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Finsler bordifications of symmetric and certain locally symmetric spaces

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We give a geometric interpretation of the maximal Satake compactification of symmetric spaces $X = G/K$ of noncompact type, showing that it arises by attaching the horofunction boundary for a suitable G–invariant Finsler metric on X. As an application, we establish the existence of natural bordifications, as orbifolds-with-corners, of locally symmetric spaces X/Γ for arbitrary discrete subgroups $\Gamma < G$. These bordifications result from attaching Γ–quotients of suitable domains of proper discontinuity at infinity. We further prove that such bordifications are compactifications in the case of Anosov subgroups. We show, conversely, that Anosov subgroups are characterized by the existence of such compactifications among uniformly regular subgroups. Along the way, we give a positive answer, in the torsion-free case, to a question of Haïssinsky and Tukia on convergence groups regarding the cocompactness of their actions on the domains of discontinuity.

1 Introduction

The goal of this paper is four-fold:

(1) We give a geometric interpretation of the maximal Satake compactification of a symmetric space $X = G/K$ of noncompact type by obtaining it as the horoclosure with respect to a suitable G–invariant Finsler metric.

(2) This compactification turns out to have good dynamical properties, better, for our purposes, than the usual visual compactification as a CAT(0) space. In it we find natural domains of proper discontinuity for discrete subgroups $\Gamma < G$. For Anosov subgroups we show that the actions on these domains are also cocompact, thereby providing natural orbifold-with-corner compactifications of the corresponding locally symmetric spaces.

(3) We use these dynamical results to establish new characterizations of Anosov subgroups.

(4) We apply our techniques for proving cocompactness to the theory of abstract convergence groups and verify the cocompactness on the domain of discontinuity for a certain class of actions.

The study of compactifications of symmetric and locally symmetric spaces has a long history. Let us mention the work of Satake [48], who was the first to define *Satake compactifications* of symmetric spaces of noncompact type and used it in [47] to construct compactifications of certain arithmetic locally symmetric spaces, Furstenberg [19], who constructed compactifications of symmetric spaces by embedding them into spaces of probability measures, Moore [44], who extended Furstenberg's work and related Furstenberg's compactifications with Satake's, Baily and Borel [2], who gave another construction of compactifications of symmetric and locally symmetric spaces and established (in the Hermitian setting) structure of normal analytic varieties for the Baily–Borel compactifications of locally symmetric spaces, Karpelevich [36], who introduced the *Karpelevich compactification* of symmetric spaces, Ash, Mumford, Rapoport and Tai [1], who defined *toroidal compactifications* of locally symmetric spaces by desingularizing Baily–Borel compactifications,[1] Marden [40], who constructed certain compactifications of hyperbolic 3–dimensional manifolds associated with geometrically finite Kleinian groups and proved that the existence of such compactifications characterizes

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[1]Analogously to our paper, the toroidal compactification depends on some auxiliary datum, which, in our paper, is a *balanced thickening* of the identity in the suitable Weyl group of G.

geometric finiteness, Borel and Serre [9], who defined compactifications (as manifolds-with-corners) of arithmetic locally symmetric spaces,[2] Gromov [21], who defined the *horofunction compactification* of general proper metric spaces, Bowditch [10] and Ratcliffe [46], who extended Marden's work to higher-dimensional hyperbolic spaces and Bowditch [11], who further extended this to negatively pinched Hadamard manifolds, and Karlsson, Metz and Noskov [35] and Walsh [52; 53], who described horofunction compactifications of normed vector spaces and of the Teichmüller spaces. We refer the reader to the monograph of Borel and Ji [8] for the in-depth discussion of compactifications of higher-rank symmetric and locally symmetric spaces. In the context of Anosov subgroups, let us mention Benoist [5], whose foundational work including a definition of limit sets in flag manifolds of arbitrary reductive Lie groups and a notion of contraction dynamics equivalent to the one used in this paper was of major influence on further developments in the field and basic for our paper, Labourie's pioneering paper [39], where he introduced Anosov subgroups, and the paper [24] by Guichard and Wienhard for the existence of an orbifold-with-boundary compactification of locally symmetric quotients by Anosov subgroups of some special classes of simple Lie groups (namely, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(n, n)$ and $\mathrm{SO}(n, n)$). In the context of convergence groups, let us mention the work of Gehring and Martin [20], who defined convergence group actions on spheres, Tukia [50; 51] and Bowditch [13], who extended the concepts of geometric finiteness from discrete group actions on ideal boundaries of hyperbolic spaces to general convergence actions and Bowditch, who in [12] gave a topological characterization of word-hyperbolic groups as uniform convergence groups.

We now describe our main results in more detail.

(1)   We prove that the *maximal Satake compactification* $\overline{X}^S_{\max}$ (see [8, Chapter 2]) is G–equivariantly homeomorphic, as a manifold-with-corners, to a *regular Finsler compactification* $\overline{X}^{\mathrm{Fins}} = \overline{X}^{\overline{\theta}}$ obtained by adding to X points at infinity represented by *Finsler horofunctions*. These horofunctions arise as limits, modulo additive constants, of distance functions

$$d^{\overline{\theta}}_x = d^{\overline{\theta}}(\,\cdot\,, x),$$

where $d^{\overline{\theta}}$ is a certain G–invariant Finsler distance on X associated with an interior point $\overline{\theta}$ of the model spherical Weyl chamber σ_{mod} of X. This *horoclosure* construction is a special case of a well-known general construction of compactifications for metric

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[2]Finding an analogue of the Borel–Serre compactifications in the context of Anosov subgroups was the main impetus for our paper.

spaces. For instance, applying it to CAT(0) spaces yields their visual compactification. The novelty here is finding the right metric on the symmetric space X which yields $\bar{X}^S_{\max}$. Our first main result, proven in Sections 5, 6 and 7, describes geometric and dynamical properties of the Finsler compactification:

Theorem 1.1 *For every regular type $\bar{\theta} \in \mathrm{int}(\sigma_{\mathrm{mod}})$,*

$$\bar{X}^{\bar{\theta}} = X \sqcup \partial^{\bar{\theta}}_\infty X$$

is a compactification of X as a G–space which satisfies the following properties:

  (i)   *There are finitely many G–orbits $S_{\tau_{\mathrm{mod}}}$ indexed by the faces τ_{mod} of σ_{mod} ($X = S_\varnothing$).*

  (ii) *The stratification of $\bar{X}^{\bar{\theta}}$ by G–orbits is a G–invariant manifold-with-corners structure.*

  (iii) *$\bar{X}^{\bar{\theta}}$ is homeomorphic to the closed ball, with X corresponding to the open ball.*

  (iv) *The compactification $\bar{X}^{\bar{\theta}}$ is independent of the regular type $\bar{\theta}$ in the sense that the identity map id_X extends to a natural homeomorphism of any two such compactifications.*

  (v)   *There exists a G–equivariant homeomorphism of manifolds-with-corners between $\bar{X}^{\bar{\theta}}$ and the maximal Satake compactification $\bar{X}^S_{\max}$ which yields a natural correspondence of strata.*

In view of (iv) we will denote the Finsler compactification from now on by $\bar{X}^{\mathrm{Fins}}$.

Remark 1.2 (i) We also give a geometric interpretation of the points in $\partial^{\mathrm{Fins}}_\infty X$ as strong asymptote classes of Weyl sectors; see Lemma 5.23.

(ii) The strata $S_{\tau_{\mathrm{mod}}} \subset \partial^{\mathrm{Fins}}_\infty X$ at infinity naturally fiber over the partial flag manifolds $\mathrm{Flag}_{\tau_{\mathrm{mod}}} \cong G/P_{\tau_{\mathrm{mod}}}$. The fibers X_τ for $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$, called *small strata*, are naturally identified with symmetric subspaces of X, namely with cross-sections of parallel sets. In the case $\tau_{\mathrm{mod}} = \sigma_{\mathrm{mod}}$ the fibration is a homeomorphism, $S_{\sigma_{\mathrm{mod}}} \cong \mathrm{Flag}_{\sigma_{\mathrm{mod}}} = \partial_{\mathrm{Fu}} X \cong G/B$, ie the Furstenberg boundary $\partial_{\mathrm{Fu}} X$ embeds at infinity as the unique closed stratum.

(iii) The Finsler viewpoint had emerged in several instances during our earlier study with Porti [30; 31; 34] of asymptotic and coarse properties of regular discrete isometry groups acting on symmetric spaces and euclidean buildings. For instance, the notion

of *flag convergence* (which we defined earlier in [29, Definition 7.4] in the context of the full flag manifold of G and in [30, Definition 5.26] in full generality; see also [31, Section 3.8]) is a special case of the Finsler convergence at infinity considered in this paper; see Proposition 5.43. Furthermore, the *Morse lemma* proven in [31] can be rephrased to the effect that regular quasigeodesics in symmetric spaces and euclidean buildings are uniformly close to Finsler geodesics; see Section 5.1.3. In the same vein, *Morse subgroups* $\Gamma < G$ can be characterized as Finsler quasiconvex; see Section 12.1.

(iv)   The maximal Satake compactification is known to carry a G–invariant real-analytic structure; see [8].

Remark 1.3   After finishing this work we learnt about work of Anne Parreau [45], where she studies the geometry of CAT(0) model spaces, ie of symmetric spaces of noncompact type and euclidean buildings, from a very natural perspective, regarding them as metric spaces with a vector-valued distance function with values in the euclidean Weyl chamber Δ (called Δ–distance in our paper). Among other things, she shows that basic properties of CAT(0) spaces persist in this setting, notably the convexity of the distance, and develops a comparison geometry for the Δ–distance function. Furthermore, she proves that the resulting Δ–valued horofunction compactifications of model spaces are naturally homeomorphic to their maximal Satake compactifications.

(2)   Our main application of Theorem 1.1 concerns discrete subgroups $\Gamma < G$. Recall that if X is a negatively curved symmetric space, then the locally symmetric space X/Γ (actually, an orbifold) admits the standard bordification

$$X/\Gamma \hookrightarrow (X \sqcup \Omega(\Gamma))/\Gamma,$$

where $\Omega(\Gamma) \subset \partial_\infty X$ is the domain of discontinuity of Γ at infinity. The quotient $(X \sqcup \Omega(\Gamma))/\Gamma$ is an orbifold with boundary $\Omega(\Gamma)/\Gamma$. Furthermore, a subgroup Γ is *convex cocompact* if and only if $(X \sqcup \Omega(\Gamma))/\Gamma$ is compact. The main purpose of this paper is to generalize these bordifications and compactifications to suitable classes of discrete subgroups of higher-rank Lie groups.

In our earlier papers [30; 31; 34], we introduced several conditions for discrete subgroups Γ of semisimple Lie groups G, generalizing the notions of discreteness and convex cocompactness in rank one; see also [27]. These properties are defined relative to faces τ_{mod} of the spherical model Weyl chamber σ_{mod}, equivalently, with respect to conjugacy classes of parabolic subgroups of G. The most important properties for the purposes of this paper are *regularity* and *asymptotic embeddedness*.

The regularity conditions capture the asymptotics of the orbits in X and are reflected by the location of their accumulation sets in $\partial_\infty^{\mathrm{Fins}} X$. A discrete subgroup $\Gamma < G$ is τ_{mod}–*regular* if its orbits $\Gamma x \subset X$ accumulate in $\partial_\infty^{\mathrm{Fins}} X$ at the closure of the stratum $S_{\tau_{\mathrm{mod}}}$. The τ_{mod}–*limit set* $\Lambda_{\tau_{\mathrm{mod}}}$ of Γ is then defined as the compact set of simplices $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that the small stratum closure $\bar{X}_\tau$ contains accumulation points. The subgroup Γ is τ_{mod}–*antipodal* if the simplices in $\Lambda_{\tau_{\mathrm{mod}}}$ are pairwise antipodal.[3] In rank one, regularity is equivalent to discreteness.

Remark 1.4 In his influential paper, Benoist [5, Section 3.6] introduced a notion of limit set Λ_Γ for Zariski dense subgroups Γ of reductive algebraic groups over local fields which in the case of real semisimple Lie groups is equivalent to our concept of σ_{mod}–limit set $\Lambda_{\sigma_{\mathrm{mod}}}$. Benoist's limit set Λ_Γ is contained in the flag manifold Y_Γ which in the case of real Lie groups is the full flag manifold G/B; see the beginning of Section 3 of his paper. It consists of the limit points of sequences contracting on G/B; see his Definitions 3.5 and 3.6. What we call the τ_{mod}–limit set $\Lambda_{\tau_{\mathrm{mod}}}$ for other face types $\tau_{\mathrm{mod}} \subsetneq \sigma_{\mathrm{mod}}$ is mentioned in his Remark 3.6(3), and his work implies that, in the Zariski dense case, $\Lambda_{\tau_{\mathrm{mod}}}$ is the image of $\Lambda_{\sigma_{\mathrm{mod}}}$ under the natural projection $\mathrm{Flag}_{\sigma_{\mathrm{mod}}} \to \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ of flag manifolds.

Asymptotically embedded subgroups form a certain subclass of regular subgroups, which turns out to coincide with the class of Anosov subgroups; see [30]. A discrete subgroup $\Gamma < G$ is τ_{mod}–*asymptotically embedded* if it is τ_{mod}–regular, τ_{mod}–antipodal and intrinsically word hyperbolic, and its Gromov boundary $\partial_\infty \Gamma$ is equivariantly homeomorphic to $\Lambda_{\tau_{\mathrm{mod}}}$.[4]

In order to obtain bordifications of locally symmetric spaces X/Γ, we construct domains of proper discontinuity for Γ in $\bar{X}^{\mathrm{Fins}}$. These domains will depend on an auxiliary combinatorial datum, namely a subset $\mathrm{Th} \subset W$ of the Weyl group, called a *thickening*. It can be thought of as a set of "sufficiently special" relative positions of pairs of chambers (full flags) in the Tits boundary. This datum is used to construct the *Finsler thickening* $\mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}) \subset \partial_\infty^{\mathrm{Fins}} X$ of $\Lambda_{\tau_{\mathrm{mod}}}$ a certain Γ–invariant saturated[5] compact subset; see Section 8.3, where the reader also finds the definitions of fat and balanced thickenings. (Balanced implies fat.)

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[3] Here we require the face τ_{mod} to be invariant under the opposition involution of σ_{mod}. For the corresponding parabolic subgroups this means that they are conjugate to their opposite parabolic subgroups.

[4] See Definition 11.1.

[5] A subset $S \subset \partial_\infty^{\mathrm{Fins}} X$ is called *saturated* if it is a union of small strata.

The following result establishes the existence of natural bordifications and compactifications (as orbifolds-with-corners) for locally symmetric spaces X/Γ by attaching Γ–quotients of suitably chosen saturated domains in the Finsler boundary of X. It is a combination of Theorems 9.16 and 11.11.

Theorem 1.5 *Let $\Gamma < G$ be a τ_{mod}–regular subgroup. Then:*

(i) *For each $W_{\tau_{\mathrm{mod}}}$–left-invariant fat thickening $\mathrm{Th} \subset W$, the action*

$$\Gamma \curvearrowright X \sqcup \Omega_{\mathrm{Th}}^{\mathrm{Fins}} := \overline{X}^{\mathrm{Fins}} - \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}})$$

is properly discontinuous. The quotient

(1.6) $$(X \sqcup \Omega_{\mathrm{Th}}^{\mathrm{Fins}})/\Gamma$$

provides a real-analytic bordification of the orbifold X/Γ as an orbifold-with-corners.

(ii) *If Γ is τ_{mod}–asymptotically embedded and Th is a $W_{\tau_{\mathrm{mod}}}$–left-invariant balanced thickening, then $(X \sqcup \Omega_{\mathrm{Th}}^{\mathrm{Fins}})/\Gamma$ is compact. In particular, the bordification is a compactification.*

Remark 1.7 (i) If Γ is τ_{mod}–antipodal, Th balanced and $\mathrm{rank}(X) \geq 2$, then the domains $\Omega_{\mathrm{Th}}^{\mathrm{Fins}}$ at infinity are nonempty; see Proposition 9.23.

(ii) The construction of the domains of proper discontinuity extends in a straightforward way to all discrete subgroups $\Gamma < G$; see Theorems 9.19 and 9.21.

(iii) The existence of an orbifold-with-boundary compactification of locally symmetric quotients by Anosov subgroups of some special classes of simple Lie groups (namely, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(n, n)$ and $\mathrm{SO}(n, n)$) appeared in [24]; see also [23].

(3) As already mentioned, the class of asymptotically embedded subgroups coincides with the class of Anosov subgroups. We also prove a converse of part (ii) of the previous theorem, thereby providing a new characterization of Anosov subgroups among uniformly regular subgroups in terms of the existence of certain compactifications of the locally symmetric spaces. To this end, we say that a discrete subgroup $\Gamma < G$ is *S-cocompact* if there exists a Γ–invariant saturated open subset $\Omega \subset \partial_{\infty}^{\mathrm{Fins}} X$ such that Γ acts properly discontinuously and cocompactly on $X \sqcup \Omega$. Theorem 1.5 shows that τ_{mod}–asymptotically embedded subgroups are S-cocompact with $\Omega = \Omega_{\mathrm{Th}}^{\mathrm{Fins}}$. Conversely, we prove for ι–invariant face types τ_{mod} (see Section 12.2):

Theorem 1.8 *Uniformly τ_{mod}–regular S-cocompact subgroups $\Gamma < G$ are τ_{mod}–Anosov.*

Combining the last two theorems, we obtain the characterization:

Corollary 1.9 *A uniformly τ_{mod}–regular subgroup $\Gamma < G$ is τ_{mod}–Anosov if and only if it is S-cocompact.*

Our cocompactness results thus provide a precise higher-rank analogue of the characterization of convex cocompact subgroups of rank-1 Lie groups in terms of compactifications of the corresponding locally symmetric spaces.

While proving Theorem 1.8, we establish yet another coarse-geometric characterization of τ_{mod}–Anosov subgroups, namely as uniformly τ_{mod}–regular subgroups which are *coarse retracts*; see Sections 2.7 and 12 for the details. This theorem is a higher-rank analogue of the characterization of quasiconvex subgroups of Gromov-hyperbolic groups as coarse retracts.

(4) In Section 10.2, as an intermediate step in the proof of Theorem 1.5, we verify a conjecture by Haïssinsky and Tukia regarding the cocompactness of convergence group actions on their domains of discontinuity under mild extra assumptions:

Theorem 1.10 *Let $\Gamma \curvearrowright \Sigma$ be a convergence group action of a virtually torsion-free hyperbolic group on a metrizable compact space Σ, and suppose that $\Lambda \subset \Sigma$ is an invariant compact subset which is equivariantly homeomorphic to $\partial_\infty \Gamma$. Then the action $\Gamma \curvearrowright \Sigma - \Lambda$ is cocompact provided that $\Sigma - \Lambda$ has finitely many path-connected components.*

2 Preliminaries

2.1 Notation and definitions

We note that for Hausdorff paracompact topological spaces (and in this paper we will be dealing only with such topological spaces), Alexander–Spanier and Čech cohomology theories are naturally isomorphic; see [49, Chapter 6.9]. Therefore, in our paper, all cohomology is Alexander–Spanier–Čech with field coefficients (the reader can assume that the field of coefficients is $\mathbb{Z}_2$). For manifolds and CW complexes, singular and cellular cohomology is naturally isomorphic to the Čech cohomology. We will use the notation H_c^* for cohomology with compact support. As for homology, we will use it again with field coefficients and only for locally finite CW complexes, where we will be using singular homology and singular locally finite homology, denoted by H_*^{lf}. By Kronecker duality, for each locally finite CW complex X,

$$(H_k^{\mathrm{lf}}(X))^* \cong H_c^k(X), \quad k \geq 0.$$

We refer the reader to [26] for the definitions of manifolds and orbifolds with corners. The only examples of orbifolds with corners which appear in this paper are the *good* ones, ie quotients of manifolds-with-corners by properly discontinuous group actions.

Throughout the paper, $\angle$ will denote the angle between vectors in a euclidean vector space and, respectively, the angular metric on spherical simplices.

2.2 Some point-set topology

Let Z and Z' be first countable Hausdorff spaces, and let $O \subset Z$ and $O' \subset Z'$ be dense open subsets. Let $f \colon Z \to Z'$ be a map such that $f(O) \subseteq O'$, and suppose that f has the following partial continuity property: If (y_n) is a sequence in O which converges to $z \in Z$, then $f(y_n) \to f(z)$ in Z'. In particular, $f|_O$ is continuous.

Lemma 2.1 *Under these assumptions, the map f is continuous.*

Proof The lemma follows from a standard diagonal subsequence argument. □

Let (A_n) be a sequence of subsets of a metrizable topological space Z. We denote by $\mathrm{Acc}((A_n))$ the closed subset consisting of the accumulation points of all sequences (a_n) of points $a_n \in A_n$.

If $\mathrm{Acc}((A_n)) \subseteq S$, we say that the sequence of subsets (A_n) *accumulates at* a subset $S \subset Z$.

If Z is compact and $C \subset Z$ is a closed subset, then the sequence (A_n) accumulates at S if and only if every neighborhood U of C contains all but finitely many of the subsets A_n.

2.3 Properness and dynamical relation

In the paper we will use the notion of *dynamical relation* between points of a topological space Z, which is an open subset of a compact metrizable space, with respect to a topological action $\Gamma \curvearrowright Z$ of a discrete group. The reader will find this definition in [18]; see also [34].

Definition 2.2 (dynamically related) Two points $z, z' \in Z$ are called *dynamically related* with respect to a topological action $\Gamma \curvearrowright Z$, written

$$z \overset{\Gamma}{\sim} z',$$

if there exists a sequence $\gamma_n \to \infty$ in Γ and a sequence $z_n \to z$ in Z such that $\gamma_n(z_n) \to z'$.

We write $z \overset{\gamma_n}{\sim} z'$ if z is dynamically related to z' with respect to the sequence $\gamma_n \to \infty$ in Γ. An action is properly discontinuous if and only if no points of Z are dynamically related to each other; see [18].

2.4 A transformation group lemma

Let K be a compact Hausdorff topological group, and let $K \curvearrowright Y$ be a continuous action on a compact Hausdorff space Y. We suppose that there exists a *cross-section* for the action, ie a compact subset $C \subset Y$ which contains precisely one point of every orbit.

Consider the natural surjective map

$$K \times C \overset{\alpha}{\to} Y$$

given by the action $\alpha(k, y) = ky$. We observe that Y carries the quotient topology with respect to α, because $K \times C$ is compact and Y is Hausdorff. The identifications by α are determined by the stabilizers of the points in C, namely $\alpha(k, y) = \alpha(k', y')$ if and only if $y = y'$ and $k^{-1}k' \in \mathrm{Stab}_K(y)$.

Consider now two such actions $K \curvearrowright Y_1$ and $K \curvearrowright Y_2$ by the same group with cross-sections $C_i \subset Y_i$, and suppose that

$$C_1 \overset{\phi}{\to} C_2$$

is a homeomorphism.

Lemma 2.3 If ϕ respects point stabilizers, ie $\text{Stab}_K(y_1) = \text{Stab}_K(\phi(y_1))$ for all $y_1 \in C_1$, then ϕ extends to a K–equivariant homeomorphism $\Phi\colon Y_1 \to Y_2$.

Proof According to the discussion above, the stabilizer condition implies that there exists a bijection $\Phi\colon Y_1 \to Y_2$ for which the diagram

$$
\begin{array}{ccc}
K \times C_1 & \xrightarrow{\ \text{id}_K \times \phi\ } & K \times C_2 \\
{\scriptstyle \alpha_1} \downarrow & & \downarrow {\scriptstyle \alpha_2} \\
Y_1 & \xrightarrow{\quad \Phi \quad} & Y_2
\end{array}
$$

commutes. Since the α_i are quotient projections, Φ is a homeomorphism. $\qquad\square$

2.5 Thom class

In this section H^{lf}_* denotes locally finite homology with $\mathbb{Z}_2$–coefficients.

Lemma 2.4 (Thom class) Let $F \xrightarrow{\iota} E \to B$ be a fiber bundle whose base B is a compact CW complex and whose fiber F is a connected m–manifold (without boundary). Suppose that there exists a section $s\colon B \to E$. Then the map

$$
\underbrace{H^{\text{lf}}_m(F)}_{\cong \mathbb{Z}_2} \xrightarrow{\iota_*} H^{\text{lf}}_m(E)
$$

induced by an inclusion of the fiber is nonzero.

