known that d is a covering onto its image iff it's not surjective ([GK], [Kam]). We restrict ourselves to the case of orientable 3-manifolds M which are nontrivial circle bundles over orientable hyperbolic surfaces. If the genus of the base $\Sigma = \Sigma_g$ is equal to g and the Euler number of the fibration is equal to $e \in \mathbb{Z}$ then manifold M will be denoted by $\mathcal{S}(g, e)$. If $e \neq 0$ and C is a flat conformal structure on $M = \mathcal{S}(e, g)$, then the developing map d is not surjective implies that the holonomy group $d_*(\pi_1(M)) = G$ is discrete [GK].

THEOREM 4. Suppose that $M = \mathcal{S}(g, e)$ admits a flat conformal structure with nonsurjective developing map. Then

$$|e| \leq \exp[\exp(10^8(g-1)]$$

PROOF. Under the hypothesis of Theorem 4 the kernel of the representation d_* must coincide with the center of $\pi_1(M)$ [Ka 7] and the manifold M is a finite covering over the circle bundle $N = \Omega(G)/G$. Therefore, the Euler number ϵ of the fibration $N \to \Sigma_g$ is not less than e. Unfortunately we can't prove that the manifold $X^4 = \mathbb{H}^4/G$ is homeomorphic to a plane bundle over Σ_g, however X^4 satisfies the conditions of Theorem 2. We shall denote by Y the manifold $X^4 \cup M$. Let T be a compact solid torus in M which is a union of fibers of $M \to \Sigma$. The solid torus T admits a homeomorphic lift $\tilde{T}$ in $\Omega(G)$. There exists a smooth embedding f of the handle D^4 to $\mathbb{H}^4$ so that $f(D^4) \cap \mathbb{S}^3 = \tilde{T}$. Apriori we can't guarantee that f projects to a an embedding $D^4 \to Y$. However, residual finiteness of the group G implies that there exists a subgroup $G_0 \subset G$ of a finite index k such that f projects to the embedding $\bar{f} : D^4 \to Y_0$ where

$$Y_0 = (\mathbb{H}^4 \cup \Omega(G))/G_0$$

We denote by S_0 the base of the circle bundle $p_0 : M_0 = \Omega(G)/G_0 \to S_0$. The projection of $\tilde{T}$ to S_0 is a disc D^2. There exists a section σ_0 of p_0 over $S_0 - D^2$. Then $\sigma_0(\partial D^2)$ is $(k\epsilon, 1)$–smooth torus knot on the boundary of D^4, therefore, there exists a smooth embedding $\tau : D^2 \to \bar{f}(D^4)$ which extends the map σ_0. Thus, we constructed an embedding the $\alpha : S_0 \to Y_0$ which coincides with σ_0 on $S_0 - D^2$ and is equal to τ on D^2. Now it's easy to see that the algebraic self-intersection number of α in Y_0 is the same as the Euler number of the fibration $M_0 \to S_0$. The last number is equal to $\epsilon \cdot k$. Therefore, for the manifold X^4 and homotopy-equivalence $\sigma : \Sigma_g \to X^4$ we have:

$$< [\sigma], [\sigma] >_{X^4} = \epsilon$$

Now, Theorem 2 implies the inequality

$$|e| \leq \epsilon \leq \exp[\exp(10^8(g-1)]$$

■

REMARK 6. The problem about existence of flat conformal structures on Seifert manifolds with surjective developing maps is highly nontrivial. Such structures do not exist if the fundamental group is almost solvable [G]. N.Kuiper and P.Waterman [KP] proved that for every triangle group

$$T = T(2,3,n) \subset PSL(2,\mathbb{R})$$

any representation $\rho : T \to SO(4,1)$ has either a fixed point in $\mathbb{H}^4 \cup \mathbb{S}^3$ or an invariant totally geodesic subspace. Let $\mathcal{O}$ be the hyperbolic orbifold with the fundamental group $T(2,3,n)$. Combined with [Ka 7] this result implies that on any Seifert manifold M^3 which is fibered over $\mathcal{O}$, there is no any flat conformal structure with surjective developing map.

5. HIGHER DIMENSIONAL FIBER BUNDLES

First we recall the result of M.Anderson [A] about metrics of negative curvature on fiber bundles over negatively curved manifolds.

THEOREM 8. Let $\xi : E \to B$ be any smooth vector bundle with the compact base B of negative sectional curvature. Then the total space E admits a Riemannian metric with strictly negatively pinched sectional curvature K_E:

$$0 > a_E \geq K_E \geq -1 \tag{5.1}$$

for some constant a_E depending on the bundle.

However, in the case when we fix the pinching constant a_E there are some restrictions on the topology of E. We consider only the case of constant sectional curvature.

THEOREM 5. There exists a function $f(B)$ of negatively curved oriented compact k-manifold B so that,

if the total space of $\mathbb{R}^k$–bundle $\xi : M^{2k} \to B$ has a complete hyperbolic metric, then

$$|e(\xi)| \leq f(B) \tag{5.2}$$

To prove this theorem we will need some fact about fundamental groups of negatively curved manifolds.

THEOREM 6. Let B be a closed manifold of negative curvature which has dimension $k > 2$. Suppose that the fundamental group $\pi_1(B) = \Gamma$ splits in a nontrivial amalgamated free product $\Gamma_1 *_A \Gamma_2$ or HNN extension $\Gamma_1 *_{A,\theta}$ with amalgamation over a finite extension of a cyclic group. Then either for $j = 1$ or $j = 2$ the group G_j is a finite extension of A.

PROOF. We shall consider only the case of amalgamated free product, the arguments in the case of HNN extension are essentially the same. There exists an aspherical CW-complex K with the fundamental group Γ so that

$$K = K_1 \cup_C K_2$$

where K_j are aspherical CW-complexes with the fundamental groups Γ_j and

$$\pi_1(C) = A$$

(see [CS]). Let $h : M \to K$ be a homotopy–equivalence inducing the isomorphism $\Gamma \to \pi_1(K)$. Then, applying the general position arguments of [CS], we conclude that K and h can be chosen so that $h^{-1}C$ contains a smooth connected hypersurface F in M which decomposes M in the union $M_1 \cup M_2$ so that images G_j of $\pi_1(M_j)$ in Γ are not finite extensions of $\mathbb{Z}$ and the image of $\pi_1(F)$ in Γ is an (almost) cyclic group Z. Consider a component $\tilde{F}$ of the lift of F to the universal cover X of B. The ideal boundary $\partial_\infty X$ of X is homeomorphic to a $k-1$ dimensional sphere. The set of accumulation points of $\tilde{F}$ at $\partial_\infty X$ is the limit set $\Lambda(Z)$ of the group Z and consists of two points. Therefore, this set doesn't separate the sphere $\partial_\infty X$. The hypersurface $\tilde{F}$ split X in two components P, Q. We conclude that one of these components (say P) is such that the set of accumulation points of P in $\partial_\infty X$ is again $\Lambda(Z)$. One component (say) $\tilde{M}_1$ of the lift of $M - F$ in X is contained in P and is adjacent to $\tilde{F}$. The stabilizer G_1 of $\tilde{M}_1$ in Γ is isomorphic to $\pi_1(M_1)$. However, the set of accumulation points of $\tilde{M}_1$ in $\partial_\infty X$ is again the two-point set $\Lambda(Z) = \{x, y\}$. Therefore, the geodesic between x and y in X must be invariant under the group G_1. Hence, either G_1 is a $\mathbb{Z}_2$-extension of Z or $G_1 = Z$. This proves Theorem. ■

If Γ is a finitely generated group, then we shall denote by

$$D(\Gamma, q)$$

the projection of the space of discrete and faithful representations of Γ to

$$Hom(\Gamma, \mathrm{Isom}(\mathbb{H}^q))/\,\mathrm{Isom}(\mathbb{H}^q) \tag{5.3}$$

We recall the following fundamental compactness theorem due to Chuckrow [C], Thurston [Th 2], Morgan [Mor 1 , Mor 2], Rips [R], Bestvina and Feighn [BF] (see also [Mart 1]).

THEOREM 9. Suppose that Γ is a group satisfying the conclusion of Theorem 6. Then $D(\Gamma, q)$ is compact.

PROOF of THEOREM 4.

Fix a k-dimensional compact manifold of negative curvature B as above. Our aim is to prove that there is a number $f(B)$ with the following property. If $\xi : M \to B$ is $\mathbb{R}^k$ bundle over B with the Euler number e and M admits a hyperbolic structure then $|e| > f(B)$.

Otherwise there exists a sequence of fiber bundles $\xi_n : M_n \to B$ with the Euler numbers $e_n \to \infty$ such that M_n are $2k$-dimensional hyperbolic manifolds. Thus, we have a sequence of discrete and faithful holonomy representations of these manifolds

$$\rho_n : \pi_1(B) = \Gamma \to \mathrm{Isom}(\mathbb{H}^{2k})$$

Theorems 6 and 9 imply that the space $D(\Gamma, 2k)$ is compact in

$$Hom(\Gamma, \mathrm{Isom}(\mathbb{H}^{2k}))/\,\mathrm{Isom}(\mathbb{H}^{2k})$$

Therefore, for any system of generators $\gamma_1, ..., \gamma_p$ of Γ there is a point $x \in \mathbb{H}^{2k}$ and a number $a <\leq \infty$ such that

$$d(\rho_n(\gamma_j)(x), x) \leq a \quad (j = 1, ..., p) \tag{5.4}$$

Fix a triangulation T of B. Then (5.4) implies that there exists a constant $r = r(T, A)$ with the property:
for each n there is a continuous map $h_n : B \to M_n$ such that the image of each simplex Δ in T is a geodesic simplex in M_n and diameter of $f(\Delta)$ is at most r.