Proof By thickening the section, one obtains a closed disk subbundle $D \to B$. Then we have the commutative diagram:

$$
\begin{array}{ccc}
H^{\text{lf}}_m(F) & \xrightarrow{\ \iota_*\ } & H^{\text{lf}}_m(E) \\
{\scriptstyle j} \downarrow & & \downarrow \\
H_m(D_F, \partial D_F) & \xrightarrow{\ \iota'_*\ } & H_m(D, \partial D)
\end{array}
$$

The map j is an isomorphism. The map

$$
H^m(D, \partial D) \xrightarrow{\iota'^*} H^m(D_F, \partial D_F)
$$

dual to ι'_* is surjective by Thom's theorem (see eg Theorem 8.1 in [41]): there is a class $u \in H^m(D, \partial D)$, the *Thom class* of the disk bundle $D \to B$, whose restriction to each fiber defines a generator of $H^m(D_F, \partial D_F)$. Dually, the map ι'_* is injective. It follows that the map ι_* is injective as well. $\qquad\square$

2.6 The horoboundary of metric spaces

We refer the reader to [21] and [3, Chapter II.1] for the definition and basic properties of horofunction compactification of metric spaces. In this section we describe these notions in the context of *nonsymmetric metrics*; compare [53].

Let (Y, d) be a metric space. We allow the distance d to be *nonsymmetric*, ie we only require that it is positive,

$$d(y, y') \geq 0 \quad \text{with equality if and only if } y = y',$$

and satisfies the triangle inequality

$$d(y, y') + d(y', y'') \geq d(y, y'').$$

The symmetrized distance

$$d^{\text{sym}}(y, y') := d(y, y') + d(y', y)$$

is a metric in the standard sense and induces a *topology* on Y. One observes that d is continuous, and the distance functions

$$d_y := d(\cdot, y)$$

are 1–Lipschitz with respect to d^{sym}. These functions satisfy the inequality

$$\textbf{(2.5)} \qquad\qquad -d(y, y') \leq d_y - d_{y'} \leq d(y', y).$$

Let $\mathcal{C}(Y)$ denote the space of continuous real-valued functions, equipped with the topology of uniform convergence on bounded subsets. Moreover, let

$$\bar{\mathcal{C}}(Y) := \mathcal{C}(Y)/\mathbb{R}$$

be the quotient space of continuous functions modulo additive constants. We will denote by $[f] \in \bar{\mathcal{C}}(Y)$ the equivalence class represented by a function $f \in \mathcal{C}(Y)$, and our notation $f \equiv g$ means that the difference $f - g$ is constant.

We consider the natural map

$$\textbf{(2.6)} \qquad\qquad Y \to \bar{\mathcal{C}}(Y), \quad y \mapsto [d_y].$$

It is continuous as a consequence of the triangle inequality. This map is a topological embedding provided that Y is a geodesic space; see [3, Chapter II.1], where this is proven for symmetric metrics, but the same proof goes through for nonsymmetric metrics as well. We assume from now on that the space Y is *geodesic*.

We identify Y with its image in $\bar{\mathcal{C}}(Y)$ and call the closure $\bar{Y}$ the *horoclosure* of Y, and $\partial_\infty Y := \bar{Y} - Y$ the *horoboundary* or *boundary at infinity*; ie we have the decomposition

$$\bar{Y} = Y \sqcup \partial_\infty Y.$$

We note that the horoclosure $\bar{Y}$ is Hausdorff and first countable since the space $\bar{\mathcal{C}}(Y)$ is.

The functions representing points in $\partial_\infty Y$ are called *horofunctions*. We write

$$y_n \to [h]$$

for a divergent sequence of points $y_n \to \infty$ in Y which converges to a point $[h] \in \partial_\infty Y$ represented by a horofunction h, ie $d_{y_n} \to h$ modulo additive constants, and say that (y_n) *converges at infinity*. Each horofunction is 1–Lipschitz with respect to the symmetrized metric.

If the metric space (Y, d^{sym}) is proper (which will be the case in this paper since we are interested in symmetric spaces), then the Arzelà–Ascoli theorem implies that the closure $\bar{Y}$ and the boundary $\partial_\infty Y$ at infinity are *compact*. In this case, $\bar{Y}$ is called the *horofunction compactification* of Y.

Suppose that

$$G \curvearrowright Y$$

is a d–isometric group action. Then the embedding (2.6) is equivariant with respect to the induced action on functions by $g \cdot f = f \circ g^{-1}$. For every $L > 0$, the subspace of L–Lipschitz functions $\overline{\mathrm{Lip}}_L(Y, d^{\mathrm{sym}}) \subset \bar{\mathcal{C}}(Y)$ is preserved by the action and contains, for $L \geq 1$, the horoclosure $\bar{Y}$. We equip G with the topology of uniform convergence on bounded subsets, using the symmetrized metric d^{sym} for both. Then the action $G \curvearrowright \overline{\mathrm{Lip}}_L(Y, d^{\mathrm{sym}})$ is continuous. In particular, the action

$$G \curvearrowright \bar{Y}$$

is continuous. We will use this fact in the situation when G is the isometry group of a Riemannian symmetric space of noncompact type. In this case the topology of uniform convergence on bounded subsets coincides with the Lie group topology.

An *oriented geodesic* in (Y, d) is a "forward" isometric embedding $c \colon I \to Y$, ie for any parameters $t_1 \leq t_2$ in I it holds that

$$d(c(t_1), c(t_2)) = t_2 - t_1.$$

In particular, c is continuous with respect to the symmetrized metric d^{sym}. The metric space (Y, d) is called a *geodesic* space if any pair of points (y, y') can be connected by an oriented geodesic from y to y'.

If (Y, d) is a geodesic space, then the horofunctions arising as limits of sequences along geodesic rays are called *Busemann functions*, and their sublevel and level sets are called *horoballs* and *horospheres*. We will denote by Hb_b a horoball for the Busemann function b, and more specifically, by $\mathrm{Hb}_{b,y}$ the horoball of b which contains the point y in its boundary horosphere.

In the situations studied in this paper, all horofunctions will turn out to be Busemann functions; see Section 5.2.3.

Suppose that $Z \subset Y$ is a closed convex subset. Then Z is a geodesic space with respect to the induced metric, and proper if Y is proper. There is a natural map

$$\text{(2.7)} \qquad\qquad\qquad \bar{Z}^Y \to \bar{Z}$$

from the extrinsic closure $\bar{Z}^Y \subset \bar{Y}$ of Z in Y to the intrinsic horoclosure $\bar{Z}$ of Z. It extends id_Z, and at infinity is the map $\partial^Y Z \to \partial_\infty Z$ from the boundary of Z in $\bar{Y}$ into the horoboundary of Z given by the restriction of horofunctions to Z. If the latter map is injective, then (2.7) is a homeomorphism and there is a natural *embedding* of horoclosures

$$\bar{Z} \to \bar{Y}$$

given by "unique extension of horofunctions".

2.7 Some notions of coarse geometry

Definition 2.8 A correspondence $f\colon (X, d) \to (X', d')$ between metric spaces is *coarse Lipschitz* if there exist constants L, A such that for all $x, y \in X$, $x' \in f(x)$ and $y' \in f(y)$, we have

$$d'(x', y') \le L d(x, y) + A.$$

Note that if (X, d) is a geodesic metric space, then in order to show that f is coarse Lipschitz it suffices to verify that there exists a constant C such that

$$d'(x', y') \le C$$

for all $x, y \in X$ with $d(x, y) \le 1$ and all $x' \in f(x)$ and $y' \in f(y)$.

Two correspondences $f_1, f_2 \colon (X, d) \to (X', d')$ are said to be *within distance D from each other*, written dist$(f, g) \leq D$, if for all $x \in X$ and $y_i \in f_i(x)$, we have

$$d'(y_1, y_2) \leq D.$$

Two correspondences f_1, f_2 are said to be within finite distance from each other if dist$(f_1, f_2) \leq D$ for some D.

A correspondence $(X, d) \to (X, d)$ is said to have *bounded displacement* if it is within finite distance from the identity map.

Definition 2.9 A coarse Lipschitz correspondence $f \colon (X, d) \to (X', d')$ is said to have a *coarse left inverse* if there exists a coarse Lipschitz correspondence $g \colon X' \to X$ such that the composition $g \circ f$ has bounded displacement.

By applying the axiom of choice, we can always replace a coarse Lipschitz correspondence $f \colon (X, d) \to (X', d')$ with a coarse Lipschitz map $f' \colon (X, d) \to (X', d')$ within bounded distance from f. With this in mind, if a coarse Lipschitz correspondence $f \colon (X, d) \to (X', d')$ admits a coarse left inverse, then f is within bounded distance from a quasiisometric embedding $f' \colon (X, d) \to (X', d')$. However, the converse is in general false, even in the setting of maps between finitely generated groups equipped with word metrics.

We now specialize these concepts to the context of group homomorphisms. We note that each continuous homomorphism of groups with left-invariant proper metrics is always coarse Lipschitz. Suppose in the remainder of this section that Γ is a finitely generated group and G is a connected Lie group equipped with a left-invariant metric.

Definition 2.10 We say that for a homomorphism $\rho \colon \Gamma \to G$, a correspondence $r \colon G \to \Gamma$ is a *coarse retraction* if r is a coarse left inverse to ρ. A subgroup $\Gamma < G$ is a *coarse retract* if the inclusion map $\Gamma \hookrightarrow G$ admits a coarse retraction.

Similarly, we say that a homomorphism $\rho \colon \Gamma \to G$ admits a *coarse equivariant retraction* if there exists a coarse Lipschitz retraction $r \colon G \to \Gamma$ such that

$$r(hg) = r(h)r(g) \quad \text{for all } h \in \rho(\Gamma).$$

Accordingly, a subgroup $\Gamma < G$ is a *coarse equivariant retract* if the inclusion homomorphism $\Gamma \hookrightarrow G$ admits a coarse equivariant retraction.

More generally, given an isometric action of $\rho\colon \Gamma \curvearrowright X$ on a metric space X, we say that a coarse retraction $r\colon X \to \Gamma$ is a *coarse equivariant retraction* if

$$r(\gamma x) = \gamma r(x) \quad \text{for all } \gamma \in \Gamma \text{ and } x \in X.$$

In the case when $X = G/K$ is the symmetric space associated with a connected semisimple Lie group G, a homomorphism $\Gamma \to G$ admits a coarse equivariant retraction if and only if the isometric action of Γ on X defined via ρ admits a coarse equivariant retraction. Similarly, a subgroup $\Gamma < G$ is a coarse retract if and only if the orbit map $\Gamma \to \Gamma x \subset X$ admits a coarse left inverse.

3 Symmetric spaces

3.1 Basics

We assume that the reader is familiar with basics of symmetric spaces of noncompact type (denoted by X throughout the paper), their isometry groups, visual boundaries and Tits boundaries. We refer the reader to [17; 37; 28] for the required background.

In what follows, G will be a connected semisimple Lie group, $K < G$ its maximal compact subgroup, the stabilizer of a basepoint in X which will be denoted by o or p. Then $X \cong G/K$. We let $B < G$ denote the minimal parabolic subgroup. All maximal flats in X are isometric to a *model* flat F_{mod}, which is isometric to a euclidean space $\mathbb{E}^n$, where n is the rank of X. The model flat comes equipped with a (finite) Weyl group, denoted by W. This group fixes the origin $0 \in F_{\mathrm{mod}}$, viewing F_{mod} as a vector space. We will use the notation a_{mod} for the visual boundary of F_{mod}; we will identify a_{mod} with the unit sphere in F_{mod} equipped with the angular metric. The sphere a_{mod} is the *model spherical apartment* for the group W. A fundamental domain for the action of W on F_{mod} is a certain convex cone $\Delta = \Delta_{\mathrm{mod}} \subset F_{\mathrm{mod}}$, the *model euclidean Weyl chamber* of W; its visual boundary is the model spherical Weyl chamber σ_{mod}, which is a spherical simplex in a_{mod}. We let $\iota\colon \sigma_{\mathrm{mod}} \to \sigma_{\mathrm{mod}}$ denote the *opposition involution*, also known as the *standard involution*, of σ_{mod}; it equals $-w_0$, where $w_0 \in W$ is the element sending σ_{mod} to the opposite chamber in the model apartment a_{mod}. We let $R \subset F_{\mathrm{mod}}^*$ denote the root system of X, and $\alpha_1, \ldots, \alpha_n$ will denote simple roots with respect to Δ:

$$\Delta = \{ x \in F_{\mathrm{mod}} : \alpha_i(x) \geq 0, i = 1, \ldots, n \}.$$

The space X has the Δ–valued "distance function" d_Δ, which is the complete G–congruence invariant of pairs of points in X; see [28].

We will denote by $\bar{X} = X \sqcup \partial_\infty X$ the *visual compactification* of X with respect to its Riemannian metric, equipped with the visual topology, and $\partial_{\text{Tits}} X$ the Tits boundary of X, which is the visual boundary together with the Tits metric $\angle_{\text{Tits}}$. The Tits boundary carries a natural structure as a piecewise spherical simplicial complex. For a simplex τ in $\partial_{\text{Tits}} X$ we will use the notation $\text{int}(\tau)$ for the open simplex in $\partial_{\text{Tits}} X$ which is the complement in τ to the union of its proper faces.

We will denote by xy the oriented geodesic segment in X connecting a point x to a point y; similarly, $x\xi$ will denote the geodesic ray from $x \in X$ asymptotic to the point $\xi \in \partial_\infty X$.

We will always use the notation $\tau, \hat{\tau}$ to indicate that the simplices τ and $\hat{\tau}$ in $\partial_{\text{Tits}} X$ are opposite (antipodal), ie are swapped by a Cartan involution of X. Each simplex, of course, has a continuum of antipodal simplices. Simplices τ and $\hat{\tau}$ are called x–*opposite* if the Cartan involution s_x fixing x sends τ to $\hat{\tau}$. Similarly, points $\xi, \hat{\xi} \in \partial_\infty X$ are x–opposite if s_x swaps ξ and $\hat{\xi}$.

We will use the notation $\theta \colon \partial_{\text{Tits}} X \to \sigma_{\text{mod}}$ for the *type map*, ie the canonical projection of the Tits building to the model chamber. For distinct points $x, y \in X$ we let $\theta(xy) \in \sigma_{\text{mod}}$ denote the *type of the direction* of the oriented segment xy, ie the unit vector in the direction of the vector $d_\Delta(x, y)$.

For each face τ_{mod} of σ_{mod} one defines the *flag manifold* $\text{Flag}_{\tau_{\text{mod}}}$, which is the set of all simplices of type τ_{mod} in $\partial_{\text{Tits}} X$. Equipped with the *visual topology*, $\text{Flag}_{\tau_{\text{mod}}}$ is a homogeneous manifold homeomorphic to G/P, where P is a parabolic subgroup of G stabilizing a face of type τ_{mod}. The full flag manifold $G/B = \text{Flag}(\sigma_{\text{mod}})$ is naturally identified with the Furstenberg boundary $\partial_{\text{Fu}} X$ of X.

The flag manifolds $\text{Flag}_{\tau_{\text{mod}}}$ and $\text{Flag}_{\iota\tau_{\text{mod}}}$ are *opposite* in the sense that the simplices opposite to simplices of type τ_{mod} have type $\iota\tau_{\text{mod}}$. To ease notation, we will denote the pair of opposite flag manifolds also by $\text{Flag}_{\pm\tau_{\text{mod}}}$ whenever convenient, ie we put $\text{Flag}_{+\tau_{\text{mod}}} := \text{Flag}_{\tau_{\text{mod}}}$ and $\text{Flag}_{-\tau_{\text{mod}}} := \text{Flag}_{\iota\tau_{\text{mod}}}$. The latter is also reasonable, because the simplices $-\tau_{\text{mod}}, \iota\tau_{\text{mod}} \subset a_{\text{mod}}$ lie in the same W–orbit, ie $-\tau_{\text{mod}}$ has type $\iota\tau_{\text{mod}}$. (Here we extend the notion of type to the model apartment, defining the type of a simplex in a_{mod} as its image under the natural quotient projection $a_{\text{mod}} \to a_{\text{mod}}/W \cong \sigma_{\text{mod}}$.)

For a simplex $\hat{\tau} \in \text{Flag}_{\iota\tau_{\text{mod}}}$, we let $C(\hat{\tau}) \subset \text{Flag}_{\tau_{\text{mod}}}$ denote the subset consisting of simplices antipodal to $\hat{\tau}$. This subset is open and dense in $\text{Flag}_{\tau_{\text{mod}}}$ and is called an *open Schubert stratum (or cell)* in $\text{Flag}_{\tau_{\text{mod}}}$.

For a point $x \in X$, we denote by $\Sigma_x X$ the *space of directions* at x, ie the unit sphere in the tangent space $T_x X$. Similarly, for a spherical building B or a subcomplex $C \subset B$, and a point $\xi \in C$, we let $\Sigma_\xi C$ denote the space of directions of C at ξ.

For a subset $Y \subset X$ we let $\partial_\infty Y$ denote the *visual boundary* of Y, ie its accumulation set in the visual boundary of X. A set $Y \subset X$ is said to be *asymptotic* to a subset $Z \subset \partial_\infty X$ if $Z \subset \partial_\infty Y$.

For a subset $Z \subset \partial_\infty X$ we let $V(x, Z) \subset X$ denote the union of geodesic rays $x\zeta$ for all $\zeta \in Z$. In the special case when $Z = \tau$ is a simplex[6] in $\partial_\infty X$, then $V(x, \tau)$ is the *Weyl sector in X with tip x and base τ*. A Weyl sector whose base is a chamber in $\partial_\infty X$ is a *(euclidean) Weyl chamber* in X.

Two Weyl sectors $V(x_1, \tau)$ and $V(x_2, \tau)$ are *strongly asymptotic* if for any $\epsilon > 0$ there exist points $y_i \in V(x_i, \tau)$ such that the subsectors $V(y_1, \tau)$ and $V(y_2, \tau)$ are ϵ–Hausdorff close.

A sequence $x_i \in V(x, \tau)$ (where τ has the type τ_{mod}) is τ_{mod}–*regular* if it diverges from the boundary of $V(x, \tau)$, ie from the subsectors $V(x, \tau')$ for all proper faces τ' of τ.

For distinct points $x, y \in X$ and $\xi \in \partial_\infty X$ we let $\angle_x(y, \xi)$ denote the angle between the geodesic segment xy and the geodesic ray $x\xi$ at the point $x \in X$.

We let b_η denote the Busemann function (defined with respect to the usual Riemannian metric on X) associated with a point η in the visual boundary of X. The gradient $\nabla b_\eta(x)$, for $x \in X$, is the unit vector tangent to the geodesic ray $x\eta$ and pointing away from η; see eg [4, page 28]. In particular, the slope of b_η along a geodesic segment xy at $z \in xy$ equals $-\cos(\angle_x(y, \xi))$.

Let d denote the standard distance function on X, and let Hb_η denote a closed horoball in X, which is a sublevel set $\{b_\eta \le t\}$ for the Riemannian Busemann function b_η.

For a chamber $\sigma \subset \partial_\infty X$ we let H_σ denote the associated *unipotent horocyclic subgroup*, the unipotent radical of the minimal parabolic subgroup B_σ of G stabilizing σ. Similarly, for a simplex τ in $\partial_{Tits} X$ we let H_τ denote the associated unipotent horocyclic subgroup, the unipotent radical in the parabolic subgroup P_τ of G stabilizing τ.

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[6]This means a simplex with respect to the spherical Tits building structure on $\partial_\infty X$.

Elements of H_τ preserve the strong asymptote classes of geodesic rays $x\xi$, $\xi \in \text{int}(\tau)$ and hence the strong asymptote classes of sectors $V(x, \tau)$. Furthermore, H_τ acts transitively on the set of sectors $V(x', \tau)$ strongly asymptotic to the given sector $V(x, \tau)$. We refer to [7, Proposition 14.21] for an algebraic argument and to [33, Section 2.10] for a geometric proof.

3.2 Parallel sets, stars and cones

3.2.1 Parallel sets Let $s \subset \partial_{\text{Tits}} X$ be an isometrically embedded (simplicial) unit sphere. We denote by $P(s) \subset X$ the *parallel set* associated to s, which can be defined as the union of maximal flats $F \subset X$ asymptotic to s, ie $s \subset \partial_\infty F$. Alternatively, one can define it as the union of flats $f \subset X$ with ideal boundary $\partial_\infty f = s$.

The parallel set is a totally geodesic subspace which splits metrically as the product

$$(3.1) \qquad\qquad P(s) \cong f \times \text{CS}(s)$$

of any of these flats and a symmetric space $\text{CS}(s)$ called its *cross-section*. Accordingly, the ideal boundary of the parallel set is a metric suspension

$$(3.2) \qquad\qquad \partial_{\text{Tits}} P(s) \cong \partial_{\text{Tits}} f \circ \partial_{\text{Tits}} \text{CS}(s).$$

It coincides with the subbuilding $\text{B}(s) \subset \partial_\infty X$ consisting of the union of all apartments $a \subset \partial_\infty X$ containing s,

$$\text{B}(s) = \partial_\infty P(s).$$

It is immediate that parallel sets are nonpositively curved symmetric spaces. However, they do not have noncompact type as their euclidean de Rham factors are nontrivial. The factor f in the splitting (3.2) of the parallel set is then the euclidean de Rham factor and the cross-section $\text{CS}(s)$ has trivial euclidean de Rham factor, ie it is a symmetric space of noncompact type.

For a pair of antipodal simplices $\tau_+, \tau_- \subset \partial_\infty X$ there exists a unique minimal singular sphere $s = s(\tau_-, \tau_+) \subset \partial_\infty X$ containing them. We let $P(\tau_-, \tau_+) := P(s(\tau_-, \tau_+))$; this parallel set is the union of (maximal) flats $F \subset X$ whose ideal boundaries contain $\tau_- \cup \tau_+$. In order to simplify the notation, we will denote $\text{B}(s(\tau_-, \tau_+))$ simply by $\text{B}(\tau_-, \tau_+)$. Given two antipodal points $\zeta_+, \zeta_- \in \partial_{\text{Tits}} X$ we let $\text{B}(\zeta_-, \zeta_+)$ denote the subbuilding $\text{B}(\tau_-, \tau_+)$, where $\tau_\pm$ are the antipodal simplices satisfying $\zeta_\pm \in \text{int}(\tau_\pm)$.

We will use the notation $\mathrm{CS}(\tau_-, \tau_+, p)$ for the cross-section $\mathrm{CS}(s)$ passing through the point $p \in P(s)$. Similarly, $f(\tau_-, \tau_+, p)$ will denote the flat in $P(\tau_-, \tau_+)$ which is parallel to the euclidean factor of $P(\tau_-, \tau_+)$ and contains p.

We will use the notation $T(s) = T(\tau_-, \tau_+)$ for the group of transvections along the flat f; this group is the same for all flats parallel to f and depends only on s.

3.2.2 Stars

Definition 3.3 (stars) Let $\tau \subset \partial_{\mathrm{Tits}} X$ be a simplex. We define the *star*[7] $\mathrm{st}(\tau)$ of the open simplex $\mathrm{int}(\tau)$ as the subcomplex of $\partial_{\mathrm{Tits}} X$ consisting of all simplices intersecting the open simplex $\mathrm{int}(\tau)$ nontrivially (ie containing τ). In other words, $\mathrm{st}(\tau)$ is the smallest subcomplex of $\partial_{\mathrm{Tits}} X$ containing all chambers σ such that $\tau \subset \sigma$.

We define the *open star* $\mathrm{ost}(\tau) \subset \partial_\infty X$ as the union of all open simplices whose closure intersects $\mathrm{int}(\tau)$ nontrivially. For the model simplex τ_{mod}, we will use the notation $\mathrm{ost}(\tau_{\mathrm{mod}})$ to denote its open star in the simplicial complex consisting of faces of σ_{mod}.

For a point $\xi \in \partial_{\mathrm{Tits}} X$ we let $\mathrm{st}(\xi) \subset \partial_\infty X$ denote the star of the simplex spanned by ξ, ie the unique simplex τ such that $\xi \in \mathrm{int}(\tau)$.