The radius of injectivity of M_n at the compact subset $f(B)$ is bounded from below by the number

$$\nu = C_1(2k, (\#T) \cdot C, \mu_{2k})$$

where $\#T$ is the number of simplices in T (see Lemma 3). Therefore, applying the same arguments as in proof of Theorem 2, we conclude that

$$| < [h_n], [h_n] > | \leq (\#T)^2 \frac{\exp((2k-1)(2r+2\nu))}{\nu^{2k-1}}$$

where $[h_n]$ is the cycle in $H_k(M_n)$ represented by h_n.

This proves Theorem. ■

6. Final Remarks

There are several other results and conjectures that seems to be similar to Theorem 2 and Conjecture 1.

6.1. If C is a smooth curve of genus g in a complex surface X and K is a canonical class of X then

$$2g - 2 = < C, C > + K \cdot C$$

(see [F]), which is a general form of the Besout theorem.

R.Kirby [Ki] conjectured that for any smooth embedded 2-manifold Σ in the same homology class as C then genus of Σ isn't less than g.

Conjecture of J.Morgan is that for any smooth oriented 4-manifold for which Donaldson's polynomials are defined and non-zero, and any smoothly embedded oriented surface $\Sigma \subset M$ with positive self-intersection one have the inequality

$$2g - 2 \geq \ < \Sigma, \Sigma >$$

P.Kronheimer and T.Mrowka [KM] proved the following

Theorem 10. Let X any smooth closed oriented simply connected 4-manifold X with nontrivial polynomial invariants and which has b^+ odd and not less than 3. Then the genus of any orientable smoothly embedded surface Σ other than sphere of self-intersection -1 or an inessential sphere of self-intersection 0 satisfies the inequality:

$$2g-2 \geq \ < \Sigma, \Sigma >$$

The reader can find further information in the paper of P.Kronheimer [Kr].

6.2. Let $\xi : \mathcal{S}(e,g) \to \Sigma$ be a circle bundle. Then the inequality

$$e = |\chi(\xi)| \leq |\chi(\Sigma)| = 2g-2$$

is necessary and sufficient condition for existence of a reduction of ξ to a flat bundle (see [W]).

6.3. Suppose that M is a complex hyperbolic surface and $f : \Sigma_g \longrightarrow M$ is a homotopy-equivalence. Let ω_M denote the Kahler form on M. Then D.Toledo has proved [To 1] that the number

$$c = \frac{1}{2\pi} \int_\Sigma f^* \omega_M$$

is an integer independent of f which satisfies

$$2-2g \leq c \leq 2g-2$$

Furthermore Toledo [To 2] proved that M is a quotient by a cocompact lattice in $\mathrm{U}(1,1)$ if and only if $|c| = 2g-2$.

In [KG] we proved that, subject to Toledo's necessary conditions, every even value of c is realized by a complex hyperbolic surface $N(c,g)$ homeomorphic to $M(e,g)$, where

$$e = e(g,c) = c/2 + 2 - 2g = c/2 + \chi(\Sigma)$$

References

[Ab] W.Abikoff, "The Real Analytic Theory of Teichmuller Space", Lecture Notes in Mathematics , vol 820. Springer Verlag, 1980.

[A] M.Anderson, *Metrics of negative curvature on vector bundles* , Proceedings of AMS, **99** (1987)

[BGS] W. Ballmann, M. Gromov and V. Schröder, "Manifolds of nonpositive curvature", Birkhäuser 1985.

[BF] M.Bestvina, M.Feighn, *Stable actions of groups on real trees,* Preprint , 1992.

[Bo] F.Bonahon, *Boutes des varietes hyperboliques dee dimension 3*, Ann. of Math. **124** (1986) 71-158.

[B] B.Bowditch, *Geometrical finiteness in variable negative curvature* , Preprint of IHES, 1990.

[C] V.Chuckrow, *Schottky groups with applications to Kleinian groups,* Ann of Math., **88** (1968) 47– 61.

[CS] M.Culler, P.Shalen, *Varieties of representations and splittings of 3-manifolds*, Ann. of Math. **117** (1983) 109– 146.

[F] W.Fulton, "Intersections Theory", Series of Modren Surveys in Mathematics, Springer, 1984.