Note that $\mathrm{ost}(\tau)$ is an open subset of the simplex σ_{mod}; it does not include any open faces of τ except for the interior of τ. Furthermore, $\partial \, \mathrm{st}(\tau) = \mathrm{st}(\tau) - \mathrm{ost}(\tau)$ is the union of all panels[8] π of type $\theta(\pi) \not\supset \tau_{\mathrm{mod}}$ which are contained in a chamber with face τ.

Lemma 3.4 *The star $\mathrm{st}(\overline{\tau})$ of a simplex $\overline{\tau} \subset a_{\mathrm{mod}}$ is a convex subset of a_{mod}. Furthermore, $\mathrm{st}(\overline{\tau})$ equals the intersection of the simplicial hemispheres $\overline{h} \subset a_{\mathrm{mod}}$ such that $\mathrm{int}(\overline{\tau}) \subset \mathrm{int}\,\overline{h}$.*

Proof If a hemisphere $\overline{h}$ contains a simplex $\overline{\tau}$, but does not contain it in its boundary, then all chambers containing this simplex as a face belong to the (closed) hemisphere. Conversely, if a chamber $\overline{\sigma}$ does not contain $\overline{\tau}$ as a face, then there exists a wall which separates $\overline{\sigma}$ from $\overline{\tau}$. □

Similarly, the star $\mathrm{st}(\tau)$ of a simplex $\tau \subset \partial_{\mathrm{Tits}} X$ is a convex subset of $\partial_{\mathrm{Tits}} X$. One can represent it as the intersection of all simplicial $\frac{\pi}{2}$–balls which contain $\mathrm{int}(\tau)$ in their interior. One can represent $\mathrm{st}(\tau)$ also as the intersection of fewer balls:

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[7]The star $\mathrm{st}(\tau)$ is also known as the *residue* of τ.

[8]*Panels* in a spherical building are codimension-one faces. The type of a panel is its projection to σ_{mod}.

Lemma 3.5 (convexity of stars)   (i) *Let $\tau \subset \partial_{\text{Tits}} X$ be a simplex. Then $\text{st}(\tau)$ equals the intersection of the simplicial $\frac{\pi}{2}$–balls whose interior contains $\text{int}(\tau)$.*

 (ii) *For any simplex $\hat{\tau}$ opposite to τ, the star $\text{st}(\tau)$ equals the intersection of the subbuilding $\text{B}(\tau, \hat{\tau}) = \partial_{\infty} P(\tau, \hat{\tau})$ with all simplicial $\frac{\pi}{2}$–balls whose interior contains $\text{int}(\tau)$ and whose center lies in this subbuilding.*

Proof   (i)   If a simplicial $\frac{\pi}{2}$–ball contains a simplex τ, but does not contain it in its boundary, then all chambers containing this simplex as a face belong to this ball. Conversely, let σ be a chamber which does not contain τ as a face. There exists an apartment $a \subset \partial_{\text{Tits}} X$ which contains σ and τ. As before in the proof of Lemma 3.4, there exists a simplicial hemisphere $h \subset a$ containing τ but not σ. Then the simplicial $\frac{\pi}{2}$–ball with the same center as h contains τ but not σ.

(ii)   Note first that $\text{st}(\tau) \subset \text{B}(\tau, \hat{\tau})$. Then we argue as in part (i), observing that if $\sigma \subset \text{B}(\tau, \hat{\tau})$ then a can be chosen inside $\text{B}(\tau, \hat{\tau})$.   □

3.2.3   Stars and ideal boundaries of cross-sections   Let $\nu \subset \partial_{\infty} X$ be a simplex. We say that two chambers $\sigma_1, \sigma_2 \supset \nu$ are *ν–antipodal* if there exists a segment connecting interior points of σ_1, σ_2 and passing through an interior point of ν.

The link $\Sigma_{\nu} \text{st}(\nu)$ carries a natural structure as a topological spherical building, and is naturally isomorphic as such to $\partial_{\infty} \text{CS}(\hat{\nu}, \nu)$ for any $\hat{\nu}$ opposite to ν. Chambers $\sigma \subset \text{st}(\nu)$ correspond to chambers in $\Sigma_{\nu} \text{st}(\nu)$, and pairs of ν–opposite chambers to pairs of opposite chambers. It follows that for every chamber $\sigma \supset \nu$, the set of ν–opposite chambers in $\text{st}(\nu)$ is open and dense as a subset of $\text{st}_{\text{Fu}}(\nu)$. Here, we denote by $\text{st}_{\text{Fu}}(\nu) \subset \partial_{\text{Fu}} X$ the subset of chambers containing ν.

3.2.4   Weyl cones   Given a simplex τ in $\partial_{\text{Tits}} X$ and a point $x \in X$, the union $V(x, \text{st}(\tau))$ of all rays $x\zeta$ with $\zeta \in \text{st}(\tau)$ is called the *Weyl cone* with the tip x and the base $\text{st}(\tau)$. Below we will prove that Weyl cones $V(x, \text{st}(\tau))$ are convex. We begin with:

Lemma 3.6 *For every $x \in P(\tau, \hat{\tau})$, the Weyl cone $V(x, \text{st}(\tau))$ is contained in the parallel set $P(\tau, \hat{\tau})$.*

Proof   Consider a chamber σ in $\partial_{\text{Tits}} X$ containing τ. The Weyl sector $V(x, \sigma)$ is contained in a (unique maximal) flat $F \subset X$. Since $\tau, \hat{\tau}$ are antipodal with respect to x, we have $\tau \cup \hat{\tau} \subset \partial_{\infty} F$. Therefore, $F \subset P(\tau, \hat{\tau})$.   □

Proposition 3.7 (convexity of Weyl cones) *Let $\hat{\tau}$ be the simplex opposite to τ with respect to x. Then the Weyl cone $V(x, \mathrm{st}(\tau))$ is the intersection of the parallel set $P(\tau, \hat{\tau})$ with the horoballs which are centered at $\partial_{\infty} P(\tau, \hat{\tau})$ and contain $V(x, \mathrm{st}(\tau))$. In particular, $V(x, \mathrm{st}(\tau))$ is a closed convex subset of X.*

Proof   One inclusion is clear. We must prove that each point $y \in P(\tau, \hat{\tau}) - V(x, \mathrm{st}(\tau))$ is not contained in one of these horoballs. There exists a maximal flat $F \subset P(\tau, \hat{\tau})$ containing x and y. (Any two points in a parallel set lie in a common maximal flat.) We extend the oriented segment xy to a ray $x\eta$ inside F.

As in the proof of Lemma 3.5, there exists $\zeta \in \partial_{\infty} F$ such that $\bar{B}\left(\zeta, \frac{\pi}{2}\right)$ contains $\mathrm{st}(\tau)$ but does not contain η. Then the horoball $\mathrm{Hb}_{\zeta,x}$ intersects F in a half-space which contains x in its boundary hyperplane but does not contain η in its ideal boundary. Therefore, it does not contain y. By convexity, $V(x, \mathrm{st}(\tau)) \subset \mathrm{Hb}_{\zeta,x}$.   □

The following consequence will be important for us.

Corollary 3.8 (nested cones) *If $x' \in V(x, \mathrm{st}(\tau))$, then $V(x', \mathrm{st}(\tau)) \subset V(x, \mathrm{st}(\tau))$.*

Let $xy \subset X$ be an oriented τ_{mod}–regular geodesic segment. Then we define the simplex $\tau = \tau(xy) \subset \partial_{\infty} X$ as follows: forward extend the segment xy to the geodesic ray $x\xi$, and let τ be the unique face of type τ_{mod} of $\partial_{\mathrm{Tits}} X$ such that $\xi \in \mathrm{st}(\tau)$.

We refer the reader to [33, Lemma 2.16] for a proof of the following:

Lemma 3.9 (open cones) *If $x \in P(\tau, \hat{\tau})$, then $V(x, \mathrm{ost}(\tau)) - \{x\}$ is an open subset of $P(\tau, \hat{\tau})$. In particular, if $x_n \in P(\tau, \hat{\tau})$ flag-converges to τ, then $x_n \in V(x, \mathrm{ost}(\tau))$ for all sufficiently large n.*

Definition 3.10 (diamond) We define the τ_{mod}–*diamond* of a τ_{mod}–regular segment $x_- x_+$ as

$$\Diamond_{\tau_{\mathrm{mod}}}(x_-, x_+) = V(x_-, \mathrm{st}(\tau_+)) \cap V(x_+, \mathrm{st}(\tau_-)) \subset P(\tau_-, \tau_+),$$

where $\tau_{\pm} = \tau(x_{\mp} x_{\pm})$.

Thus, every diamond is a convex subset of X.

3.2.5   Shadows at infinity and strong asymptoticity of Weyl cones   This material is taken from [34, Section 4.1].

For a simplex $\tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ and a point $x \in X$, we consider the function

(3.11) $$\tau \mapsto d(x, P(\tau_-, \tau))$$

on the open Schubert stratum $C(\tau_-) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$. We denote by $\tau_+ \in C(\tau_-)$ the simplex x–opposite to τ_-.

Lemma 3.12 *The function* (3.11) *is continuous and proper.*

Proof This follows from the fact that $C(\tau_-)$ and X are homogeneous spaces for the parabolic subgroup P_{τ_-}. Indeed, continuity follows from the continuity of the function

$$g \mapsto d(x, P(\tau_-, g\tau_+)) = d(g^{-1}x, P(\tau_-, \tau_+))$$

on P_{τ_-} which factors through the orbit map $P_{\tau_-} \to C(\tau_-)$, $g \mapsto g\tau_+$.

Regarding properness, note that a simplex $\tau \in C(\tau_-)$ is determined by any point y contained in the parallel set $P(\tau_-, \tau)$, namely as the simplex y–opposite to τ_-. Thus, if $P(\tau_-, \tau) \cap B(x, R) \neq \varnothing$ for some fixed $R > 0$, then there exists $g \in P_{\tau_-}$ such that $\tau = g\tau_+$ and $d(x, gx) < R$. In particular, g lies in a compact subset. This implies properness. $\square$

Moreover, the function (3.11) has a unique minimum zero in τ_+.

We define the following open subsets of $C(\tau_-)$ which can be regarded as *shadows* of balls in X with respect to τ_-. For $x \in X$ and $r > 0$, we put

$$U_{\tau_-, x, r} := \{\tau \in C(\tau_-) \mid d(x, P(\tau_-, \tau)) < r\}.$$

The next fact expresses the uniform strong asymptoticity of asymptotic Weyl cones.

Lemma 3.13 *For $r, R > 0$, there exists $d = d(r, R) > 0$ such that if $y \in V(x, \mathrm{st}(\tau_-))$ with $d(y, \partial V(x, \mathrm{st}(\tau_-))) \geq d(r, R)$, then $U_{\tau_-, x, R} \subset U_{\tau_-, y, r}$.*

Proof If $U_{\tau_-, x, R} \not\subset U_{\tau_-, y, r}$ there exists $x' \in B(x, R)$ such that $d(y, V(x', \mathrm{st}(\tau_-))) \geq r$. Thus, if the assertion is wrong, there exist a sequence $x_n \to x_\infty$ in $\bar{B}(x, R)$ and a sequence $y_n \to \infty$ in $V(x, \mathrm{st}(\tau_-))$ such that $d(y_n, \partial V(x, \mathrm{st}(\tau_-))) \to +\infty$ and $d(y_n, V(x_n, \mathrm{st}(\tau_-))) \geq r$.

Let $\rho \colon [0, +\infty) \to V(x, \tau_-)$ be a geodesic ray with initial point x and asymptotic to an interior point of τ_-. Then the sequence (y_n) eventually enters every Weyl

cone $V(\rho(t), \operatorname{st}(\tau_-))$. Since the distance function $d(\cdot, V(x_n, \operatorname{st}(\tau_-)))$ is convex and bounded, and hence nonincreasing along rays asymptotic to $\operatorname{st}(\tau_-)$, we have that

$$R \ge d(x, V(x_n, \operatorname{st}(\tau_-))) \ge d(\rho(t), V(x_n, \operatorname{st}(\tau_-))) \ge d(y_n, V(x_n, \operatorname{st}(\tau_-))) \ge r$$

for $n \ge n(t)$. It follows that, for all t,

$$R \ge d(\rho(t), V(x_\infty, \operatorname{st}(\tau_-))) \ge r.$$

However, the ray ρ is strongly asymptotic to $V(x_\infty, \operatorname{st}(\tau_-))$, a contradiction. $\qquad\square$

3.2.6 Some spherical building facts We discuss some facts from spherical building geometry. In this paper, they are applied to the visual boundary $\partial_\infty X$ equipped with its structure of a thick spherical Tits building. We recall that a building is *thick* if every wall in the building is the intersection of three half-apartments.

First recall the following lemma; see the first part of [37, Lemma 3.10.2].[9]

Lemma 3.14 *In a spherical building* B, *every point* $\xi \in$ B *has an antipode in every apartment* $a \subset$ B, *and hence for every simplex* $\tau \subset B$, *there is an opposite simplex* $\hat{\tau} \subset a$.

We need the more precise statement that a point has *several* antipodes in an apartment unless it lies in the apartment itself:

Lemma 3.15 (cf [30, Sublemma 5.20]) *Let* ξ *be a point in a spherical building* B *and let* $a \subset$ B *be an apartment. If* ξ *has only one antipode in* a, *then* $\xi \in a$.

Proof Suppose that $\xi \notin a$ and let $\hat{\xi} \in a$ be an antipode of ξ. We choose a "generic" segment $\xi\hat{\xi}$ of length π tangent to a at $\hat{\xi}$ as follows. The suspension $B(\xi, \hat{\xi}) \subset$ B contains an apartment a' with the same unit tangent sphere at $\hat{\xi}$, $\Sigma_{\hat{\xi}} a' = \Sigma_{\hat{\xi}} a$. Inside a' there exists a segment $\xi\hat{\xi}$ whose interior does not meet simplices of codimension ≥ 2. Hence $\hat{\xi}\xi$ leaves a at an interior point $\eta \ne \xi, \hat{\xi}$ of a panel $\pi \subset a$, ie $a \cap \xi\hat{\xi} = \eta\hat{\xi}$ and $\pi \cap \xi\hat{\xi} = \eta$, and $\eta\xi$ initially lies in a chamber adjacent to π but not contained in a. Let $s \subset a$ be the wall (codimension-one singular sphere) containing π. By reflecting $\hat{\xi}$ at s, one obtains a second antipode for ξ in a, contradiction. $\qquad\square$

We will also need the following fact:

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[9]The statement of the second part contains a typo: H should be replaced by $\bar{B}\left(\eta, \frac{\pi}{2}\right) \cap A$.

Lemma 3.16 *Suppose that the spherical building* B *is thick. Then for any simplex* $\tau \subset$ B, *the intersection of all apartments containing* τ *equals* τ.

Proof Suppose first that τ is a chamber. Let $\xi \in$ B $-\tau$. For a generic point $\eta \in \text{int}(\tau)$, the segment $\eta\xi$ leaves τ through an interior point of a panel. By thickness, there exists an apartment $a \subset$ B such that $a \cap \eta\xi = \tau \cap \eta\xi$. Then $\xi \notin a$. This shows the assertion in the case when τ is a chamber. If τ is an arbitrary simplex, it follows that the intersection of all apartments containing τ is contained in the intersection of all chambers containing τ, which equals τ. $\qquad\square$

4 Regularity and contraction

In this section, we discuss a class of discrete subgroups of semisimple Lie groups which will be the framework for most of our investigations in this paper. In particular, it contains Anosov subgroups. The class of subgroups will be distinguished by an asymptotic *regularity* condition which in rank one just amounts to discreteness, but in higher rank is strictly stronger. The condition will be formulated in two equivalent ways. First dynamically in terms of the action on a flag manifold, then geometrically in terms of the orbits in the symmetric space.

Much of the material in this section can be found in some form already in the foundational work of Benoist (see [5, Section 3]) in the setting of Zariski dense subgroups of reductive algebraic groups over local fields, notably the notions of *regularity* and *contraction*, their essential equivalence, and the notion of *limit set*. For the sake of completeness we give independent proofs in our setting of discrete subgroups of semisimple Lie groups. Also our methods are rather different. We give here a geometric treatment and present the material in a form suitable to serve as a basis for the further development of our theory of discrete isometry groups acting on Riemannian symmetric spaces and euclidean buildings of higher rank, such as in our papers [34; 30; 31].

4.1 Contraction

Consider the action

$$G \curvearrowright \text{Flag}_{\tau_{\text{mod}}}$$

on the flag manifold of type τ_{mod}. Recall that for a simplex τ_- of type $\iota\tau_{\text{mod}}$ we denote by $C(\tau_-) \subset \text{Flag}_{\tau_{\text{mod}}}$ the open dense P_{τ_-}–orbit; it consists of the simplices opposite to τ_-.

We introduce the following dynamical conditions for sequences and subgroups in G:

Definition 4.1 (contracting sequence) A sequence (g_n) in G is τ_{mod}–*contracting* if there exist simplices $\tau_+ \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ and $\tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ such that

$$(4.2) \qquad\qquad g_n|_{C(\tau_-)} \to \tau_+$$

uniformly on compacts as $n \to +\infty$.

Definition 4.3 (convergence type dynamics) A subgroup $\Gamma < G$ is a τ_{mod}–*convergence subgroup* if every sequence (γ_n) of distinct elements in Γ contains a τ_{mod}–contracting subsequence.

Note that τ_{mod}–contracting sequences diverge to infinity and therefore τ_{mod}–convergence subgroups are necessarily *discrete*.

A notion for sequences in G equivalent to τ_{mod}–contraction had been introduced by Benoist in his fundamental paper [5]; see in particular his Lemma 3.5(5).

The contraction property exhibits a symmetry:

Lemma 4.4 (symmetry) *Property* (4.2) *is equivalent to the dual property that*

$$(4.5) \qquad\qquad g_n^{-1}|_{C(\tau_+)} \to \tau_-$$

uniformly on compacts as $n \to +\infty$.

Proof Suppose that (4.2) holds but (4.5) fails. Equivalently, after extraction there exists a sequence $\xi_n \to \xi \neq \tau_-$ in $\mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ such that $g_n\xi_n \to \xi' \in C(\tau_+)$. Since $\xi \neq \tau_-$, there exists $\hat\tau_- \in C(\tau_-)$ not opposite to ξ. (For instance, take an apartment in $\partial_\infty X$ containing τ_- and ξ, and let $\hat\tau_-$ be the simplex opposite to τ_- in this apartment.) Hence there is a sequence $\tau_n \to \hat\tau_-$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that τ_n is not opposite to ξ_n for all n. (It can be obtained eg by taking a sequence $h_n \to e$ in G such that $\xi_n = h_n\xi$ and putting $\tau_n = h_n\hat\tau_-$.) Since $\hat\tau_- \in C(\tau_-)$, condition (4.2) implies that $g_n\tau_n \to \tau_+$. It follows that τ_+ is not opposite to ξ', because $g_n\tau_n$ is not opposite to $g_n\xi_n$ and being opposite is an open condition. This contradicts $\xi' \in C(\tau_+)$. Therefore, condition (4.2) implies (4.5). The converse implication follows by replacing the sequence (g_n) with (g_n^{-1}). $\square$

Lemma 4.6 (uniqueness) *The simplices* $\tau_\pm$ *in* (4.2) *are uniquely determined.*

Proof Suppose that besides (4.2) we also have $g_n|_{C(\tau'_-)} \to \tau'_+$ with simplices $\tau'_\pm \in$ Flag$_{\pm\tau_{\text{mod}}}$. Since the subsets $C(\tau_-)$ and $C(\tau'_-)$ are open dense in Flag$_{\tau_{\text{mod}}}$, their intersection is nonempty and hence $\tau'_+ = \tau_+$. Using the equivalent dual conditions (4.5) we similarly obtain that $\tau'_- = \tau_-$. □

4.2 Regularity

The second set of asymptotic properties concerns the geometry of the orbits in X.

We first consider sequences in the euclidean model Weyl chamber Δ. Recall that $\partial_{\tau_{\text{mod}}}\Delta = V(0, \partial_{\tau_{\text{mod}}}\sigma_{\text{mod}}) \subset \Delta$ is the union of faces of Δ which do not contain the sector $V(0, \tau_{\text{mod}})$. Note that $\partial_{\tau_{\text{mod}}}\Delta \cap V(0, \tau_{\text{mod}}) = \partial V(0, \tau_{\text{mod}}) = V(0, \partial\tau_{\text{mod}})$.

Definition 4.7 A sequence (δ_n) in Δ is

(i) τ_{mod}–*regular* if it drifts away from $\partial_{\tau_{\text{mod}}}\Delta$:

$$d(\delta_n, \partial_{\tau_{\text{mod}}}\Delta) \to +\infty.$$

(ii) τ_{mod}–*pure* if it is contained in a tubular neighborhood of the sector $V(0, \tau_{\text{mod}})$ and drifts away from its boundary:

$$d(\delta_n, \partial V(0, \tau_{\text{mod}})) \to +\infty.$$

Note that (δ_n) is τ_{mod}–regular/pure if and only if $(\iota\delta_n)$ is $\iota\tau_{\text{mod}}$–regular/pure.

We extend these notions to sequences in X and G:

Definition 4.8 (regular and pure) (i) A sequence (x_n) in X is τ_{mod}–*regular* (respectively, τ_{mod}–*pure*) if for some (any) basepoint $o \in X$ the sequence of Δ–distances $d_\Delta(o, x_n)$ in Δ has this property.

(ii) A sequence (g_n) in G is τ_{mod}–*regular* (respectively, τ_{mod}–*pure*) if for some (any) point $x \in X$ the orbit sequence $(g_n x)$ in X has this property.

(iii) A subgroup $\Gamma < G$ is τ_{mod}–*regular* if all sequences of distinct elements in Γ have this property.

That these properties are independent of the basepoint and stable under bounded perturbation of the sequences is due to the triangle inequality $|d_\Delta(x, y) - d_\Delta(x', y')| \le d(x, x') + d(y, y')$.

Subsequences of τ_{mod}–regular/pure sequences are again τ_{mod}–regular/pure.

Clearly, τ_{mod}–pureness is a strengthening of τ_{mod}–regularity; a sequence in Δ is τ_{mod}–pure if and only if it is τ_{mod}–regular and contained in a tubular neighborhood of $V(0, \tau_{\mathrm{mod}})$.

The face type of a pure sequence is uniquely determined. Moreover, a τ_{mod}–regular sequence is τ'_{mod}–regular for every face type $\tau'_{\mathrm{mod}} \subset \tau_{\mathrm{mod}}$, because $\partial_{\tau'_{\mathrm{mod}}} \Delta \subset \partial_{\tau_{\mathrm{mod}}} \Delta$.

A sequence (g_n) is τ_{mod}–regular/pure if and only if the inverse sequence (g_n^{-1}) is $\iota\tau_{\mathrm{mod}}$–regular/pure, because $d_\Delta(x, g_n^{-1}x) = d_\Delta(g_n x, x) = \iota d_\Delta(x, g_n x)$.

Note that τ_{mod}–regular subgroups are in particular *discrete*. If $\mathrm{rank}(X) = 1$, then discreteness is equivalent to (σ_{mod}–)regularity. In higher rank, *regularity* can be considered as a *strengthening of discreteness*: a discrete subgroup $\Gamma < G$ may not be τ_{mod}–regular for any face type τ_{mod}; this can happen eg for free abelian subgroups of transvections of rank ≥ 2.

A property for sequences in G equivalent to regularity had appeared in the influential work of Benoist, [5, Lemma 3.5(1)].

Lemma 4.9 (pure subsequences) *Every sequence which diverges to infinity contains a τ_{mod}–pure subsequence for some face type $\tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$.*

Proof In the case of sequences in Δ, take τ_{mod} to be a minimal face type so that a subsequence is contained in a tubular neighborhood of $V(0, \tau_{\mathrm{mod}})$.   □

Note also that a sequence which diverges to infinity is τ_{mod}–regular if and only if it contains ν_{mod}–pure subsequences only for face types $\nu_{\mathrm{mod}} \supseteq \tau_{\mathrm{mod}}$.

The lemma implies in particular that every sequence $\gamma_n \to \infty$ in a discrete subgroup $\Gamma < G$ contains a subsequence which is τ_{mod}–regular, even τ_{mod}–pure, for some face type τ_{mod}.

Remark 4.10 Regularity has a natural Finsler geometric interpretation: a sequence in X is τ_{mod}–regular if and only if, in the Finsler compactification $\overline{X}^{\mathrm{Fins}} = X \sqcup \partial_\infty^{\mathrm{Fins}} X$ of X, it accumulates at the closure of the stratum $S_{\tau_{\mathrm{mod}}} \subset \partial_\infty^{\mathrm{Fins}} X$ at infinity. See Proposition 5.42.

4.3 Contraction implies regularity

In this section and the next, we relate contractivity and regularity for sequences and, as a consequence, establish the equivalence between τ_{mod}–regularity and the τ_{mod}–convergence property for discrete subgroups.

To relate contraction and regularity, it is useful to consider the G–action on flats. We let $\mathcal{F}_{\tau_{\mathrm{mod}}}$ denote the space of flats $f \subset X$ of type τ_{mod}. Two flats $f_\pm \in \mathcal{F}_{\tau_{\mathrm{mod}}}$ are *dynamically related* with respect to a sequence (g_n) in G, written

$$f_- \overset{(g_n)}{\sim} f_+,$$

if there exists a sequence of flats $f_n \to f_-$ in $\mathcal{F}_{\tau_{\mathrm{mod}}}$ such that $g_n f_n \to f_+$. The action of (g_n) on $\mathcal{F}_{\tau_{\mathrm{mod}}}$ is *proper* if and only if there are no dynamical relations with respect to subsequences; see Section 2.3.

Dynamical relations between singular flats yield dynamical relations between maximal ones:

Lemma 4.11 *If $f_\pm \in \mathcal{F}_{\tau_{\mathrm{mod}}}$ are flats such that $f_- \overset{(g_n)}{\sim} f_+$, then for every maximal flat $F_+ \supseteq f_+$ there exist a maximal flat $F_- \supseteq f_-$ and a subsequence (g_{n_k}) such that*

$$F_- \overset{(g_{n_k})}{\sim} F_+.$$

Proof Let $f_n \to f_-$ be a sequence in $\mathcal{F}_{\tau_{\mathrm{mod}}}$ such that $g_n f_n \to f_+$. Then there exists a sequence of maximal flats $F_n \supseteq f_n$ such that $g_n F_n \to F_+$. The sequence (F_n) is bounded because the sequence (f_n) is, and hence (F_n) subconverges to a maximal flat $F_- \supseteq f_-$. □

For pure sequences there are dynamical relations between singular flats of the corresponding type with respect to suitable subsequences:

Lemma 4.12 *If (g_n) is τ_{mod}–pure, then the action of (g_n) on $\mathcal{F}_{\tau_{\mathrm{mod}}}$ is not proper.*

More precisely, there exist simplices $\tau_\pm \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that for every flat $f_+ \in \mathcal{F}_{\tau_{\mathrm{mod}}}$ asymptotic to τ_+ there exist a flat $f_- \in \mathcal{F}_{\tau_{\mathrm{mod}}}$ asymptotic to τ_- and a subsequence (g_{n_k}) such that

$$f_- \overset{(g_{n_k})}{\sim} f_+.$$

Proof By pureness, there exists a sequence (τ_n) in $\text{Flag}_{\tau_{\text{mod}}}$ such that

(4.13) $$\sup_n d(g_n x, V(x, \tau_n)) < +\infty$$

for any point $x \in X$. There exists a subsequence (g_{n_k}) such that $\tau_{n_k} \to \tau_+$ and $g_{n_k}^{-1} \tau_{n_k} \to \tau_-$.

Let $f_+ \in \mathcal{F}_{\tau_{\text{mod}}}$ be asymptotic to τ_+. We choose $x \in f_+$ and consider the sequence of flats $f_k \in \mathcal{F}_{\tau_{\text{mod}}}$ through x asymptotic to τ_{n_k}. Then $f_k \to f_+$. The sequence of flats $(g_{n_k}^{-1} f_k)$ is bounded as a consequence of (4.13). Therefore, after further extraction, we obtain convergence $g_{n_k}^{-1} f_k \to f_-$. The limit flat f_- is asymptotic to τ_- because the f_k are asymptotic to $g_{n_k}^{-1} \tau_{n_k}$. $\square$

By a diagonal argument one can also show that the subsequences (g_{n_k}) in Lemmas 4.11 and 4.12 can be made independent of the flats F_+ and f_+, respectively.

For contracting sequences, the possible dynamical relations between maximal flats are restricted as follows:

Lemma 4.14 *Suppose that* (g_n) *is* τ_{mod}*–contracting with* (4.2)*, and that* $F_- \overset{(g_n)}{\sim} F_+$ *for maximal flats* $F_\pm \in \mathcal{F}$*. Then* $\tau_\pm \subset \partial_\infty F_\pm$*.*

Proof Suppose that $\tau_- \not\subset \partial_\infty F_-$. Then the visual boundary sphere $\partial_\infty F_-$ contains at least two different simplices $\hat{\tau}_-, \hat{\tau}'_-$ opposite to τ_-; see Lemma 3.15.

Let $F_n \to F_-$ be a sequence in $\mathcal{F}$ such that $g_n F_n \to F_+$. Due to $F_n \to F_-$, there exist sequences of simplices $\tau_n, \tau'_n \subset \partial_\infty F_n$ such that $\tau_n \to \hat{\tau}_-$ and $\tau'_n \to \hat{\tau}'_-$. In particular, $\tau_n \neq \tau'_n$ for large n. After extraction, we also obtain convergence $g_n \tau_n \to \hat{\tau}_+$ and $g_n \tau'_n \to \hat{\tau}'_+$. Moreover, since $g_n F_n \to F_+$, it follows that the limits $\hat{\tau}_+, \hat{\tau}'_+$ are *different* simplices in $\partial_\infty F_+$.

This is however in conflict with the contraction property (4.2). In view of $\hat{\tau}_-, \hat{\tau}'_- \in C(\tau_-)$, the contraction property implies that $g_n \tau_n \to \tau_+$ and $g_n \tau'_n \to \tau_+$, convergence to the same simplex, a contradiction. Thus, $\tau_- \subset \partial_\infty F_-$.

Considering the inverse sequence (g_n^{-1}) yields $\tau_+ \subset \partial_\infty F_+$; see Lemma 4.4. $\square$

Combining the previous lemmas, we obtain:

Lemma 4.15 *If a sequence in G is τ_{mod}–contracting and ν_{mod}–pure, then $\tau_{\text{mod}} \subseteq \nu_{\text{mod}}$.*

Proof We denote the sequence by (g_n) and assume (4.2). According to Lemmas 4.12 and 4.11, by ν_{mod}–purity, there exist simplices $\nu_\pm \in \mathrm{Flag}_{\nu_{\mathrm{mod}}}$ such that for *every* maximal flat F_+ with $\partial_\infty F_+ \supset \nu_+$ there exist a maximal flat F_- with $\partial_\infty F_- \supset \nu_-$ and a subsequence (g_{n_k}) such that

$$F_- \overset{(g_{n_k})}{\sim} F_+.$$

By Lemma 4.14, always $\tau_+ \subset \partial_\infty F_+$. Varying F_+, it follows that $\tau_+ \subseteq \nu_+$; see Lemma 3.16. $\qquad\square$

From these observations, we conclude:

Proposition 4.16 (contracting implies regular) *If a sequence in G is τ_{mod}–contracting then it is τ_{mod}–regular.*

Proof Consider a sequence in G which is not τ_{mod}–regular. Then a subsequence is ν_{mod}–pure for some face type $\nu_{\mathrm{mod}} \subseteq \partial_{\tau_{\mathrm{mod}}}\sigma_{\mathrm{mod}}$; compare Lemma 4.9. The condition on the face type is equivalent to $\nu_{\mathrm{mod}} \not\supseteq \tau_{\mathrm{mod}}$. By the last lemma, the subsequence cannot be τ_{mod}–contracting. $\qquad\square$

4.4 Regularity implies contraction

We now prove a converse to Proposition 4.16. Since contractivity involves a convergence condition, we can expect regular sequences to be contracting only after extraction.

Consider a τ_{mod}–regular sequence (g_n) in G. After fixing a point $x \in X$, there exist simplices $\tau_n^\pm \in \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$ (unique for large n) such that

(4.17) $$g_n^{\pm 1}x \in V(x, \mathrm{st}(\tau_n^\pm)).$$

Note that the sequence (g_n^{-1}) is $\iota\tau_{\mathrm{mod}}$–regular; see the comment after Definition 4.8.

Lemma 4.18 *If $\tau_n^\pm \to \tau_\pm$ in $\mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$, then (g_n) is τ_{mod}–contracting with (4.2).*

Proof Since $x \in g_n V(x, \mathrm{st}(\tau_n^-)) = V(g_nx, \mathrm{st}(g_n\tau_n^-))$, it follows together with $g_nx \in V(x, \mathrm{st}(\tau_n^+))$ that the Weyl cones $V(g_nx, \mathrm{st}(g_n\tau_n^-))$ and $V(x, \mathrm{st}(\tau_n^+))$ lie in the same parallel set, namely in $P(g_n\tau_n^-, \tau_n^+)$, and face in opposite directions. In particular, the simplices $g_n\tau_n^-$ and τ_n^+ are x–opposite, and thus $g_n\tau_n^-$ converges to the simplex $\hat{\tau}_+$ which is x–opposite to τ_+:

$$g_n\tau_n^- \to \hat{\tau}_+.$$

Since the sequence $(g_n^{-1} x)$ is $\iota\tau_{\mathrm{mod}}$–regular, it holds that

$$d(g_n^{-1} x, \partial V(x, \mathrm{st}(\tau_n^-))) \to +\infty.$$

By Lemma 3.13, for any $r, R > 0$, one has for $n \ge n(r, R)$ the inclusion of shadows:

$$U_{\tau_n^-, x, R} \subset U_{\tau_n^-, g_n^{-1} x, r}.$$

Thus, there exist sequences of positive numbers $R_n \to +\infty$ and $r_n \to 0$ such that

$$U_{\tau_n^-, x, R_n} \subset U_{\tau_n^-, g_n^{-1} x, r_n}$$

for large n; equivalently,

$$\textbf{(4.19)} \qquad g_n U_{\tau_n^-, x, R_n} \subset U_{g_n \tau_n^-, x, r_n}.$$

Since $\tau_n^- \to \tau_-$ and $R_n \to +\infty$, the shadows $U_{\tau_n^-, x, R_n} \subset C(\tau_n^-) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ *exhaust* $C(\tau_-)$ in the sense that every compact in $C(\tau_-)$ is contained in $U_{\tau_n^-, x, R_n}$ for large n.[10] On the other hand, since $g_n \tau_n^- \to \hat{\tau}_+$ and $r_n \to 0$, the $U_{g_n \tau_n^-, x, r_n}$ *shrink*, ie Hausdorff converge to the point τ_+.[11] Therefore, (4.19) implies that

$$g_n|_{C(\tau_-)} \to \tau_+$$

uniformly on compacts, ie (g_n) is τ_{mod}–contracting. $\qquad\square$

With the lemma, we can add the desired converse to Proposition 4.16 and obtain a characterization of regularity in terms of contraction:

Proposition 4.20 *The following properties are equivalent for sequences in G:*

(i) *Every subsequence contains a τ_{mod}–contracting subsequence.*

(ii) *The sequence is τ_{mod}–regular.*

Proof This is a direct consequence of the lemma. For the implication (ii) $\Longrightarrow$ (i) one uses the compactness of flag manifolds. The implication (i) $\Longrightarrow$ (ii) is obtained as follows (compare the proof of Proposition 4.16): If a sequence is not τ_{mod}–regular,

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[10]Indeed, for fixed $R > 0$ we have Hausdorff convergence $U_{\tau_n^-, x, R} \to U_{\tau_-, x, R}$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$, which follows eg from the transitivity of the action $K_x \curvearrowright \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ of the maximal compact subgroup $K_x < G$ fixing x. Furthermore, the shadows $U_{\tau_-, x, R}$ exhaust $C(\tau_-)$ as $R \to +\infty$; see the continuity part of Lemma 3.12.

[11]Indeed, $U_{g_n \tau_n^-, x, r} \to U_{\hat{\tau}_+, x, r}$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ for fixed $r > 0$, and $U_{\hat{\tau}_+, x, r} \to \tau_+$ as $r \to 0$, using again the continuity part of Lemma 3.12 and the fact that the function (3.11) assumes the value zero only in τ_+.

then it contains a ν_{mod}–pure subsequence for some face type $\nu_{\mathrm{mod}} \not\supseteq \tau_{\mathrm{mod}}$. Every subsequence of this subsequence is again ν_{mod}–pure and hence not τ_{mod}–contracting by Lemma 4.15. □

A version of Proposition 4.20 had already been proven by Benoist in his fundamental work; see [5, Lemma 3.5].

We conclude for subgroups:

Theorem 4.21 *A subgroup $\Gamma < G$ is τ_{mod}–regular if and only if it is a τ_{mod}–convergence subgroup.*

Proof By definition, Γ is τ_{mod}–regular if and only if every sequence (γ_n) of distinct elements in Γ is τ_{mod}–regular, and τ_{mod}–convergence if and only if every such sequence (γ_n) has a τ_{mod}–contracting subsequence. According to the proposition, both conditions are equivalent. □

4.5 Convergence at infinity and limit sets

The discussion in the preceding two sections leads to a natural notion of convergence at infinity for regular sequences in X and G. As regularity, it can be expressed both in terms of orbit geometry in X and dynamics on flag manifolds.

We first consider a τ_{mod}–regular sequence (g_n) in G. Flexibilizing condition (4.17), we choose points $x, x' \in X$ and consider a sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that

$$(4.22) \qquad \sup_n d(g_n x, V(x', \mathrm{st}(\tau_n))) < +\infty.$$

Note that the condition is independent of the choice of the points x and x'.[12]

Lemma 4.23 *The accumulation set of (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ depends only on (g_n).*

Proof Let (τ_n') be another sequence in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that $d(g_n x, V(x', \mathrm{st}(\tau_n')))$ is uniformly bounded. Assume that after extraction $\tau_n \to \tau$ and $\tau_n' \to \tau'$. We must show that $\tau = \tau'$.

We may suppose $x' = x$. There exist bounded sequences (b_n) and (b_n') in G such that

$$g_n b_n x \in V(x, \mathrm{st}(\tau_n)) \quad \text{and} \quad g_n b_n' x \in V(x, \mathrm{st}(\tau_n'))$$

---

[12]Recall that the Hausdorff distance of asymptotic Weyl cones $V(y, \mathrm{st}(\tau))$ and $V(y', \mathrm{st}(\tau))$ is bounded by the distance $d(y, y')$ of their tips.

for all n. Note that the sequences $(g_n b_n)$ and $(g_n b_n')$ in G are again τ_{mod}–regular. By Lemma 4.18, after further extraction, they are τ_{mod}–contracting with

$$g_n b_n|_{C(\tau_-)} \to \tau \quad \text{and} \quad g_n b_n'|_{C(\tau_-')} \to \tau'$$

uniformly on compacts for some $\tau_-, \tau_-' \in \mathrm{Flag}_{\iota\tau_{mod}}$. Moreover, we may assume convergence $b_n \to b$ and $b_n' \to b'$. Then

$$g_n|_{C(b\tau_-)} \to \tau \quad \text{and} \quad g_n|_{C(b'\tau_-')} \to \tau'$$

uniformly on compacts. With Lemma 4.6 it follows that $\tau = \tau'$. □

In view of the lemma, we can define the following notion of convergence:

Definition 4.24 (flag convergence of sequences in G) A τ_{mod}–regular sequence (g_n) in G τ_{mod}–*flag converges* to a simplex $\tau \in \mathrm{Flag}_{\tau_{mod}}$,

$$g_n \to \tau,$$

if $\tau_n \to \tau$ in $\mathrm{Flag}_{\tau_{mod}}$ for some sequence (τ_n) in $\mathrm{Flag}_{\tau_{mod}}$ satisfying (4.22).

We can now characterize contraction in terms of flag convergence. We rephrase Lemma 4.18 and show that its converse holds as well:

Lemma 4.25 *For a sequence* (g_n) *in* G *and simplices* $\tau_\pm \in \mathrm{Flag}_{\pm\tau_{mod}}$, *the following are equivalent:*

(i) (g_n) *is* τ_{mod}–*contracting with* $g_n|_{C(\tau_-)} \to \tau_+$ *uniformly on compacts.*

(ii) (g_n) *is* τ_{mod}–*regular and* $g_n^{\pm 1} \to \tau_\pm$.

In part (ii), the sequence (g_n^{-1}) is $\iota\tau_{mod}$–regular and $g_n^{-1} \to \tau_-$ means $\iota\tau_{mod}$–flag convergence.

Proof The implication (ii) $\Longrightarrow$ (i) is Lemma 4.18.

Conversely, suppose that (i) holds. Since the sequence (g_n) is τ_{mod}–contracting, it is τ_{mod}–regular by Proposition 4.16. Let $(\tau_n^\pm)$ be sequences satisfying (4.17). We must show that $\tau_n^\pm \to \tau_\pm$. Otherwise, after extraction we obtain that $\tau_n^\pm \to \tau_\pm'$ with $\tau_+' \neq \tau_+$ or $\tau_-' \neq \tau_-$. Then also $g_n|_{C(\tau_-')} \to \tau_+'$ by Lemma 4.18, and Lemma 4.6 implies that $\tau_\pm' = \tau_\pm$, a contradiction. □

Conversely, we can characterize flag convergence in terms of contraction and thus give an alternative *dynamical definition* of it:

Lemma 4.26 *For a sequence (g_n) in G, the following are equivalent:*

(i) *(g_n) is τ_{mod}–regular and $g_n \to \tau$.*

(ii) *There exists a bounded sequence (b_n) in G and $\tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ such that $g_n b_n|_{C(\tau_-)} \to \tau$ uniformly on compacts.*

(iii) *There exists a bounded sequence (b_n') in G such that $b_n' g_n^{-1}|_{C(\tau)}$ converges to a constant map uniformly on compacts.*

Proof (ii) $\Longrightarrow$ (i) According to the previous lemma the sequence $(g_n b_n)$ is τ_{mod}–regular and τ_{mod}–flag converges, $g_n b_n \to \tau$. Since $d(g_n x, g_n b_n x)$ is uniformly bounded, this is equivalent to (g_n) being τ_{mod}–regular and $g_n \to \tau$.

(i) $\Longrightarrow$ (ii) The sequence (g_n^{-1}) is $\iota\tau_{\mathrm{mod}}$–regular. There exists a bounded sequence (b_n') in G such that $(b_n' g_n^{-1})$ $\iota\tau_{\mathrm{mod}}$–flag converges, $b_n' g_n^{-1} \to \tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$. We put $b_n = b_n'^{-1}$. Since also $(g_n b_n)$ is τ_{mod}–regular and $g_n b_n \to \tau$, it follows from the previous lemma that $g_n b_n|_{C(\tau_-)} \to \tau$ uniformly on compacts.

The equivalence (ii) $\Longleftrightarrow$ (iii) with $b_n' = b_n^{-1}$ follows from [Lemma 4.4](#). $\qquad\square$

We carry over the notion of flag convergence to sequences in X.

Consider now a τ_{mod}–regular sequence (x_n) in X. We choose again a basepoint $x \in X$ and consider a sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that

$$(4.27) \qquad\qquad \sup_n d(x_n, V(x, \mathrm{st}(\tau_n))) < +\infty,$$

analogous to [(4.22)](#). As before, the condition is independent of the choice of the point x, and we obtain a version of [Lemma 4.23](#):

Lemma 4.28 *The accumulation set of (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ depends only on (x_n).*

Proof Let (g_n) be a sequence in G such that the sequence $(g_n^{-1} x_n)$ in X is bounded. Then (g_n) is τ_{mod}–regular and [(4.27)](#) becomes equivalent to [(4.22)](#). This reduces the claim to [Lemma 4.23](#). $\qquad\square$

We therefore can define, analogous to [Definition 4.24](#) above:

Definition 4.29 (flag convergence of sequences in X) A τ_{mod}–regular sequence (x_n) in X τ_{mod}–*flag converges* to a simplex $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$,

$$x_n \to \tau,$$

if $\tau_n \to \tau$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ for some sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ satisfying (4.27).

For any τ_{mod}–regular sequence (g_n) in G and any point $x \in X$, we have $g_n \to \tau$ if and only if $g_n x \to \tau$.

Flag convergence and flag limits are stable under bounded perturbations of sequences:

Lemma 4.30   (i) *For any τ_{mod}–regular sequence (g_n) and any bounded sequence (b_n) in G, the sequences (g_n) and $(g_n b_n)$ have the same τ_{mod}–flag accumulation sets in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$.*

  (ii) *If (x_n) and (x_n') are τ_{mod}–regular sequences in X such that $d(x_n, x_n')$ is uniformly bounded, then both sequences have the same τ_{mod}–flag accumulation set in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$.*

Proof (i) The sequence $(g_n b_n)$ is also τ_{mod}–regular and satisfies condition (4.22) if and only if (g_n) does.

(ii) The sequence (x_n') satisfies condition (4.27) if and only if (x_n') does.   □

Remark 4.31 There is a natural topology on the bordification $X \sqcup \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ which induces τ_{mod}–flag convergence. Moreover, the bordification embeds into a natural Finsler compactification of X; compare Remark 4.10.

Flag convergence leads to a notion of limit sets in flag manifolds for subgroups:

Definition 4.32 (flag limit set) For a subgroup $\Gamma < G$, the τ_{mod}–*limit set*

$$\Lambda_{\tau_{\mathrm{mod}}}(\Gamma) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

is the set of possible limit simplices of τ_{mod}–flag converging τ_{mod}–regular sequences in Γ, equivalently, the set of simplices τ_+ as in (4.2) for all τ_{mod}–contracting sequences in Γ.

The limit set is Γ–invariant and closed, as a diagonal argument shows.

Remark 4.33 In Section 3.6 of his groundbreaking work [5], Benoist introduced a notion of limit set Λ_Γ for Zariski dense subgroups Γ of reductive algebraic groups over local fields which in the case of real semisimple Lie groups is equivalent to (the dynamical version of) our concept of σ_{mod}–limit set $\Lambda_{\sigma_{\mathrm{mod}}}$.[13] What we call the τ_{mod}–limit set $\Lambda_{\tau_{\mathrm{mod}}}$ for other face types $\tau_{\mathrm{mod}} \subsetneq \sigma_{\mathrm{mod}}$ is mentioned in his Remark 3.6(3), and his work implies that, in the Zariski dense case, $\Lambda_{\tau_{\mathrm{mod}}}$ is the image of $\Lambda_{\sigma_{\mathrm{mod}}}$ under the natural projection $\mathrm{Flag}_{\sigma_{\mathrm{mod}}} \to \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ of flag manifolds.

4.6 Uniform regularity

In this section we introduce stronger forms of the regularity conditions discussed in Section 4.2.

We first consider sequences in the euclidean model Weyl chamber Δ.

Definition 4.34 A sequence $\delta_n \to \infty$ in Δ is *uniformly τ_{mod}–regular* if it drifts away from $\partial_{\tau_{\mathrm{mod}}}\Delta$ at a linear rate with respect to its norm:

$$\liminf_{n \to +\infty} \frac{d(\delta_n, \partial_{\tau_{\mathrm{mod}}}\Delta)}{\|\delta_n\|} > 0.$$

We extend these notions to sequences in X and G (compare Definition 4.8):

Definition 4.35 (uniformly regular) (i) A sequence (x_n) in X is *uniformly τ_{mod}–regular* if for some (any) basepoint $o \in X$ the sequence of Δ–distances $d_\Delta(o, x_n)$ in Δ has this property.

   (ii) A sequence (g_n) in G is *uniformly τ_{mod}–regular* if for some (any) point $x \in X$ the orbit sequence $(g_n x)$ in X has this property.

   (iii) A subgroup $\Gamma < G$ is *uniformly τ_{mod}–regular* if all sequences of distinct elements in Γ have this property.

For a subgroup $\Gamma < G$, uniform τ_{mod}–regularity is equivalent to the visual limit set $\Lambda(\Gamma) \subset \partial_\infty X$ being contained in the union of the open τ_{mod}–stars.