[G] W.Goldman, *Conformally flat manifolds with nilpotent holonomy*, Trans. Amer. Math. Soc., **278** (1983), 573– 583.

[GLT] M.Gromov, H.B.Lawson, W.Thurston, *Hyperbolic structures on 4-manifolds and flat conformal structures on 3-manifolds* , Math. Publications of IHES, vol. **68** (1988).

[G K] N.Gusevskii, M.Kapovich, *Conformal structures on 3- manifolds*, Soviet Math. Dokl. 34 (1987) 314- 318.

[Ki] R.Kirby, *Problems in low-dimensional topology.* In: Algebraic and Geometric Topology. (Proc. Symp. Pure Math. v. 22) AMS, Providence, 1978, p. 273-312

[Ku 1] N.Kuiper, *Hyperbolic 4-manifolds and tesselations (variations on (GLT))* , Math. Publications of IHES, vol. **68** (1988), 47- 76

[Ku 2] N.Kuiper, *Fairly symmetric hyperbolic manifolds* , In: Geometry and Topology of Submanifolds, II (1990) p. 165- 203. World Sci. Publisher, Singapore, New Jersey, London, Hongkong.

[Ku 3] N.Kuiper, 1988 (unpublished)

[K W] N.Kuiper, P.Waterman (in preparation)

[Kam] Y.Kamishima, *Conformally flat manifolds whose development* maps are not sur*jective*, Trans. Amer. Math. Soc., **294** (1986), 607- 621.

[Ka 1] M.Kapovich, *Flat conformal structures on 3-manifolds. The existence problem, I.* Siberian Math. Journal, vol **30** (1989) N 5 p. 60-73.

[Ka 2] M.Kapovich, *Flat conformal structures on 3-manifolds (survey)*, In: Proceedings of International Conference dedicated to A.Maltsev, Novosibirsk 1989. Contemporary Mathematics, 1992, Vol. 131.1, p. 551– 570.

[Ka 3] M.Kapovich, *Deformation spaces of flat conformal structures*, In: Proceedings of Soviet- Japanese Topology Symposium held in Khabarovsk 1989. Answers and Questions in General Topology, vol. **8** (1990) 253-264.

[Ka 4] M.Kapovich, *Flat conformal structures on 3-manifolds. I,* J. Diff. Geom. Vol. **38**, N 1, (1993)

[Ka 5] M.Kapovich, *Intersection pairing on hyperbolic 4-manifolds*, Preprint of MSRI, 1992.

[Ka 6] M.Kapovich, *Intersection pairing on hyperbolic 4-manifolds*, (in preparation)

[Ka 7] M.E.Kapovich, *Some properties of developments of conformal structures on 3-manifolds*, Soviet. Math. Dokl. **35** (1987), 146– 149.

[K G] M.Kapovich, W.Goldman, *Complex hyperbolic surfaces homotopy-equivalent to a Riemann surface*, Preprint of MSRI, 1992.

[Kr] P.Kronheimer, *Embedded Surfaces in 4-manifolds*, Proceedings of ICM-90, vol. 1, 529- 538.

[KM] P.Kronheimer, T.Mrowka, *Gauge theorey for embedded surfaces,* Preprint , 1992

[L] F.Luo, *Constructing flat conformal structures on some Seifert fibered 3-manifolds*, Math. Annalen, **294** (1992) N 3, 449– 458.

[Mar] V.Marenich (1988) (unpublished)

[Mart 1] G.Martin, *On discrete Mobius groups in all dimensions: A generalization of Jorgensen's inequality*, Acta Math. 163 (1989) 3:4, p. 253-289.

[Mart 2] G.Martin, *Balls in Hyperbolic Manifolds*, J. London Math Soc (2) **40** (1989) 257–264

[Mor 1] J.Morgan, *Group actions on trees and the compactification of the space of classes of* **SO(n, 1)** *representations*, Topology, 1986, V. **25**, N 1, p. 1-33.

[Mor 2] J.Morgan, Λ-*trees and their applications*, Bull. AMS, 1992, v. **26**, N 1, p. 87-112.

[R] E.Rips,

[Th] W.Thurston, "Geometry and Topology of 3-manifolds", Lecture Notes, Princeton University, 1978.

[Th 1] W.Thurston, *Hyperbolic structures on 3-manifolds, I*, Ann. of Math., **124** (1986) 203– 246

[To 1] Toledo, D., *Harmonic maps from surfaces to certain Kaehler manifolds,* Math. Scand. **45** (1979), 13– 26

[To 2] Toledo, D., *Representations of surface groups on complex hyperbolic space,* J. Diff. Geo. **29** (1989), 125–133

[W] J.Wood, *Bundles with totally disconnecyted structure group,* Comm. Math. Helv. **51** (1976) 183– 199

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