A subgroup $\Gamma < G$ is uniformly τ_{mod}–regular if and only if it is uniformly $\iota\tau_{\mathrm{mod}}$–regular.

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[13]Benoist's limit set Λ_Γ is contained in the flag manifold Y_Γ which in the case of real Lie groups is the full flag manifold G/B; see the beginning of Section 3 of his paper. It consists of the limit points of sequences contracting on G/B; see his Definitions 3.5 and 3.6.

5   Finsler compactifications of symmetric spaces

Let $X = G/K$ be a symmetric space of noncompact type.

5.1   Finsler metrics

5.1.1   The Riemannian distance   We denote by d^{Riem} the G–invariant Riemannian distance on X.

Let $xy \subset X$ be an oriented geodesic segment. The Busemann functions b_ξ for $\xi \in \partial_\infty X$ have slope ≥ -1 along xy, because they are 1–Lipschitz. Therefore,

$$(5.1) \qquad\qquad b_\xi(x) - b_\xi(y) \leq d^{\mathrm{Riem}}(x, y)$$

with equality if and only if $y \in x\xi$. Therefore, the Riemannian distance can be represented in the form

$$(5.2) \qquad\qquad d^{\mathrm{Riem}}(x, y) = \max_{\xi \in \partial_\infty X} (b_\xi(x) - b_\xi(y)).$$

5.1.2   Finsler distances   We fix a type $\bar\theta \in \sigma_{\mathrm{mod}}$ and now work only with Busemann functions b_ξ of this type, $\theta(\xi) = \bar\theta$. There is the following sharper bound for the slopes of such Busemann functions along segments:

Lemma 5.3   *The slope of a Busemann function b_ξ of type $\theta(\xi) = \bar\theta$ along a nondegenerate oriented segment $xy \subset X$ is at least $-\cos \angle(\theta(xy), \bar\theta)$ with equality in some point, equivalently, along the entire segment, if and only if $y \in V(x, \mathrm{st}(\xi))$.*

Proof   As we noted in Section 3.1, the slope of $b_\xi|_{xy}$ in an interior point $z \in xy$ equals $-\cos \angle_z(y, \xi)$. The angle $\angle_z(y, \xi)$ assumes its minimal value $\angle(\theta(zy), \bar\theta) = \angle(\theta(xy), \bar\theta)$ if and only if the segment zy and the ray $z\xi$ are contained in a euclidean Weyl chamber with tip at z, equivalently, if xy and $x\xi$ are contained in a euclidean Weyl chamber with tip at x, equivalently, if $y \in V(x, \mathrm{st}(\xi))$. In this case, the slope of b_ξ equals $\angle(\theta(xy), \bar\theta)$ along the entire segment xy. $\square$

We define the G–invariant $\bar\theta$–*Finsler distance* $d^{\bar\theta} \colon X \times X \to [0, +\infty)$ by

$$(5.4) \qquad\qquad d^{\bar\theta}(x, y) := \max_{\theta(\xi) = \bar\theta} (b_\xi(x) - b_\xi(y)),$$

where the maximum is taken over all ideal points $\xi \in \partial_\infty X$ with type $\theta(\xi) = \bar{\theta}$. By analogy with (5.1), we have the inequality

(5.5)
$$b_\xi(x) - b_\xi(y) \leq d^{\bar{\theta}}(x, y)$$

for all $\xi \in \partial_\infty X$ with $\theta(\xi) = \bar{\theta}$. According to the lemma, equality holds if and only if $y \in V(x, \mathrm{st}(\xi))$.

The triangle inequality is clearly satisfied for $d^{\bar{\theta}}$. In view of $\mathrm{diam}(\sigma_{\mathrm{mod}}) \leq \frac{\pi}{2}$ and the lemma we have semipositivity, ie $d^{\bar{\theta}} \geq 0$. Regarding symmetry, we have the identity

(5.6)
$$d^{\iota\bar{\theta}}(y, x) = d^{\bar{\theta}}(x, y),$$

and hence $d^{\bar{\theta}}$ is symmetric if and only if $\iota\bar{\theta} = \bar{\theta}$. To see (5.6) we note that, according to the lemma, b_ξ has maximal decay along xy if and only if $b_{\hat{\xi}}$ has maximal decay along yx, where $\hat{\xi} \in \partial_\infty X$ denotes the ideal point which is x–opposite to ξ and has type $\iota\bar{\theta}$.

The distance $d^{\bar{\theta}}$ can be derived from the vector-valued Δ–distance d_Δ by composing it with the linear functional $l_{\bar{\theta}} = -b_{\bar{\theta}}$ on $F_{\mathrm{mod}} \supset \Delta_{\mathrm{mod}}$ (normalized at the origin):

$$d^{\bar{\theta}} = l_{\bar{\theta}} \circ d_\Delta.$$

Let $F \subset X$ be a maximal flat. The restriction of the distance $d^{\bar{\theta}}$ to F can be written intrinsically as

(5.7)
$$d^{\bar{\theta}}(x, y) = \max_{\xi \in \partial_\infty F,\ \theta(\xi) = \bar{\theta}} (b_\xi(x) - b_\xi(y))$$

for $x, y \in F$, because equality holds in (5.5) if ξ lies in a chamber $\sigma \subset \partial_\infty F$ with $y \in V(x, \sigma)$. The restriction of $d^{\bar{\theta}}$ to a maximal flat is thus the translation invariant pseudometric associated to the W–invariant polyhedral seminorm on F_{mod} given by

(5.8)
$$\|\cdot\|_{\bar{\theta}} = \max_{w \in W}(l_{\bar{\theta}} \circ w^{-1}).$$

If the seminorm $\|\cdot\|_{\bar{\theta}}$ is a norm, equivalently, if $d^{\bar{\theta}}$ is a (nonsymmetric) metric, then $d^{\bar{\theta}}$ is equivalent to the Riemannian distance d^{Riem}. We describe when this is the case:

Lemma 5.9 (positivity) *The following are equivalent:*

   (i) $d^{\bar{\theta}}$ *is a (nonsymmetric) metric.*

   (ii) *The radius of σ_{mod} with respect to $\bar{\theta}$ is $< \frac{\pi}{2}$.*

   (iii) $\bar{\theta}$ *is not contained in a factor of a nontrivial spherical join decomposition of σ_{mod}.*

Proof The equivalence (i) $\iff$ (ii) is immediate in view of (5.8).

To see (ii) $\iff$ (iii), consider the spherical join decomposition of σ_{mod} into its irreducible factors σ_{mod}^i. These have diameter $< \frac{\pi}{2}$. We work now in the model apartment and represent directions by unit vectors which we orthogonally decompose into their σ_{mod}^i–components. Any two vectors in Δ_{mod} with nontrivial σ_{mod}^i–components for some i have angle $< \frac{\pi}{2}$. This yields the implication (iii) $\implies$ (ii). The converse direction is clear. $\square$

In particular, $d^{\bar{\theta}}$ is a (nonsymmetric) metric if $\bar{\theta}$ is regular or if X is irreducible.

If $d^{\bar{\theta}}$ is only a pseudometric, then X splits as a product $X_1 \times X_2$ such that $d^{\bar{\theta}}$ is degenerate precisely in the X_2–direction and induces an honest (nonsymmetric) metric on X_1.

5.1.3 Geodesics We first analyze when equality holds in the triangle inequality for the Finsler distance. Let τ_{mod} denote the face type spanned by $\bar{\theta}$, ie $\bar{\theta} \in \mathrm{int}(\tau_{\mathrm{mod}})$.

Lemma 5.10 *A triple of points $x, y, z \in X$ satisfies*

$$d^{\bar{\theta}}(x, z) + d^{\bar{\theta}}(z, y) = d^{\bar{\theta}}(x, y)$$

if and only if it is contained in a parallel set $P(\tau_-, \tau_+)$ for a pair of opposite simplices $\tau_\pm \in \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$ and z lies in the τ_{mod}–diamond determined by x, y:

$$(5.11) \qquad z \in \Diamond_{\tau_{\mathrm{mod}}}(x, y) = V(x, \mathrm{st}(\tau_+)) \cap V(y, \mathrm{st}(\tau_-)).$$

Proof Assume that the equality holds. Let $\tau_+ \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ be a simplex such that $y \in V(x, \mathrm{st}(\tau_+))$, and let $\xi_+ \in \tau_+$ be the ideal point with type $\theta(\xi_+) = \bar{\theta}$. Then b_{ξ_+} has maximal decay along xy; see Lemma 5.3. From

$$\begin{aligned} d^{\bar{\theta}}(x, y) &= d^{\bar{\theta}}(x, z) + d^{\bar{\theta}}(z, y) \\ &\geq \big(b_{\xi_+}(x) - b_{\xi_+}(z)\big) + \big(b_{\xi_+}(z) - b_{\xi_+}(y)\big) \\ &= b_{\xi_+}(x) - b_{\xi_+}(y) = d^{\bar{\theta}}(x, y), \end{aligned}$$

it follows that b_{ξ_+} must have maximal decay also along the segments xz and zy. This implies that $z \in V(x, \mathrm{st}(\tau_+))$, again by the same lemma. Furthermore, b_{ξ_-} has maximal decay along yx for the ideal point ξ_- which is x–opposite to ξ_+ and therefore contained in the simplex τ_- x–opposite to τ_+. It follows that also $z \in V(y, \mathrm{st}(\tau_-))$.

Conversely, if (5.11) holds, then b_{ξ_+} has maximal decay along xy, yz and xz, and hence the equality is satisfied. $\square$

It follows that the (pseudo)metric space $(X, d^{\bar{\theta}})$ is a *geodesic* space. The Riemannian geodesics in X are also $d^{\bar{\theta}}$–geodesics, but besides these there are other $d^{\bar{\theta}}$–geodesics, due to the nonstrict convexity of balls for the norm $\|\cdot\|_{\bar{\theta}}$.

The lemma yields a precise description of all $d^{\bar{\theta}}$–geodesics: a path $c\colon I \to X$ is an (unparametrized) $d^{\bar{\theta}}$–geodesic if and only if it is contained in a parallel set $P(\tau_-, \tau_+)$ with $\tau_\pm \in \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$ and

$$c(t') \in V(c(t), \mathrm{st}(\tau_+)) \quad \text{(equivalently, } c(t) \in V(c(t'), \mathrm{st}(\tau_-)))$$

for all $t < t'$ in I, ie c *drifts* towards τ_+ and away from τ_-. As a consequence, a geodesic $c\colon [t_-, t_+] \to X$ is contained the diamond $\Diamond_{\tau_{\mathrm{mod}}}(c(t_-), c(t_+))$ determined by its endpoints. Moreover, we obtain the *Finsler geometric interpretation of diamonds*, namely the diamond $\Diamond_{\tau_{\mathrm{mod}}}(x, y)$ is the union of all $d^{\bar{\theta}}$–geodesics xy.

The most relevant case for this paper is when $\bar{\theta}$ is *regular*, $\bar{\theta} \in \mathrm{int}(\sigma_{\mathrm{mod}})$. The above discussion then specializes as follows: The pair of simplices $\tau_\pm$ in the lemma becomes a pair of opposite chambers $\sigma_\pm$, the parallel set $P(\tau_-, \tau_+)$ becomes a maximal flat $F(\sigma_-, \sigma_+)$, the Weyl cones $V(\cdot, \mathrm{st}(\tau_\pm))$ become euclidean Weyl chambers $V(\cdot, \sigma_\pm)$. Thus (5.11) simplifies to

$$z \in V(x, \sigma_+) \cap V(y, \sigma_-),$$

and a $d^{\bar{\theta}}$–geodesic $c\colon I \to X$ is contained in a maximal flat $F(\sigma_-, \sigma_+)$ and

$$c(t') \in V(c(t), \sigma_+) \quad \text{(equivalently, } c(t) \in V(c(t'), \sigma_-))$$

for all $t < t'$ in I.

5.2 Finsler compactifications

Throughout this section we assume that the type $\bar{\theta}$ is regular, $\bar{\theta} \in \mathrm{int}(\sigma_{\mathrm{mod}})$. In particular, $d^{\bar{\theta}}$ is a metric.

5.2.1 Definition If one applies the horoboundary construction (see Section 2.6) to the Riemannian distance d^{Riem} on X, one obtains the visual compactification

$$(5.12) \qquad\qquad \bar{X} = X \sqcup \partial_\infty X.$$

The ideal boundary points are represented by Busemann functions, ie the horofunctions are in this case precisely the Busemann functions.

We define the $\bar{\theta}$–*Finsler compactification* of X as the compactification

(5.13) $$\bar{X}^{\bar{\theta}} = X \sqcup \partial^{\bar{\theta}}_{\infty} X$$

obtained by applying the horoboundary construction to the Finsler distance $d^{\bar{\theta}}$.

5.2.2 Horofunctions For a chamber $\sigma \subset \partial_{\infty} X$, let $\theta_{\sigma} \in \sigma$ denote the point of type $\bar{\theta}$. The associated Busemann function $b_{\theta_{\sigma}}$ is well defined up to additive constant, and the Busemann function $b_{\theta_{\sigma}} - b_{\theta_{\sigma}}(x)$ normalized in a point $x \in X$ is well defined.

According to our definition (5.4) of the $d^{\bar{\theta}}$–distance, we have

$$d_x^{\bar{\theta}} := d^{\bar{\theta}}(\,\cdot\,, x) = \max_{\sigma}(b_{\theta_{\sigma}} - b_{\theta_{\sigma}}(x)),$$

where the maximum is taken over all chambers σ. For a simplex $\tau \subset \partial_{\infty} X$ and a point $x \in X$, we consider the "mixed" Busemann function

(5.14) $$b_{\tau,x}^{\bar{\theta}} := \max_{\sigma \supset \tau}(b_{\theta_{\sigma}} - b_{\theta_{\sigma}}(x))$$

normalized in x, the maximum being taken only over the chambers which contain τ as a face. We will see that these are precisely the horofunctions for $\bar{X}^{\bar{\theta}}$.

On a euclidean Weyl chamber with tip at x, the function $d_x^{\bar{\theta}}$ agrees with one of the Busemann functions occurring in the maximum: If $\sigma, \hat{\sigma} \subset \partial_{\infty} X$ are x–opposite chambers, equivalently, if $x \in F(\hat{\sigma}, \sigma)$, then

$$d^{\bar{\theta}}(\,\cdot\,, x) = b_{\theta_{\sigma}} - b_{\theta_{\sigma}}(x)$$

on $V(x, \hat{\sigma})$; see Lemma 5.3. Thus, on a Weyl cone with tip at x, the function $d_x^{\bar{\theta}}$ reduces to a maximum over a subfamily of Busemann functions: if $\tau, \hat{\tau} \subset \partial_{\infty} X$ are x–opposite simplices, equivalently, if $x \in P(\hat{\tau}, \tau)$, then

$$d^{\bar{\theta}}(\,\cdot\,, x) = \max_{\sigma \supset \tau}(b_{\theta_{\sigma}} - b_{\theta_{\sigma}}(x)) = b_{\tau,x}^{\bar{\theta}}$$

on $V(x, \mathrm{st}(\hat{\tau}))$. For the normalized distance functions, we observe that if $o \in X$ is a basepoint and if x lies in the Weyl sector $V(o, \tau)$, equivalently, if $o \in V(x, \hat{\tau}) = \bigcap_{\hat{\sigma} \supset \hat{\tau}} V(x, \hat{\sigma})$, then the difference $b_{\theta_{\sigma}}(o) - b_{\theta_{\sigma}}(x)$ has the same value for all chambers $\sigma \supset \tau$, and hence the function $d_x^{\bar{\theta}}$ normalized in o is given by

(5.15) $$d_x^{\bar{\theta}} - d_x^{\bar{\theta}}(o) = \max_{\sigma \supset \tau}(b_{\theta_{\sigma}} - b_{\theta_{\sigma}}(o)) = b_{\tau,o}^{\bar{\theta}}$$

on the Weyl cone $V(x, \mathrm{st}(\hat{\tau}))$.

With these observations we are prepared for understanding the horofunctions, ie the limits of (normalized) distance functions $d_x^{\bar{\theta}}$ as $x \to \infty$. We first show that the mixed Busemann functions $b_{\tau,x}^{\bar{\theta}}$ are horofunctions.

Lemma 5.16 *Let $o \in X$ and $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$. If (x_n) is a τ_{mod}–regular sequence in the Weyl sector $V(o, \tau)$, then*

$$d_{x_n}^{\bar{\theta}} - d_{x_n}^{\bar{\theta}}(o) \to b_{\tau,o}^{\bar{\theta}}$$

uniformly on compacts in X.

Proof We proceed in two steps.

Step 1 Let $\hat{\tau}$ be the simplex o–opposite to τ. We first note that the claimed convergence holds uniformly on compacts in $P(\hat{\tau}, \tau)$. In fact, on every such compact, eventually equality holds. This follows from (5.15) and because the τ_{mod}–regularity of the sequence (x_n) implies that the Weyl cones $V(x_n, \mathrm{st}(\hat{\tau}))$ exhaust $P(\hat{\tau}, \tau)$ as $n \to +\infty$.

Step 2 To verify the convergence on all of X, we use the action of the unipotent horocyclic subgroup H_τ. The Busemann functions b_ξ centered at ideal points $\xi \in \mathrm{st}(\tau)$ are H_τ–invariant, $b_\xi \circ h^{-1} = b_{h\xi} = b_\xi$ for $h \in H_\tau$, and hence also the mixed Busemann functions,

$$b_{\tau,x}^{\bar{\theta}} \circ h^{-1} = b_{\tau,x}^{\bar{\theta}}.$$

By step 1, it holds for $(h, x) \in H_\tau \times P(\hat{\tau}, \tau)$ that

$$(5.17) \qquad \underbrace{d_{hx_n}^{\bar{\theta}}(hx)}_{d_{x_n}^{\bar{\theta}}(x)} - d_{x_n}^{\bar{\theta}}(o) \to \underbrace{b_{\tau,o}^{\bar{\theta}}(hx)}_{b_{\tau,o}^{\bar{\theta}}(x)},$$

and the convergence is uniform on $H_\tau \times A$ for $A \subset P(\hat{\tau}, \tau)$ compact. Note that

$$d^{\bar{\theta}}(x_n, hx_n) = d^{\bar{\theta}}(h^{-1}x_n, x_n) \to 0$$

as $n \to +\infty$ locally uniformly in h, because (x_n) is a τ_{mod}–regular sequence in $V(o, \tau)$ and therefore drifts away from $\partial V(o, \tau)$. Hence

$$d_{hx_n}^{\bar{\theta}} - d_{x_n}^{\bar{\theta}} \to 0$$

uniformly on X due to the triangle inequality; compare (2.5). It follows from (5.17) that

$$d_{x_n}^{\bar{\theta}}(hx) - d_{x_n}^{\bar{\theta}}(o) \to b_{\tau,o}^{\bar{\theta}}(hx)$$

locally uniformly in (h, x), ie

$$d_{x_n}^{\bar{\theta}} - d_{x_n}^{\bar{\theta}}(o) \to b_{\tau,o}^{\bar{\theta}}$$

locally uniformly on X, as claimed. $\qquad \square$

We show next that, conversely, there are no other horofunctions besides the mixed Busemann functions $b_{\tau,x}^{\bar{\theta}}$:

Lemma 5.18 *Let $x_n \to \infty$ be a divergent sequence in X. Then, after extraction, there exist a simplex $\tau \subset \partial_\infty X$ and a point $p \in X$ such that*

$$d_{x_n}^{\bar{\theta}} - d_{x_n}^{\bar{\theta}}(p) \to b_{\tau,p}^{\bar{\theta}}$$

uniformly on compacts in X.

Proof We reduce the assertion to the previous lemma using the action of the maximal compact subgroup $K < G$ fixing a basepoint $o \in X$. There exists a sequence (k_n) in K such that the sequence $(k_n x_n)$ is contained in a fixed euclidean Weyl chamber $V(o, \sigma)$. After extraction, we may assume that $k_n \to e$. We may assume moreover that the sequence (x_n), equivalently, $(k_n x_n)$ is τ_{mod}–pure for some face type τ_{mod}. Let $\tau \subset \sigma$ be the face of type τ_{mod}. After further extraction, there exists a point $p \in V(o, \sigma)$ such that $(k_n x_n)$ approaches the Weyl sector $V(p, \tau) \subset V(o, \sigma)$, ie there exists a sequence (y_n) in $V(p, \tau)$ such that $d(k_n x_n, y_n) \to 0$. The sequence (y_n) is then also τ_{mod}–pure, and in particular τ_{mod}–regular. By the previous lemma, $d_{y_n}^{\bar{\theta}} - d_{y_n}^{\bar{\theta}}(p) \to b_{\tau,p}^{\bar{\theta}}$ uniformly on compacts, and hence

$$d_{k_n x_n}^{\bar{\theta}} - d_{k_n x_n}^{\bar{\theta}}(p) \to b_{\tau,p}^{\bar{\theta}}$$

uniformly on compacts. Since $b_{\tau,p}^{\bar{\theta}} \circ k_n \to b_{\tau,p}^{\bar{\theta}}$ uniformly on compacts, the assertion follows. $\qquad \square$

Thus, the horofunctions for $\bar{X}^{\bar{\theta}}$ are precisely the mixed Busemann functions $b_{\tau,p}^{\bar{\theta}}$.

We now discuss some of their properties. As already mentioned, the functions $b_{\tau,p}^{\bar{\theta}}$ are invariant under the unipotent horocyclic subgroup H_τ. Moreover, they are invariant up to additive constants under transvections towards τ:

Lemma 5.19 *For a transvection t with axes asymptotic to $\xi \in \tau$ it holds that*[14]

$$b_{\tau,p}^{\bar{\theta}} \circ t^{-1} \equiv b_{\tau,p}^{\bar{\theta}}.$$

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[14]Recall that the notation $f \equiv g$ for functions f, g means that $f - g$ is a constant.

Proof For every chamber $\sigma \supset \tau$ the function $b_{\theta_\sigma} \circ t^{-1} - b_{\theta_\sigma}$ is constant, because t fixes $\mathrm{st}(\tau) \supset \sigma$ and therefore $b_{\theta_\sigma} \circ t^{-1} \equiv b_{\theta_\sigma}$. Furthermore, the difference $b_{\theta_\sigma} \circ t^{-1} - b_{\theta_\sigma}$ is independent of σ because, along a t–axis, b_{θ_σ} is linear with slope $-\cos \angle(\theta(\xi), \overline{\theta})$ independent of σ. It follows that $b^{\overline{\theta}}_{\tau,p} \circ t^{-1} - b^{\overline{\theta}}_{\tau,p} = b_{\theta_\sigma} \circ t^{-1} - b_{\theta_\sigma}$ is constant. $\qquad\square$

Our next aim is to distinguish the functions $b^{\overline{\theta}}_{\tau,p}$ from each other.

Let $\hat{\tau}$ be the simplex p–opposite to τ, and let $\mathrm{CS}(p) = \mathrm{CS}(\tau, \hat{\tau}, p)$ denote the cross-section of the parallel set $P(\tau, \hat{\tau})$ through p.

Lemma 5.20 *We have $b^{\overline{\theta}}_{\tau,p} > 0$ on $\mathrm{CS}(p) - \{p\}$.*

Proof Let $p \neq q \in \mathrm{CS}(p)$. We need to find a chamber $\sigma \supset \tau$ such that

$$b_{\theta_\sigma}(q) > b_{\theta_\sigma}(p).$$

The latter holds if

$$\angle_p(q, \theta_\sigma) > \tfrac{\pi}{2}$$

because then the convex function b_{θ_σ} strictly increases along pq.

Let $F \subset P(\tau, \hat{\tau})$ be a maximal flat containing pq. Then $\tau \subset \partial_\infty F$, and we denote by $\eta \in \partial_\infty F$ the ideal point with $q \in p\eta$. We will show that $\partial_\infty F$ contains a chamber $\sigma \supset \tau$ with the desired property, equivalently, with the property that

(5.21) $$\angle_{\mathrm{Tits}}(\eta, \theta_\sigma) > \tfrac{\pi}{2}.$$

Let $\overline{\theta}' \in \mathrm{int}(\tau_{\mathrm{mod}})$ be the nearest point projection of $\overline{\theta} \in \mathrm{int}(\sigma_{\mathrm{mod}})$ to τ_{mod}. We denote by $\theta'_\tau \in \tau$ the point of type $\overline{\theta}'$. The arcs $\theta'_\tau \theta_\sigma$ in $\partial_\infty F$ for $\sigma \supset \tau$ are perpendicular to τ. Note that $\tau \subset \partial B(\eta, \tfrac{\pi}{2})$, because $\eta \in \partial_\infty \mathrm{CS}(p)$, and hence also the arc $\theta'_\tau \eta$ is perpendicular to τ. Property (5.21) is equivalent to

(5.22) $$\angle_{\theta'_\tau}(\eta, \theta_\sigma) > \tfrac{\pi}{2}.$$

Since the type $\overline{\theta}$ is regular, the directions $\overrightarrow{\theta'_\tau \theta_\sigma}$ for the chambers $\sigma \supset \tau$ correspond to a regular Weyl orbit in the spherical Coxeter complex associated to the link of τ in $\partial_{\mathrm{Tits}} X$. If (5.22) would fail for all chambers $\sigma \supset \tau$, then this regular Weyl orbit would be contained in a closed hemisphere, which is impossible.[15] $\qquad\square$

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[15] In a spherical Coxeter complex without sphere factor, no regular Weyl orbit is contained in a closed hemisphere; compare Lemma 5.9.

Based on these properties, we can now distinguish the functions $b^{\overline{\theta}}_{\tau,p}$ from each other.

Lemma 5.23 (distinguishing horofunctions) *We have $b^{\overline{\theta}}_{\tau,p} \equiv b^{\overline{\theta}}_{\tau',p'}$ if and only if $\tau = \tau'$ and the sectors $V(p,\tau)$ and $V(p',\tau)$ are strongly asymptotic.*[16]

Proof We first show that τ can be read off the asymptotics of $b^{\overline{\theta}}_{\tau,p}$.

For a chamber σ and an ideal point $\xi \in \partial_\infty X$ it holds that $\angle_{\mathrm{Tits}}(\xi, \theta_\sigma) = \angle(\theta(\xi), \overline{\theta})$ if and only if $\xi \in \sigma$, since $\overline{\theta} \in \mathrm{int}(\sigma_{\mathrm{mod}})$. Lemma 5.3 therefore implies that along a ray $x\xi$, the function b_{θ_σ} has slope $\equiv -\cos\angle(\theta(\xi), \overline{\theta})$ if $\xi \in \sigma$, and slope greater than $-\cos\angle(\theta(\xi), \overline{\theta})$ everywhere if $\xi \notin \sigma$. It follows that $b^{\overline{\theta}}_{\tau,p}$ has slope $\equiv -\cos\angle(\theta(\xi), \overline{\theta})$ along $x\xi$ if and only if this is the case for the b_{θ_σ} for all chambers $\sigma \supset \tau$, ie if and only if $\xi \in \bigcap_{\sigma \supset \tau} \sigma = \tau$. Hence, $b^{\overline{\theta}}_{\tau,p} \equiv b^{\overline{\theta}}_{\tau',p'}$ implies that $\tau = \tau'$.

Due to H_τ–invariance, $b^{\overline{\theta}}_{\tau,hp} = b^{\overline{\theta}}_{\tau,p}$ for $h \in H_\tau$, we may replace p, p' by points in their H_τ–orbits. We can therefore assume that $p, p' \in P = P(\widehat{\tau}, \tau)$ with $\widehat{\tau} \in C(\tau)$, because every H_τ–orbit intersects the parallel set P (exactly once). In view of Lemma 5.19, we may furthermore replace p, p' by their images under transvections along lines parallel to the euclidean factor of P, and can thus assume that they lie in the same cross-section of P, that is, $p' \in \mathrm{CS}(p)$. Now Lemma 5.20 implies that $p = p'$.

The converse follows from the invariance of the equivalence classes $[b^{\overline{\theta}}_{\tau,\cdot}]$ under H_τ and the transvections along lines asymptotic to τ.   □

Corollary 5.24 *The points in $\partial^{\overline{\theta}}_\infty X$ are in one-to-one correspondence with the strong asymptote classes of Weyl sectors in X.*

Note that all Finsler boundary points are limits of sequences along Weyl sectors, and in particular limits of sequences along Finsler geodesic rays. Hence all horofunctions are *Busemann functions*, as defined in Section 2.6.

Corollary 5.25 *The Furstenberg boundary $\partial_{\mathrm{Fu}} X$ embeds G–equivariantly into $\partial^{\overline{\theta}}_\infty X$ as the set of equivalence classes $[b^{\overline{\theta}}_{\sigma,\cdot}]$ of chambers $\sigma \subset \partial_{\mathrm{Fu}} X$.*

Proof Note that $b^{\overline{\theta}}_{\sigma,p} \equiv b^{\overline{\theta}}_{\sigma,p'}$ for any pair of points $p, p' \in X$. Thus, the equivariant embedding is given by the map

$$\sigma \mapsto [b^{\overline{\theta}}_{\sigma,\cdot}].$$ □

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[16]Note that two asymptotic Weyl sectors $V(x,\tau)$ and $V(x',\tau)$ are strongly asymptotic if and only if their tips x and x' lie in the same orbit of the closed subgroup of P_τ which is generated by H_τ and the transvections along lines asymptotic to τ.

5.2.3 Convergence at infinity We fix a basepoint $o \in X$ and denote by $K < G$ the maximal compact subgroup fixing o.

We first study the convergence at infinity of divergent sequences in X. Since a divergent sequence always contains pure subsequences of some face type, we can restrict to this case.

Let (x_n) be a τ_{mod}–pure sequence in X. There exists a sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that $x_n \in V(o, \mathrm{st}(\tau_n))$. Let $\hat{\tau}_n \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ denote the simplices o–opposite to τ_n. Then $V(o, \mathrm{st}(\tau_n)) \subset P(\hat{\tau}_n, \tau_n)$. Due to pureness, there exists a bounded sequence of points $p_n \in \mathrm{CS}(\hat{\tau}_n, \tau_n, o)$ such that x_n belongs to the flat $f(\hat{\tau}_n, \tau_n, p_n)$ through p_n with visual boundary $\partial_\infty f(\hat{\tau}_n, \tau_n, p_n) = s(\hat{\tau}_n, \tau_n)$, the singular sphere spanned by $\hat{\tau}_n$ and τ_n; see Section 3.2.1.

Proposition 5.26 (convergence at infinity) *Under these assumptions, (x_n) converges in $\overline{X}^{\overline{\theta}}$ if and only if there is convergence $\tau_n \to \tau$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ and $p_n \to p \in \mathrm{CS}(\hat{\tau}, \tau, o)$ in X. In this case,*
$$x_n \to [b_{\tau,p}^{\overline{\theta}}].$$

Proof Suppose that $\tau_n \to \tau$ and $p_n \to p$. We write $\tau_n = k_n \tau$ and $\hat{\tau}_n = k_n \hat{\tau}$ with $k_n \to e$ in K. The sequence of points
$$k_n^{-1} x_n \in f(\hat{\tau}, \tau, \underbrace{k_n^{-1} p_n}_{\to p}) \subset P(\hat{\tau}, \tau)$$
is also τ_{mod}–pure and contained in a tubular neighborhood of the sector $V(o, \tau)$. It follows that $d(k_n^{-1} x_n, V(p, \tau)) \to 0$. Lemma 5.16 then implies that
$$d_{k_n^{-1} x_n}^{\overline{\theta}} - d_{k_n^{-1} x_n}^{\overline{\theta}}(p) \to b_{\tau,p}^{\overline{\theta}}$$
uniformly on compacts, and furthermore that
$$d_{x_n}^{\overline{\theta}} - d_{x_n}^{\overline{\theta}}(p) \to b_{\tau,p}^{\overline{\theta}},$$
ie $x_n \to [b_{\tau,p}^{\overline{\theta}}]$. The converse direction follows from this direction, Lemma 5.23 distinguishing horofunctions and the compactness of flag manifolds. $\qquad \square$

Finsler and flag convergence for divergent sequences in X are related as follows:

Corollary 5.27 (Finsler and flag convergence) *If $x_n \to [b_{\tau,p}^{\overline{\theta}}]$ with $\theta(\tau) = \tau_{\mathrm{mod}}$, then (x_n) is τ_{mod}–pure and τ_{mod}–flag converges $x_n \to \tau$.*

Proof If (x_n) were not τ_{mod}–pure, we could extract a ν_{mod}–pure subsequence for a different face type $\nu_{\mathrm{mod}} \neq \tau_{\mathrm{mod}}$. By Proposition 5.26, after further extraction, (x_n) Finsler converges to a boundary point $[b_{\nu,q}^{\bar{\theta}}]$ with $\theta(\nu) = \nu_{\mathrm{mod}}$. However, $[b_{\nu,q}^{\bar{\theta}}] \neq [b_{\tau,p}^{\bar{\theta}}]$ according to Lemma 5.23, a contradiction. Hence (x_n) must be τ_{mod}–pure. Since $x_n \in V(o, \mathrm{st}(\tau_n))$ and $\tau_n \to \tau$, again due to Proposition 5.26 and Lemma 5.23, the definition of flag convergence implies that $x_n \to \tau$. $\qquad\square$

We will also use the following fact:

Lemma 5.28 Let (x_n) and (x_n') be sequences in X which are bounded distance apart and converge at infinity $x_n \to [b_{\tau,p}^{\bar{\theta}}]$ and $x_n' \to [b_{\tau',p'}^{\bar{\theta}}]$. Then $\tau = \tau'$.

Proof Since the functions $d_{x_n}^{\bar{\theta}} - d_{x_n'}^{\bar{\theta}}$ are uniformly bounded independently of n, also $b_{\tau,p}^{\bar{\theta}} - b_{\tau',p'}^{\bar{\theta}}$ is bounded. This implies that $\tau = \tau'$; compare the first part of the proof of Lemma 5.23. $\qquad\square$

Remark 5.29 If $\theta(\tau) \neq \sigma_{\mathrm{mod}}$, then the limit points $[b_{\tau,p}^{\bar{\theta}}]$ and $[b_{\tau',p'}^{\bar{\theta}}]$ are in general different.

We now discuss the convergence of sequences *at* infinity.

For face types $\tau_{\mathrm{mod}} \subset \nu_{\mathrm{mod}}$, every boundary point of type ν_{mod} is a limit of boundary points of type τ_{mod}:

Lemma 5.30 Let $\tau \subset \nu$ be faces in $\partial_\infty X$, and let (x_n) be a $\theta(\nu)$–regular sequence in a sector $V(p, \nu)$. Then $[b_{\tau,x_n}^{\bar{\theta}}] \to [b_{\nu,p}^{\bar{\theta}}]$.

Proof Using Lemma 5.16, we can approximate the boundary points $[b_{\tau,x_n}^{\bar{\theta}}]$ by points in X: There exist points $y_n \in V(x_n, \tau) \subset V(p, \nu)$ such that

$$d_{y_n}^{\bar{\theta}} - d_{y_n}^{\bar{\theta}}(x_n) - b_{\tau,x_n}^{\bar{\theta}} \to 0$$

uniformly on compacts. The sequence (y_n) is also $\theta(\nu)$–regular, and the same lemma yields that

$$d_{y_n}^{\bar{\theta}} - d_{y_n}^{\bar{\theta}}(p) \to b_{\nu,p}^{\bar{\theta}}$$

uniformly on compacts. It follows that $[b_{\tau,x_n}^{\bar{\theta}}] \to [b_{\nu,p}^{\bar{\theta}}]$. $\qquad\square$

The next result yields necessary conditions for the convergence of sequences at infinity:

Lemma 5.31 *If*

(5.32)
$$[b^{\overline{\theta}}_{\tau_n,x_n}] \to [b^{\overline{\theta}}_{\nu,p}]$$

and $\theta(\tau_n) = \tau_{\mathrm{mod}}$ *for all* n, *then* $\tau_{\mathrm{mod}} \subseteq \theta(\nu)$ *and* $\tau_n \to \tau \subseteq \nu$.

Proof We may assume without loss of generality that $x_n \in \mathrm{CS}(\widehat{\tau}_n, \tau_n, o)$ and $p \in \mathrm{CS}(\widehat{\nu}, \nu, o)$, where $\widehat{\tau}_n$ is o–opposite to τ_n and $\widehat{\nu}$ is o–opposite to ν.

As in the proof of the previous lemma, we use Lemma 5.16 to approximate the points $[b^{\overline{\theta}}_{\tau_n,x_n}]$ at infinity by points $y_{n,i} \in V(x_n, \tau_n)$:

$$\lim_{i \to \infty} y_{n,i} = [b^{\overline{\theta}}_{\tau_n,x_n}],$$

where $(y_{n,i})_{i \in \mathbb{N}}$ is a τ_{mod}–regular sequence which flag-converges to τ_n for each n. Taking into account the limit (5.32), we find a diagonal subsequence $y_n \in V(x_n, \tau_n)$ such that

$$y_n \to [b^{\overline{\theta}}_{\nu,p}].$$

In view of openness of the cones $V(o, \mathrm{ost}(\tau_n))$ in $P(\widehat{\tau}_n, \tau_n)$ (see Lemma 3.9), without loss of generality, $y_n \in V(o, \mathrm{st}(\tau_n))$, and hence there exist chambers $\sigma_n \supseteq \tau_n$ such that $y_n \in V(o, \sigma_n)$. After extraction, we may assume that (y_n) is τ'_{mod}–pure for some face type $\tau'_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$. In view of τ_{mod}–regularity of the sequence (y_n), it follows that $\tau'_{\mathrm{mod}} \supseteq \tau_{\mathrm{mod}}$.

Consider the faces $\tau_n \subseteq \tau'_n \subseteq \sigma_n$ of type $\theta(\tau'_n) = \tau'_{\mathrm{mod}}$, and denote by $\widehat{\tau}'_n$ the simplices o–opposite to τ'_n. There exists a bounded sequence (x'_n) of points $x'_n \in \mathrm{CS}(\widehat{\tau}'_n, \tau'_n, o)$ such that $y_n \in f(\widehat{\tau}'_n, \tau'_n, x'_n)$. After further extraction, we may assume convergence $\tau'_n \to \tau'$ and $x'_n \to x'$. Then $y_n \to [b^{\overline{\theta}}_{\tau',x'}]$ by Proposition 5.26, and hence $[b^{\overline{\theta}}_{\tau',x'}] = [b^{\overline{\theta}}_{\nu,p}]$. In particular, $\tau'_{\mathrm{mod}} = \theta(\nu)$ and $\tau' = \nu$. It follows that $\tau_n \to \tau \subseteq \nu$, ie the assertion holds for the subsequence.

Returning to the original sequence of points $[b^{\overline{\theta}}_{\tau_n,x_n}]$, our argument shows that every subsequence has a subsequence for which the assertion holds. Thus $\tau_{\mathrm{mod}} \subseteq \theta(\nu)$ and the sequence of simplices τ_n can only accumulate at the face $\tau \subseteq \nu$ of type τ_{mod}. In view of the compactness of $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$, it follows that $\tau_n \to \tau$. $\square$

Our discussion of sequential convergence implies that the Finsler compactification does not depend on the regular type $\overline{\theta}$.

Proposition 5.33 (type-independence of Finsler compactification) *For any two regular types $\bar\theta, \bar\theta' \in \mathrm{int}(\sigma_{\mathrm{mod}})$, the identity map id_X extends to a G–equivariant homeomorphism*

$$\bar X^{\bar\theta} \to \bar X^{\bar\theta'}$$

sending $[b^{\bar\theta}_{\tau,p}] \mapsto [b^{\bar\theta'}_{\tau,p}]$ at infinity.

Proof The extension of id_X sending $[b^{\bar\theta}_{\tau,p}] \mapsto [b^{\bar\theta'}_{\tau,p}]$ is a G–equivariant bijection $\bar X^{\bar\theta} \to \bar X^{\bar\theta'}$. The conditions given in Proposition 5.26 for sequences $x_n \to \infty$ in X to converge at infinity do not depend on the type $\bar\theta$, ie $x_n \to [b^{\bar\theta}_{\tau,p}]$ in $\bar X^{\bar\theta}$ if and only if $x_n \to [b^{\bar\theta'}_{\tau,p}]$ in $\bar X^{\bar\theta'}$. A general point-set topology argument now implies that the extension is a homeomorphism; see Lemma 2.1. □

We therefore will from now on mostly use the notation $\bar X^{\mathrm{Fins}}$ for $\bar X^{\bar\theta}$.

5.2.4 Stratification and G–action For every face type $\tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$, we define the *stratum* at infinity

(5.34) $$S_{\tau_{\mathrm{mod}}} = \{[b^{\bar\theta}_{\tau,p}] : \theta(\tau) = \tau_{\mathrm{mod}}, \ p \in X\}.$$

Furthermore, we put $S_\varnothing = X$. We define the *stratification* of $\bar X^{\bar\theta}$ as

$$\bar X^{\bar\theta} = \bigsqcup_{\varnothing \subseteq \tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}} S_{\tau_{\mathrm{mod}}}.$$

In the sequel, when talking about the stratification, we will also admit $\varnothing$ as a face type.

Lemmas 5.30 and 5.31 yield for the closures of strata

(5.35) $$\bar S_{\tau_{\mathrm{mod}}} = \bigsqcup_{\nu_{\mathrm{mod}} \supseteq \tau_{\mathrm{mod}}} S_{\nu_{\mathrm{mod}}}.$$

The stratum $S_\varnothing = X$ is open dense, while the stratum $S_{\sigma_{\mathrm{mod}}} \cong \partial_{\mathrm{Fu}} X$ (see Corollary 5.25) is closed and contained in the closure of every other stratum.

The continuous extension of the G–*action* on X to $\bar X^{\bar\theta}$ is given at infinity by

$$g \cdot [b^{\bar\theta}_{\tau,p}] = [b^{\bar\theta}_{\tau,p} \circ g^{-1}] = [b^{\bar\theta}_{g\tau,gp}].$$

The G–orbits are precisely the strata $S_{\tau_{\mathrm{mod}}}$.

The stabilizer of a boundary point $[b^{\bar\theta}_{\tau,p}]$ is the semidirect product

$$H_\tau \rtimes (T(\hat\tau,\tau) \times K_{f(\hat\tau,\tau,p)}),$$

where $H_\tau \subset P_\tau$ is the unipotent horocyclic subgroup, $\hat{\tau} \in C(\tau)$ a simplex opposite to τ, $f(\hat{\tau}, \tau, p)$ the singular flat through p with visual boundary sphere $s(\hat{\tau}, \tau)$, $K_{f(\hat{\tau},\tau,p)} < G$ its pointwise stabilizer, and $T(\hat{\tau}, \tau)$ the group of transvections along $f(\hat{\tau}, \tau, p)$; see Section 5.2.2.

We will use the following observation concerning the dynamics of $G \curvearrowright \bar{X}^{\bar{\theta}}$:

Lemma 5.36 *Every open subset $O \subset \bar{X}^{\bar{\theta}}$ which intersects the closed stratum $\partial_{\mathrm{Fu}} X$ sweeps out the entire space: $G \cdot O = \bar{X}^{\bar{\theta}}$.*

Proof The G–orbit $\partial_{\mathrm{Fu}} X$ is in the closure of every G–orbit $S_{\tau_{\mathrm{mod}}}$. $\qquad\square$

For the strata at infinity, there are the natural G–equivariant *fibrations* of homogeneous G–spaces

$$(5.37) \qquad\qquad\qquad S_{\tau_{\mathrm{mod}}} \to \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

by the forgetful maps $[b^{\bar{\theta}}_{\tau,p}] \mapsto \tau$. The fiber

$$(5.38) \qquad\qquad\qquad X_\tau = \{[b^{\bar{\theta}}_{\tau,p}] : p \in X\}$$

over $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ is naturally identified with the space of strong asymptote classes of Weyl sectors $V(x, \tau)$ (see Lemma 5.23), which is in turn naturally identified with the cross-section of the parallel set $P(\tau, \hat{\tau})$ for any simplex $\hat{\tau} \in C(\tau)$. We will refer to the fibers X_τ as *small strata*. Again according to Lemmas 5.30 and 5.31, we have that

$$(5.39) \qquad\qquad\qquad \bar{X}_\tau = \bigsqcup_{\nu \supseteq \tau} X_\nu.$$

Note that for different simplices τ_1, τ_2 of the same type τ_{mod}, it holds that

$$(5.40) \qquad\qquad\qquad \bar{X}_{\tau_1} \cap \bar{X}_{\tau_2} = \varnothing$$

because every simplex in $\partial_\infty X$ has at most one face of type τ_{mod}.

Remark 5.41 One can show that the closure $\bar{X}_\tau$ is naturally identified with the regular Finsler compactification of X_τ.

The discussion of convergence at infinity in the previous section yields the following *characterization of pureness and regularity* for divergent sequences in X in terms of their accumulation set in the Finsler boundary:

Proposition 5.42 (pureness and regularity) *Let $x_n \to \infty$ be a divergent sequence.*

(i) *(x_n) is τ_{mod}–pure if and only if it accumulates at a compact subset of the stratum $S_{\tau_{\mathrm{mod}}}$.*

(ii) *(x_n) is τ_{mod}–regular if and only if it accumulates at the stratum closure $\bar{S}_{\tau_{\mathrm{mod}}}$.*

Proof (i) If (x_n) is τ_{mod}–pure, then Proposition 5.26 implies that it accumulates at a compact subset of $S_{\tau_{\mathrm{mod}}}$. Otherwise, if (x_n) is not τ_{mod}–pure, then it contains a ν_{mod}–pure subsequence for another face type $\nu_{\mathrm{mod}} \neq \tau_{\mathrm{mod}}$ and therefore has, by the same proposition, accumulation points in $S_{\nu_{\mathrm{mod}}}$, ie outside $S_{\tau_{\mathrm{mod}}}$.

(ii) If (x_n) is τ_{mod}–regular, then all ν_{mod}–pure subsequences have type $\nu_{\mathrm{mod}} \supseteq \tau_{\mathrm{mod}}$, and the assertion therefore follows from (i) and (5.35). □

Similarly, we can characterize *flag convergence* (compare Corollary 5.27 above):

Proposition 5.43 (flag convergence) *A τ_{mod}–regular sequence (x_n) τ_{mod}–flag converges $x_n \to \tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ if and only if it accumulates at the small stratum closure $\bar{X}_\tau$.*

Proof By the previous proposition, (x_n) accumulates at $\bar{S}_{\tau_{\mathrm{mod}}}$.

Suppose that we have Finsler convergence $x_n \to [b_{\nu,p}^{\bar\theta}]$. By Corollary 5.27, (x_n) is ν_{mod}–pure with $\nu_{\mathrm{mod}} = \theta(\nu)$ and ν_{mod}–flag converges, $x_n \to \nu$. Necessarily $\nu_{\mathrm{mod}} \supseteq \tau_{\mathrm{mod}}$, because (x_n) is τ_{mod}–regular. It follows that we also have τ_{mod}–flag convergence $x_n \to \tau_\nu \subset \nu$ to the face τ_ν of type τ_{mod}. Furthermore, $[b_{\nu,p}^{\bar\theta}] \in \bar{X}_{\tau_\nu}$; see (5.39).

Thus, if $x_n \to \tau$ then all accumulation points of (x_n) in $\partial_\infty^{\mathrm{Fins}} X$ must lie in $\bar{X}_\tau$. On the other hand, if (x_n) does not τ_{mod}–flag converge to τ, then after extraction it τ_{mod}–flag converges to some other $\tau' \neq \tau$ and has Finsler accumulation points in the small stratum closure $\bar{X}_{\tau'}$ disjoint from $\bar{X}_\tau$; see (5.40). □

5.2.5 Maximal flats and Weyl sectors We first discuss maximal flats $F \subset X$. We start by showing that their extrinsic closure in $\bar{X}^{\bar\theta}$ coincides with their intrinsic Finsler compactification; compare the general discussion in the end of Section 2.6.

According to Proposition 5.26,

$$\partial^{\bar{X}^{\bar\theta}} F = \{[b_{\nu,x}^{\bar\theta}] : \nu \subset \partial_\infty F,\ x \in F\} \subset \partial_\infty^{\bar\theta} X,$$

where $\partial^{\bar{X}^{\bar\theta}} F \subset \partial_\infty^{\bar\theta} X$ means the "boundary" $\bar{F} - F$ of F inside $\bar{X}^{\bar\theta}$.

Lemma 5.44 *For simplices $v, v' \subset \partial_\infty F$ and points $x, x' \in F$, if $b^{\bar\theta}_{v,x}|_F \equiv b^{\bar\theta}_{v',x'}|_F$, then $b^{\bar\theta}_{v,x} \equiv b^{\bar\theta}_{v',x'}$.*

Proof One proceeds as in the proof of Lemma 5.23. The asymptotics of $b^{\bar\theta}_{v,x}|_F$ and $b^{\bar\theta}_{v',x'}|_F$ allow us to read off v and v', and thus $v = v'$. Furthermore, Lemma 5.20 implies that the singular flats spanned by the Weyl sectors $V(x, v)$ and $V(x', v)$ coincide, equivalently, these Weyl sectors intersect. Hence they are strongly asymptotic, which implies that $b^{\bar\theta}_{v,x} \equiv b^{\bar\theta}_{v',x'}$. □

It follows that there is a natural inclusion of Finsler compactifications

$$\overline{F}^{\bar\theta} \subset \overline{X}^{\bar\theta}.$$

The stratification of $\overline{X}^{\bar\theta}$ induces the *stratification*

$$\overline{F}^{\bar\theta} = \bigsqcup_{\tau_{\mathrm{mod}}} S^F_{\tau_{\mathrm{mod}}}$$

by the stratum $S^F_{\varnothing} = F$ and the strata at infinity

$$S^F_{\tau_{\mathrm{mod}}} = S_{\tau_{\mathrm{mod}}} \cap \partial^{\bar\theta}_\infty F = \{[b^{\bar\theta}_{\tau,x}] : \tau \subset \partial_\infty F, \ \theta(\tau) = \tau_{\mathrm{mod}}, \ x \in F\}.$$

The fibration (5.37) restricts to the finite decomposition

$$S^F_{\tau_{\mathrm{mod}}} = \bigsqcup_{\tau \subset \partial_\infty F, \theta(\tau) = \tau_{\mathrm{mod}}} X^F_\tau$$

into the small strata

(5.45) $$X^F_\tau = \{[b^{\bar\theta}_{\tau,x}] : x \in F\}$$

at infinity. They are euclidean spaces which are canonically identified with the affine subspaces of F perpendicular to the Weyl sectors $V(x, \tau)$. By analogy with (5.39), the closures of small strata decompose as

(5.46) $$\overline{X}^F_\tau = \bigsqcup_{\tau \subseteq v \subseteq \partial_\infty F} X^F_v.$$

Dynamics at infinity The subgroup $T_F < G$ of transvections along F restricts to the group of translations on F. Unlike for the visual boundary, the induced action

$$T_F \curvearrowright \partial^{\bar\theta}_\infty F$$

on the Finsler boundary is nontrivial in higher rank. Its orbits are the small strata X^F_τ, since $t[b^{\bar\theta}_{x,\tau}] = [b^{\bar\theta}_{tx,\tau}]$ for $t \in T_F$. The stabilizer $\mathrm{Stab}_G(F)$ of F in G acts on F by the affine Weyl group. Its orbits at infinity are the big strata $S^F_{\tau_{\mathrm{mod}}}$.

It is worth pointing out how the description of the *convergence at infinity* simplifies for divergent sequences contained in maximal flats. Proposition 5.26 and Corollary 5.27 reduce to:

Proposition 5.47 (convergence at infinity for maximal flats) (i) *Suppose that* (x_n) *is a sequence in* F *convergent to some* $[b^{\bar{\theta}}_{\tau,p}] \in \bar{F}^{\theta}$. *Then the sequence* (x_n) *is* τ_{mod}–*pure,* $\tau_{\mathrm{mod}} = \theta(\tau)$.

(ii) *Suppose that* (x_n) τ_{mod}–*pure sequence in* F *which for some simplex* $\tau \subset \partial_{\infty} F$ *of type* τ_{mod} *is contained in* R–*tubular neighborhoods of the sectors* $V(\cdot, \tau)$ *for a fixed* $R < \infty$. *Then* (x_n) *converges in* $\bar{F}^{\theta}$ *if and only if the flats* $f(\hat{\tau}, \tau, x_n) \subset F$ *Hausdorff converge. In this case,*

$$x_n \to [b^{\bar{\theta}}_{\tau,p}]$$

with $p \in F$ *such that* $f(\hat{\tau}, \tau, x_n) \to f(\hat{\tau}, \tau, p)$.

The discussion for *Weyl sectors* $V = V(p, \tau)$ is analogous: We have

(5.48)
$$\partial^{\bar{X}^{\bar{\theta}}} V = \{[b^{\bar{\theta}}_{\nu,x}] : \nu \subseteq \tau, x \in V\} \subset \partial^{\bar{\theta}}_{\infty} X.$$

Again, horofunctions uniquely extend from V to X:

Lemma 5.49 *For simplices* $\nu, \nu' \subseteq \tau$ *and points* $x, x' \in V$, *if* $b^{\bar{\theta}}_{\nu,x}|_V \equiv b^{\bar{\theta}}_{\nu',x'}|_V$, *then* $b^{\bar{\theta}}_{\nu,x} \equiv b^{\bar{\theta}}_{\nu',x'}$.

Proof As in the case of maximal flats, compare the previous lemma, one can recognize ν and ν' from the asymptotics of $b^{\bar{\theta}}_{\nu,x}|_V$ and $b^{\bar{\theta}}_{\nu',x'}|_V$, and thus see that $\nu = \nu'$. Then Lemma 5.20 implies that $V(x, \nu)$ and $V(x', \nu)$ intersect and hence are strongly asymptotic. □

We thus have the natural inclusion

$$\bar{V}^{\theta} \subset \bar{X}^{\theta}.$$

Furthermore, we have the *stratification*

(5.50)
$$\bar{V}^{\theta} = \bigsqcup_{\nu \subseteq \tau} X^V_{\nu}$$

by $X^V_{\varnothing} = V$ and the (small) strata at infinity

$$X^V_{\nu} = X_{\nu} \cap \partial^{\bar{\theta}}_{\infty} V = \{[b^{\bar{\theta}}_{\nu,x}] : x \in V\}.$$

The stratum closures decompose as

$$\bar{X}_\nu^V = \bigsqcup_{\nu \subseteq \nu' \subseteq \tau} X_{\nu'}^V.$$

5.2.6 Action of maximal compact subgroups Let $o \in X$ be a basepoint and $K < G$ the maximal compact subgroup fixing it. Let $V = V(o, \sigma)$ be a euclidean Weyl chamber in X with tip at o. We recall some facts about the action $K \curvearrowright X$:

(i) V is a *cross-section* for the action, ie every K–orbit intersects V exactly once.

(ii) **Point stabilizers** The fixed-point set in V of any element $k \in K$ is a Weyl sector $V(o, \tau)$, where $\varnothing \subseteq \tau \subseteq \sigma$ is the face fixed by k. In other words, if k fixes a point $p \in V$, then it fixes the smallest Weyl sector $V(o, \tau)$ containing it. (Here, we put $V(o, \varnothing) := \{o\}$.)

We now establish analogous properties for the action $K \curvearrowright \bar{X}^{\bar\theta}$.

Lemma 5.51 (cross-section) $\bar{V}^{\bar\theta} \subset \bar{X}^{\bar\theta}$ *is a cross-section for the action of* $K \curvearrowright \bar{X}^{\bar\theta}$.

Proof Since $K \cdot \bar{V}^{\bar\theta}$ is compact and contains $K \cdot V = X$, and since X is dense in $\bar{X}^{\bar\theta}$, it holds that $K \cdot \bar{V}^{\bar\theta} = \bar{X}^{\bar\theta}$, ie every K–orbit in $\bar{X}^{\bar\theta}$ intersects $\bar{V}^{\bar\theta}$. We must show that every K–orbit in $\partial_\infty^{\bar\theta} X$ intersects $\partial_\infty^{\bar\theta} V$ only once. Suppose that

(5.52)
$$k \cdot [b_{\tau, p}^{\bar\theta}] = [b_{k\tau, kp}^{\bar\theta}] = [b_{\tau', p'}^{\bar\theta}]$$

for $k \in K$ and Weyl sectors $V(p, \tau), V(p', \tau') \subset V$; see (5.48). Then, in particular, $k\tau = \tau'$. Since $\tau, \tau' \subseteq \sigma$, this implies that $\tau = \tau'$ and $k\tau = \tau$.

It follows that k preserves the parallel set $P(\hat{\tau}, \tau)$, where $\hat{\tau}$ denotes the simplex o–opposite to τ. The sectors $V(kp, \tau), V(p', \tau) \subset P(\hat{\tau}, \tau)$ are strongly asymptotic, because $[b_{\tau, kp}^{\bar\theta}] = [b_{\tau, p'}^{\bar\theta}]$ This means that they intersect. Let $q \in V(kp, \tau) \cap V(p', \tau)$. Then $q, k^{-1}q \in V$ and hence $k^{-1}q = q$, because V is a cross-section for the action $K \curvearrowright X$. It follows that the sectors $V(p, \tau)$ and $V(kp, \tau)$ intersect and hence are strongly asymptotic. Thus, $k \cdot [b_{\tau, p}^{\bar\theta}] = [b_{\tau, kp}^{\bar\theta}] = [b_{\tau, p}^{\bar\theta}]$. □

Lemma 5.53 *Let* $k \in K$ *and* $V(p, \tau) \subset V$. *The following are equivalent:*

(i) k *fixes* $[b_{\tau, p}^{\bar\theta}] \in \bar{V}^{\bar\theta}$.

(ii) k *fixes* $V(p, \tau)$ *pointwise.*

(iii) k *fixes pointwise the smallest Weyl sector* $V(o, \nu)$ *containing* $V(p, \tau)$.

Proof (i) $\Longrightarrow$ (ii) In the proof of the previous lemma, we saw that the sectors $V(p, \tau)$ and $V(kp, \tau)$ intersect. Since k also preserves the cross-sections of $P(\hat{\tau}, \tau)$, it follows that k fixes p.

The converse direction (ii) $\Longrightarrow$ (i) is trivial.

(ii) $\Longleftrightarrow$ (iii) This is clear, because the fixed-point set of k on V is a sector $V(o, \nu)$. $\square$

Let $K_\tau = P_\tau \cap K$ denote the stabilizer of the simplex τ in K, and put $K_\varnothing = K$.

Corollary 5.54 (point stabilizers in compactified euclidean Weyl chambers)

(i) *The points in $\overline{V}^{\bar{\theta}}$ fixed by K_τ are precisely the points in $\overline{V(o, \tau)}^{\bar{\theta}}$.*

(ii) *The points with stabilizer equal to K_τ are precisely the points in*

$$\overline{V(o, \tau)}^{\bar{\theta}} - \bigcup_{\varnothing \subseteq \nu \subsetneq \tau} \overline{V(o, \nu)}^{\bar{\theta}}.$$

Notation 5.55 In view of Proposition 5.33 we will from now on denote the Finsler compactification $\overline{X}^{\bar{\theta}}$ for $\bar{\theta} \in \text{int}(\sigma_{\text{mod}})$ by $\overline{X}^{\text{Fins}}$.

6 Coxeter groups and their regular polytopes

6.1 Basics of polytopes

We refer the readers to [22; 54] for a detailed treatment of polytopes. In what follows, V will denote a euclidean vector space, ie a finite-dimensional real vector space equipped with an inner product (x, y). We will use the notation V^* for the dual vector space, and for $\lambda \in V^*$ and $x \in V$ we let $\langle \lambda, x \rangle = \lambda(x)$. The inner product on V defines the inner product, again denoted by (λ, μ), on the dual space.

A *polytope* B in V is a compact convex subset equal to the intersection of finitely many closed half-spaces. Note that we do not require B to have nonempty interior. The *affine span* $\langle B \rangle$ of B is the minimal affine subspace of V containing B. The topological frontier of B in its affine span is the boundary ∂B of B. A *facet* of B is a codimension-one face of ∂B.

Each polytope B has a *face poset* $\mathcal{F}_B$. It is the poset whose elements are the faces of B with the order given by the inclusion relation. Two polytopes are *combinatorially*

isomorphic if there is an isomorphism of their posets. Such an isomorphism necessarily preserves the dimension of faces. Two polytopes B and B' are *combinatorially homeomorphic* if there exists a (piecewise linear) homeomorphism $h\colon B \to B'$ which sends faces to faces.

Given a polytope B whose dimension equals $n = \dim(V)$, the *polar* (or *dual*) polytope of B is defined as the following subset of the dual vector space:

$$B^* = \{\lambda \in V^* : \lambda(x) \le 1 \text{ for all } x \in B\}.$$

Thus, $\lambda \in B^* \subset V^*$ implies that the affine hyperplane $H_\lambda = \{\lambda = 1\}$ is disjoint from the interior of B. Moreover, $\lambda \in \partial B^*$ if and only if H_λ has nonempty intersection with B. Each face φ of B determines the *dual face* φ^* of B^*, consisting of the elements $\lambda \in B^*$ which are equal to 1 on the entire face φ. This defines a natural bijection between the faces of B and B^*:

$$\star \colon \varphi \mapsto \varphi^*.$$

Under this bijection, faces have complementary dimensions:

$$\dim(\varphi) + \dim(\varphi^*) = n - 1.$$

The bijection $\star$ also reverses the face inclusion:

$$\varphi \subset \psi \iff \varphi^* \supset \psi^*.$$

In particular, the face poset of ∂B^* is dual to the face poset of ∂B. If W is a group of linear transformations preserving B, its dual action

$$w^*(\lambda) = \lambda \circ w^{-1}$$

on V^* preserves B^*. The naturality of $\star$ implies that it is W–equivariant.

A polytope B is called *simplicial* if its faces are simplices. It is called *simple* if it has a natural structure of a manifold-with-corners: Each vertex v of B is contained in exactly d facets, where d is the dimension of B. Equivalently, the affine functionals defining these facets in $\langle B \rangle$ have linearly independent linear parts. For each simplicial polytope, its dual is a simple polytope, and vice versa.

Lemma 6.1 *Two polytopes are combinatorially isomorphic if and only if they are combinatorially homeomorphic.*

Proof One direction is trivial. Conversely, given an isomorphism of posets, one constructs a combinatorial homeomorphism by induction over skeleta and coning off. □

6.2 Root systems

In this and the following sections, the euclidean vector space V is identified with the model maximal flat F_{mod} for the symmetric space X; the root system $R \subset V^*$ is the root system of X. Accordingly, the Coxeter group W defined via R is the Weyl group of X. Since the symmetric space X has noncompact type, R spans V^*, ie W fixes only the origin 0 in V.

Given a face τ of the spherical Coxeter complex $\partial_\infty V$, we define the root subsystem

$$R_\tau \subset R$$

consisting of all roots which vanish identically on $V(0, \tau)$.

Each root $\alpha \in R$ corresponds to a *coroot* $\alpha^\vee \in V$, which is a vector such that the reflection $s_\alpha \colon V \to V$ corresponding to α acts on V by the formula

$$(6.2) \qquad\qquad s_\alpha(x) = x - \langle \alpha, x \rangle \alpha^\vee.$$

The group W also acts isometrically on the dual space V^*; each reflection $s_\alpha \in W$ acts on V^* as a reflection. The corresponding wall is given by the equation

$$\{\lambda \in V^* : \langle \lambda, \alpha^\vee \rangle = 0\};$$

equivalently, this wall is $\alpha^\perp$, the orthogonal complement of α in V^*.

From now on, we fix a Weyl chamber $\Delta = \Delta_{\mathrm{mod}} \subset V$ for the action of W on V. The visual boundary of Δ is the model spherical chamber σ_{mod}.

Notation 6.3 We let $[n]$ denote the set $\{1, \ldots, n\}$.

The choice of Δ determines the set of positive roots $R^+ \subset R$ and the set of *simple roots* $\alpha_1, \ldots, \alpha_n \in R^+$, where $n = \dim(V)$:

$$\Delta = \{x \in V : \alpha_i(x) = \langle \alpha_i, x \rangle \geq 0, \ i \in [n]\}.$$

We will use the notation $s_i = s_{\alpha_i}$ for the *simple reflections*. They generate W.

The *dual chamber* to Δ is

$$\Delta^* \subset V^*, \quad \Delta^* = \{\lambda \in V^* : (\alpha_i, \lambda) \geq 0, \ i \in [n]\}.$$

Remark 6.4 There is another notion of a dual cone to Δ in V^*, namely the *root cone* $\Delta^\vee$, consisting of all $\lambda \in V^*$ such that the restriction of λ to Δ is nonnegative. The root cone consists of the nonnegative linear combinations of simple roots. The root cone contains the dual chamber but, is, with few exceptions, strictly larger.

Let B be a W–invariant polytope in V with nonempty interior. We will use the notation $\Delta_B = \Delta \cap B$ and $\Delta^*_{B^*} = \Delta^* \cap B^*$.

Lemma 6.5 *Suppose that $\lambda \in \Delta^*$ is such that $\lambda(x) \le 1$ for all $x \in \Delta_B$. Then $\lambda \in B^*$.*

Proof Let $\lambda \in V^*$ and let $v \in \text{int}(\Delta) \subset V$. Then $\lambda|_{Wv}$ is maximal in v if and only if $\lambda \in \Delta^*$. The assertion follows. □

6.3 Geometry of the dual ball

We assume now that $B \subset V$ is a W–invariant polytope in V with nonempty interior, such that

$$\Delta_B = \{x \in \Delta : l(x) \le 1\},$$

where $l = l_{\bar\theta} \in \text{int}(\Delta^*)$ is a *regular linear functional*. The gradient vector of l gives a direction $\bar\theta$, which is a regular point of σ_{mod}.

Set $l_w = w^* l = l \circ w^{-1}$, where $w \in W$. Then,

$$B = \bigcap_{w \in W} \{x \in V : l_w(x) \le 1\};$$

ie the facets of B are carried by the affine hyperplanes $l_w = 1$ for $w \in W$.

The polytope B defines a (possibly nonsymmetric) norm on V, namely the norm for which B is the unit ball:

(6.6) $$\|x\| = \|x\|_{\bar\theta} = \max_{w \in W} (l_w(x)).$$

We let $\omega_1, \ldots, \omega_n$ denote the nonzero vertices of the n–simplex Δ_B. We will label these vertices consistently with the labeling of the simple roots: ω_i is the unique vertex of Δ_B on which α_i does not vanish. Geometrically speaking, ω_i is opposite to the facet A_i of Δ_B carried by the wall $\alpha_i = 0$.

The regularity of l implies:

Lemma 6.7 *The polytope B is simplicial. Its facets are the simplices*

$$\{x \in w\Delta : l_w(x) = 1\}.$$

For each reflection $s_i = s_{\alpha_i}$, the line segment $\omega_i s_i(\omega_i)$ is not contained in the boundary of B.

Proof We will prove the last statement. The segment $\omega_i s_i(\omega_i)$ is parallel to the vector $\alpha_i^\vee$. If $\alpha_i^\vee$ were to be parallel to the face $l = 1$ of B, then $\langle l, \alpha_i^\vee \rangle = 0$, which implies that l is singular. □

Corollary 6.8 *Since the polytope B is simplicial, the dual polytope B^* is simple.*

The chamber Δ^* contains a distinguished vertex of $\Delta_{B^*}^*$, namely the linear functional l; this is the only vertex of $\Delta_{B^*}^*$ contained in the interior of Δ^*. (The other vertices of $\Delta_{B^*}^*$ are not vertices of B^*.)

We now analyze the geometry of $\Delta_{B^*}^*$ in more detail.

Lemma 6.9 $\Delta_{B^*}^*$ *is given by the set of $2n$ inequalities $(\cdot, \alpha_i) \geq 0$ and $\langle \cdot, \omega_i \rangle \leq 1$ for $i \in [n]$.*

Proof It is clear that these inequalities are necessary for $\lambda \in V^*$ to belong to $\Delta_{B^*}^*$. In order to prove that they are sufficient, we have to show that each λ satisfying these inequalities belongs to B^*. The inequalities $\langle \lambda, \omega_i \rangle \leq 1$ show that the restriction of λ to Δ_B is ≤ 1. Now, [Lemma 6.5](#) shows that $\lambda(x) \leq 1$ for all $x \in B$. □

Close to the origin, $\Delta_{B^*}^*$ is given by the n inequalities $(\cdot, \alpha_i) \geq 0$, while the other n inequalities are strict. Close to l, it is given by the n inequalities $\langle \cdot, \omega_i \rangle \leq 1$, while the other n inequalities are strict.

We define the *exterior facet* $E_i \subset \Delta_{B^*}^*$ by the equation

$$\langle \cdot, \omega_i \rangle = 1,$$

and the *interior facet* $F_j \subset \Delta_{B^*}^*$ as the fixed-point set of the reflection s_j, equivalently, by the equation

$$(\cdot, \alpha_j) = 0.$$

For subsets $I, J \subset [n] = \{1, \dots, n\}$ we define the *exterior faces*

$$E_I := \bigcap_{i \in I} E_i$$

containing l, and the *interior faces*

$$F_J := \bigcap_{j \in J} F_j$$

containing the origin. These are nonempty faces of $\Delta^*_{B^*}$ of the expected dimensions, due to the linear independence of the ω_i, respectively, the α_j.

As a consequence of the last lemma, every face of the polytope $\Delta^*_{B^*}$ has the form

$$E_I \cap F_J$$

for some subsets $\varnothing \subseteq I, J \subseteq [n]$.

We now describe the combinatorics of the polytope $\Delta^*_{B^*}$.

Lemma 6.10 $E_i \cap F_i = \varnothing$ for each $i = 1, \ldots, n$.

Proof   Suppose that $\lambda \in \Delta^*_{B^*}$ is a point of intersection of these faces. Then λ is a linear function fixed by the reflection s_i and satisfying the equation $\langle \lambda, \omega_i \rangle = 1$. Then $\lambda(s_i(\omega_i)) = 1$ as well. Thus, $\lambda = 1$ on the entire segment connecting the vertices ω_i and $s_i(\omega_i)$ of B. Since λ belongs to B^*, this segment has to be contained in the boundary of B. But this contradicts Lemma 6.7. Therefore, such a λ cannot exist. $\square$

We denote by $W_J < W$ the subgroup generated by the reflections s_j for $j \in J$. The fixed-point set of W_J on $\Delta^*_{B^*}$ equals F_J.

Furthermore, we define ω_I as the face of B, as well as of Δ_B, which is the convex hull of the vertices ω_i for $i \in I$. The dual face ω^*_I of B^* is given, as a subset of B^*, by the equations $\langle \cdot, \omega_i \rangle = 1$. It is contained in $W_J \cdot \Delta^*_{B^*}$, where we put $J = [n] - I$. Indeed, the vertices of ω^*_I are the functionals l_w for which the dual facet $l_w = 1$ of B contains ω_I, equivalently, for $w \in W_J$.

Note that W_J preserves ω_I and therefore also ω^*_I (and acts on it as a reflection group). The fixed-point set of W_J on $W_J \cdot \Delta^*_{B^*}$ is contained in the intersection

$$\bigcap_{w \in W_J} w \Delta^*_{B^*}$$

and in particular in $\Delta^*_{B^*}$. This implies that

$$\varnothing \neq \mathrm{Fix}_{W_J}(\omega^*_I) \subset \Delta^*_{B^*}.$$

Note that $E_I = \omega_I^* \cap \Delta_{B^*}^*$. It follows that

$$E_I \cap F_J \supseteq \mathrm{Fix}_{W_J}(\omega_I^*) \neq \varnothing.$$

In combination with the previous lemma, we conclude:

Lemma 6.11 *For arbitrary subsets* $\varnothing \subseteq I, J \subseteq [n]$, *it holds that* $E_I \cap F_J \neq \varnothing$ *if and only if* $I \cap J = \varnothing$.

Next, we prove the uniqueness of the labeling of the faces.

Lemma 6.12 *If* $E_I \cap F_J = E_{I'} \cap F_{J'} \neq \varnothing$, *then* $I = I'$ *and* $J = J'$.

Proof Since $E_I \cap E_{I'} = E_{I \cup I'}$ and $F_J \cap F_{J'} = F_{J \cup J'}$, the proof reduces to the case of containment $I \subseteq I'$ and $J \subseteq J'$.

Suppose that $j' \in J' - J$. Then, intersecting both sides of the equality $E_I \cap F_J = E_{I'} \cap F_{J'}$ with $E_{j'}$, the previous lemma yields that

$$\varnothing \neq E_{I \cup \{j'\}} \cap F_J = E_{I' \cup \{j'\}} \cap F_{J'} = \varnothing,$$

a contradiction. Thus $J = J'$, and similarly $I = I'$. □

For the n–cube $[0,1]^n$, we define similarly facets $E_i' = \{t_i = 1\}$ and $F_j' = \{t_j = 0\}$. They satisfy the same intersection properties as in Lemmas 6.11 and 6.12. Hence the correspondence

$$E_I \cap F_J \xmapsto{c} E_I' \cap F_J'$$

is a combinatorial isomorphism between the polytopes $\Delta_{B^*}^*$ and $[0,1]^n$. Lemma 6.1 now yields:

Theorem 6.13 *The polytope* $\Delta_{B^*}^*$ *is combinatorially homeomorphic to the* n–*cube* $[0,1]^n$; *ie there exists a combinatorial homeomorphism*

$$\Delta_{B^*}^* \xrightarrow{h} [0,1]^n$$

inducing the bijection c *of face posets.*

6.4 Cube structure of the compactified Weyl chamber

In this section we construct a canonical homeomorphism from the Finsler compactification $\overline{\Delta}^{\mathrm{Fins}}$ of the model Weyl chamber $\Delta \subset V$, to the cube $[0, \infty]^n$. Recall that $\alpha_1, \ldots, \alpha_n$ are the simple roots with respect to Δ. Each intersection

$$\Delta_i = \ker(\alpha_i) \cap \Delta$$

is a facet of Δ.

For $x \in \Delta$ define

$$\vec{\alpha}(x) := (\alpha_1(x), \ldots, \alpha_n(x)) \in [0, \infty)^n.$$

This map is clearly a homeomorphism from Δ to $[0, \infty)^n$. We wish to extend the map $\vec{\alpha}$ to a homeomorphism of the compactifications.

We recall the description of sequential convergence at infinity; see Proposition 5.47. A sequence $x_k \to \infty$ in Δ converges at infinity if and only if the following properties hold:

(a)　By parts (i) and (ii) of the proposition, there exists a face $\tau = \tau_{\mathrm{mod}}$ of $\sigma_{\mathrm{mod}} = \partial_\infty \Delta$ such that for every $\alpha_i \in R_\tau$ the sequence $(\alpha_i(x_k))$ converges.

(b)　Since every τ_{mod}–pure sequence is also τ_{mod}–regular, for the other simple roots $\alpha \notin R_\tau$, we have divergence $\alpha(x_k) \to +\infty$.

In other words, the sequence (x_k) converges at infinity if and only if the limit

$$\lim_{k \to +\infty} \vec{\alpha}(x_k) \in [0, \infty]^n$$

in the closed cube exists. Moreover, Proposition 5.47 combined with Lemma 5.23 implies that the extension

$$\overline{\Delta}^{\mathrm{Fins}} \xrightarrow{\vec{\alpha}} [0, \infty]^n$$

sending

$$\lim_{k \to +\infty} x_k \mapsto \lim_{k \to +\infty} \vec{\alpha}(x_k)$$

for sequences (x_k) converging at infinity is well defined and bijective. Now, Lemma 2.1 implies that the extension is a homeomorphism. Composing with the homeomorphism

$$\kappa \colon [0, \infty]^n \to [0, 1]^n, \quad (t_1, \ldots, t_n) \mapsto \left(1 - \frac{1}{t_1 + 1}, \ldots, 1 - \frac{1}{t_n + 1}\right),$$

we obtain:

Lemma 6.14 *The map $\kappa \circ \vec{\alpha}$ is a homeomorphism from $\overline{\Delta}^{\mathrm{Fins}}$ to the cube $[0, 1]^n$. It sends the compactification of each face $\overline{\Delta}_i^{\mathrm{Fins}}$, $i \in [n]$, to the face F_i' of the cube $[0, 1]^n$.*

For a partition $[n] = I \sqcup J$, we define $\varnothing \subseteq \tau_I \subseteq \sigma_{\mathrm{mod}}$ as the face fixed by the reflections s_j for $j \in J$. Equivalently, the vertices of τ are the directions of the vectors ω_i for $i \in I$.

Conversely, for a face $\varnothing \subseteq \tau = \tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$, we define the partition $[n] = I_\tau \sqcup J_\tau$ such that $\tau_{I_\tau} = \tau$, ie I_τ indexes the vertices of τ.

Moreover, we have the sector $\Delta_I = \bigcap_{i \in I} \Delta_i = V(0, \tau_I) \subset \Delta$ and its compactification

$$\overline{\Delta}_I^{\mathrm{Fins}} = \bigcap_{i \in I} \overline{\Delta}_i^{\mathrm{Fins}};$$

compare (5.50).

Recall that our vector space V is the underlying vector space of the model maximal flat $F = F_{\mathrm{mod}}$. We can now combine the above lemma with the homeomorphism constructed in Theorem 6.13:

Theorem 6.15 *There exists a homeomorphism*

$$\overline{\Delta}^{\mathrm{Fins}} \xrightarrow{\phi} \Delta_{B^*}^* \subset B^*$$

satisfying the following:

(1) *For each partition $[n] = I \sqcup J$,*

$$\phi(\overline{X}_{\tau_I}^\Delta) = E_I \quad \text{and} \quad \phi(\overline{\Delta}_J^{\mathrm{Fins}}) = F_J.$$

In particular, $\phi(0) = 0$.

(2) *The map ϕ preserves W–stabilizers: $x \in \overline{\Delta}^{\mathrm{Fins}}$ is fixed by $w \in W$ if and only if $\phi(x)$ is fixed by w.*

(3) *As a consequence, ϕ extends to a W–equivariant homeomorphism of the compactified model flat to the dual ball:*

$$\Phi_{F_{\mathrm{mod}}} \colon \overline{F}_{\mathrm{mod}}^{\mathrm{Fins}} \to B^*.$$

Proof Combining Theorem 6.13 and Lemma 6.14, we define

$$\phi = h^{-1} \circ \kappa \circ \vec{\alpha}.$$

The polytope $\Delta_{B^*}^*$ is a cross-section for the action of W on B^* because Δ^* is a cross-section for its action on V^*. By Lemma 5.51, the compactified chamber $\overline{\Delta}^{\mathrm{Fins}}$ is a cross-section for the action of W on $\overline{F}^{\mathrm{Fins}}$. We also note that for $J = [n] - I$, the fixed-point sets of the subgroup $W_J < W$ in $\overline{\Delta}^{\mathrm{Fins}}$ and $\Delta_{B^*}^*$ are precisely $\overline{\Delta}_I^{\mathrm{Fins}}$ and F_I; see Corollary 5.54. The last assertion of the theorem follows using Lemma 2.3. $\square$

Remark 6.16 One can also derive this theorem from [8, Proposition I.18.11]. Our proof is a direct argument which avoids symplectic geometry.

Remark 6.17 The paper [35] computes horofunctions on finite-dimensional vector spaces V equipped with polyhedral norms, but does not address the question about the global topology of the associated compactification of V. See also [14; 52].

Question 6.18 Suppose that $\|\cdot\|$ is a polyhedral norm on a finite-dimensional real vector space V. Is it true that the horoclosure $\overline{V}$ of V with respect to this norm, with its natural stratification, is homeomorphic to the closed unit ball for the dual norm? Is it homeomorphic to a closed ball for arbitrary norms?

After this work was completed, this question was answered in the affirmative by Ji and Schilling [25].

7 Manifold-with-corners structure on the Finsler compactification

In this section, we assume that $\overline{\theta} \in \operatorname{int}(\sigma_{\mathrm{mod}})$. We recall that the Finsler compactification is independent of the choice of $\overline{\theta}$.

In Theorem 6.15, we proved the existence of a W–equivariant homeomorphism $\Phi_F \colon \overline{F}^{\mathrm{Fins}} \to B^*$. Since B^* is a simple polytope, it has a natural structure of a manifold-with-corners, whose strata are the faces of B^*. Via the homeomorphism Φ_F, we then endow $\overline{F}^{\mathrm{Fins}}$ with the structure of a manifold-with-corners as well. The homeomorphism Φ_F^{-1} sends each face τ^* of B^* (dual to the face τ of B, which we will identify with the corresponding face of the Coxeter complex at infinity a_{mod}) to the ideal boundary

$$\partial_\infty^{\mathrm{Fins}} V(0, \tau).$$

The latter can be described as the set of strong asymptote classes of sectors $V(x, \tau)$:

$$[V(x, \tau)] = [V(x', \tau)] \iff x \equiv x' \in F / \operatorname{Span}(V(0, \tau)),$$

see Lemma 5.23. In other words, this is the stratum X_τ^F of $\overline{F}^{\mathrm{Fins}}$; see (5.45). The goal of this section is to extend this manifold-with-corners structure from $\overline{F}^{\mathrm{Fins}}$ to $\overline{X}^{\mathrm{Fins}}$. We will also see that this structure matches the one of the maximal Satake compactification of X.

7.1 Manifold-with-corners

Let $\sigma \subset \partial_\infty X$ be a chamber which we view as a point in the closed stratum $\partial_{\mathrm{Fu}} X$ of $\bar{X}^{\mathrm{Fins}}$. Let $o \in X$ be the fixed point of K.

Lemma 7.1 *For every neighborhood U of σ in $\overline{V(o, \sigma)}^{\mathrm{Fins}}$ and every neighborhood U' of the identity e in K, the subset $U' \cdot U$ is a neighborhood of σ in $\bar{X}^{\mathrm{Fins}}$.*

Proof Suppose that there exist a neighborhood U of σ and a neighborhood U' of $e \in K$ such that $U' \cdot U$ is not a neighborhood of σ in $\bar{X}^{\mathrm{Fins}}$. Then there exists a sequence $\xi_n \to \sigma$ in $\bar{X}^{\mathrm{Fins}}$ outside $U' \cdot U$. There exist chambers σ_n such that $\xi_n \in \overline{V(o, \sigma_n)}^{\mathrm{Fins}}$, and points $y_n \in V(o, \sigma_n)$ approximating ξ_n such that $y_n \to \sigma$. Our description of sequential convergence (Proposition 5.26) implies that the sequence (y_n) is σ_{mod}–regular and $\sigma_n \to \sigma$. Hence there exist elements $k_n \to e$ in K such that $k_n \sigma = \sigma_n$. Then, due to the continuity of the K–action, the points $k_n^{-1} \xi_n \in \overline{V(o, \sigma)}^{\mathrm{Fins}}$ converge to σ. Hence they enter the neighborhood U, and (k_n) enters U' for large n. This is a contradiction. $\square$

Suppose now that the neighborhood $U \subset \overline{V(o, \sigma)}^{\mathrm{Fins}}$ of σ is sufficiently small, say, disjoint from the union of the compactified sectors $\overline{V(o, \tau)}^{\mathrm{Fins}}$ over all proper faces $\tau \subsetneq \sigma$. Then the stabilizer of every point in U equals the pointwise stabilizer $K_\sigma = K_F$ of the maximal flat $F \supset V(o, \sigma)$; see Corollary 5.54. We consider the bijective continuous map

$$K/K_F \times U \to KU \subset \bar{X}^{\mathrm{Fins}}$$

given by the K–action. By the previous lemma, its image KU is a neighborhood of the closed stratum $S_{\sigma_{\mathrm{mod}}} = \partial_{\mathrm{Fu}} X$. After shrinking U to a compact neighborhood of σ, the map becomes a homeomorphism. After further shrinking U to an open neighborhood, the map becomes a homeomorphism onto an open neighborhood of $\partial_{\mathrm{Fu}} X$.

Since U is a manifold-with-corners (see Theorem 6.15) and K/K_F is a manifold, we conclude, via Lemma 5.36:

Theorem 7.2 (manifold-with-corners) $\bar{X}^{\mathrm{Fins}}$ *is a manifold-with-corners with respect to the stratification by the strata $S_{\tau_{\mathrm{mod}}}$. In particular, the manifold-with-corners structure is G–invariant.*

This means that the k–dimensional stratum of the manifold with corner structure equals the union of the k–dimensional strata $S_{\tau_{\mathrm{mod}}}$.

7.2 Homeomorphism to a ball

At last, we can now prove that the Finsler compactification of the symmetric space X is K–equivariantly homeomorphic to a closed ball. Let B^* be the dual ball to the unit ball $B \subset F_{\mathrm{mod}}$ of the norm (6.6) on the vector space F_{mod}, defined via the regular vector $\bar{\theta}$. We will identify the dual vector space of F_{mod} with F_{mod} itself using the euclidean metric on F_{mod}. Hence, B^* becomes a unit ball in F_{mod} for the *dual norm*

$$\| \cdot \|^* = \| \cdot \|_{\bar{\theta}}^*$$

of our original norm.

Since $B^* \subset F_{\mathrm{mod}}$ is W–invariant, the dual norm extends from F_{mod} to a G–invariant function $d_{\bar{\theta}}^*$ on $X \times X$ by

$$d_{\bar{\theta}}^*(x, y) = \|d_\Delta(x, y)\|_{\bar{\theta}}^*.$$

We call the set

$$B^*(o, 1) = \{q \in X : d_{\bar{\theta}}^*(o, q) \le 1\}$$

the *dual ball*. It is preserved by the group K since K fixes the point o. As a compact star-like subset of (X, d^{Riem}), it is homeomorphic to the closed ball. We can now prove:

Theorem 7.3 *There exists a K–equivariant homeomorphism*

$$\bar{X}^{\mathrm{Fins}} \xrightarrow{\Phi} B^*(o, 1)$$

which restricts to the homeomorphism $\phi \colon \bar{\Delta}^{\mathrm{Fins}} \to \Delta_{B^*}^*$ from Theorem 6.15. In particular, $\bar{X}^{\mathrm{Fins}}$ is homeomorphic to the closed ball.

Proof We will use Lemma 2.3 to construct Φ. In order to do so, we have to know that $\bar{\Delta}^{\mathrm{Fins}}$ and $\Delta_{B^*}^*$ are cross-sections for the actions of K on $\bar{X}^{\mathrm{Fins}}$ and $B^*(o, 1)$, and that ϕ respects the K–stabilizers.

(1) According to Lemma 5.51, $\bar{\Delta}^{\mathrm{Fins}}$ is a cross-section for the action of K on $\bar{X}^{\mathrm{Fins}}$. Since K preserves the dual ball $B^*(o, 1)$ and

$$\Delta_{B^*}^* = \Delta \cap B^*(o, 1),$$

while Δ is a cross-section for the action $K \curvearrowright X$, it follows that $\Delta_{B^*}^*$ is a cross-section for the action $K \curvearrowright B^*(o, 1)$.

(2) The faces τ with $\varnothing \subseteq \tau \subseteq \sigma$ correspond to index sets J with $\varnothing \subseteq J_\tau \subseteq [n]$, where $j \in J_\tau$ if and only if the reflection s_j fixes τ. According to Corollary 5.54, the

fixed-point set of K_τ on $\overline{\Delta}^{\mathrm{Fins}}$ equals $\overline{V(o,\tau)}^{\mathrm{Fins}}$. On the other hand, the fixed-point set of K_τ on $\Delta^*_{B^*}$ equals the interior face F_{J_τ}. By Theorem 6.15, the homeomorphism ϕ carries $\overline{V(o,\tau)}^{\mathrm{Fins}}$ to F_{J_τ}. Therefore, ϕ respects the point stabilizers. $\qquad\square$

7.3 Relation to the maximal Satake compactification

It turns out that the compactification $\overline{X}^{\mathrm{Fins}}$ constructed in this paper is naturally isomorphic to the *maximal Satake compactification* $\overline{X}^S_{\max}$. To this end, we will use the *dual-cell* interpretation of the maximal Satake compactification; see [8, Chapter I.19].

Theorem 7.4 *There is a G–equivariant homeomorphism of manifolds-with-corners $\overline{X}^{\mathrm{Fins}} \to \overline{X}^S_{\max}$ which extends the identity map $X \to X$.*

Proof We first observe that the group K acts on both compactifications so that the cross-sections for the actions are the respective compactifications of the model euclidean Weyl chamber $\Delta = \Delta_{\mathrm{mod}} \subset F = F_{\mathrm{mod}}$. We therefore compare the W–invariant compactifications of F_{mod}. On the side of $\overline{X}^{\mathrm{Fins}}$, the ideal boundary of F is the union of small strata X^F_τ as in Section 5.2.5. Elements of X^F_τ are equivalence classes $[V(x,\tau)]$ of sectors $V(x,\tau)$ in F. Two sectors $V(x,\tau), V(x',\tau)$ with the same base τ are equivalent if and only if x, x' project to the same vector in $F/\operatorname{Span}(V(0,\tau))$. These are exactly the strata, denoted by $e(C)$, in the maximal Satake compactification of F, denoted by $\overline{F}^S_{\max}$ (see [8, Chapter I.19]): For each sector $C = V(0,\tau)$, the stratum $e(C)$ is $F/\operatorname{Span}(C)$. We then have a W–equivariant bijection

$$h\colon \overline{F}^{\mathrm{Fins}} \to \overline{F}^S_{\max}$$

defined via the collection of maps

$$[V(x,\tau)] \mapsto [x] \in e(C).$$

For $\tau = \varnothing$, this is just the identity map $F \to F$.

In order to show that this map is a homeomorphism we note that the topology on $\overline{F}^S_{\max}$ is defined via roots (see [8, Chapter I.19]) and on the Weyl chamber Δ in F this topology is exactly the topology on $\overline{\Delta}^{\mathrm{Fins}}$ described in terms of simple roots; compare the proof of Lemma 6.14.

Lastly, we note that the map h we described respects the stabilizers in the group K. Therefore, by Lemma 2.3, we obtain a K–equivariant homeomorphic extension

$$\overline{X}^{\mathrm{Fins}} \to \overline{X}^S_{\max}$$

of h, which is also an extension of the identity map $X \to X$. Since the identity is G–equivariant, the same holds for the extension. □

Remark 7.5 The maximal Satake compactification is a real-analytic manifold with corners on which the group G acts real-analytically; see [8, Chapter I.19]. Therefore, the same conclusion holds for the compactification $\overline{X}^{\text{Fins}}$.

7.4 Proof of Theorem 1.1

The theorem is the combination of the following results:

Part (i) is proven in Section 5.2.4, where we established that $\overline{X}^{\text{Fins}}$ is a union of strata $S_{\tau_{\text{mod}}}$, each of which is a single G–orbit. Thus, G acts on $\overline{X}^{\text{Fins}}$ with finitely many orbits.

Part (ii) is proven in Theorem 7.2, part (iii) is proven in Theorem 7.3, part (iv) is the content of Proposition 5.33, and lastly, part (v) is established in Theorem 7.4. □

8 Relative position and thickenings

8.1 Relative position at infinity and folding order

In this section, we review some combinatorial concepts from [34] related to the geometry of Tits buildings. We will discuss here only the relative position of chambers with respect to simplices, which is the case needed in this paper, and refer the reader to [34] for more general treatment.

Let $\sigma_0, \sigma \subset \partial_\infty X$ be chambers. We view them also as points $\sigma_0, \sigma \in \partial_{\text{Fu}} X$. There exists an (in general nonunique) apartment $a \subset \partial_\infty X$ containing these chambers, $\sigma_0, \sigma \subset a$, and a unique apartment chart $\alpha \colon a_{\text{mod}} \to a$ such that $\sigma_0 = \alpha(\sigma_{\text{mod}})$. We define the *position of σ relative to σ_0* as the chamber

$$\text{pos}(\sigma, \sigma_0) := \alpha^{-1}(\sigma) \subset a_{\text{mod}}.$$

Abusing notation, it can be regarded algebraically as the unique element

$$\text{pos}(\sigma, \sigma_0) \in W$$

such that

$$\sigma = \alpha(\text{pos}(\sigma, \sigma_0)\sigma_{\text{mod}});$$

see [34, Section 3.3]. It does not depend on the choice of the apartment a. To see this, choose regular points $\xi_0 \in \text{int}(\sigma_0)$ and $\xi \in \text{int}(\sigma)$ which are not antipodal, $\angle_{\text{Tits}}(\xi, \xi_0) < \pi$. Then the segment $\xi_0\xi$ is contained in a by convexity, and its image $\alpha^{-1}(\xi_0\xi)$ in a_{mod} is independent of the chart α because its initial portion $\alpha^{-1}(\xi_0\xi \cap \sigma_0)$ in σ_{mod} is.

The level sets of $\text{pos}(\cdot, \sigma_0)$ in $\partial_{\text{Fu}} X$ are the *Schubert cells* relative σ_0, ie the orbits of the minimal parabolic subgroup $B_{\sigma_0} \subset G$ fixing σ_0.

More generally, we define the position of a chamber $\sigma \subset \partial_\infty X$ relative to a *simplex* $\tau_0 \subset \partial_\infty X$ as follows. Let $\tau_{\text{mod}} = \theta(\tau_0)$. Let again $a \subset \partial_\infty X$ be an apartment containing τ_0 and σ, and let $\alpha \colon a_{\text{mod}} \to a$ be a chart such that $\tau_0 = \alpha(\tau_{\text{mod}})$. We define the *position* $\text{pos}(\sigma, \tau_0)$ *of* σ *relative to* τ_0 as the $W_{\tau_{\text{mod}}}$–orbit of the chamber

$$\alpha^{-1}(\sigma) \subset a_{\text{mod}}.$$

It can be interpreted algebraically as a coset

$$\text{pos}(\sigma, \tau_0) \in W_{\tau_{\text{mod}}} \backslash W.$$

The (strong) *Bruhat order* $\preceq$ on the Weyl group W has the following geometric interpretation as *folding order*; see [34, Definition 3.2]. For distinct elements $w_1, w_2 \in W$, it holds that

$$w_1 \prec w_2$$

if and only if there exists a folding map $a_{\text{mod}} \to a_{\text{mod}}$ fixing σ_{mod} and mapping $w_2\sigma_{\text{mod}} \mapsto w_1\sigma_{\text{mod}}$; see [34, Section 3.2]. Here, by a *folding map* $a_{\text{mod}} \to a_{\text{mod}}$ we mean a type-preserving continuous map which sends chambers isometrically onto chambers.

The folding order on relative positions coincides with the inclusion order on Schubert cycles, ie $w_1 \preceq w_2$ if and only if the Schubert cell $\{\text{pos}(\cdot, \sigma_0) = w_1\}$ is contained in the closure of the Schubert cell $\{\text{pos}(\cdot, \sigma_0) = w_2\}$, and the *Schubert cycles* relative σ_0 are the sublevel sets of $\text{pos}(\cdot, \sigma_0)$. In the case of complex semisimple Lie groups G this inclusion relation is a classical result of Chevalley [15]; for the case of general semisimple Lie groups we refer the reader to [34, Proposition 3.14] or, alternatively, to [42; 43].

We also need to define the folding order more generally on positions of chambers relative to simplices τ_0 of an arbitrary face type τ_{mod}. We say that

$$W_{\tau_{\text{mod}}}\overline{\sigma}_1 \preceq_{\tau_{\text{mod}}} W_{\tau_{\text{mod}}}\overline{\sigma}_2$$

for chambers $\bar{\sigma}_1, \bar{\sigma}_2 \subset a_{\mathrm{mod}}$ if and only if there exist $\bar{\sigma}_i' \in W_{\tau_{\mathrm{mod}}} \bar{\sigma}_i$ such that

$$\bar{\sigma}_1' \preceq \bar{\sigma}_2',$$

equivalently, geometrically, if for some (any) chambers $\bar{\sigma}_i' \in W_{\tau_{\mathrm{mod}}} \bar{\sigma}_i$ there exists a folding map $a_{\mathrm{mod}} \to a_{\mathrm{mod}}$ fixing τ_{mod} and mapping $\bar{\sigma}_2'$ to $\bar{\sigma}_1'$. (Note that the elements in $W_{\tau_{\mathrm{mod}}}$ are such folding maps.)

Lemma 8.1 *The relation $\prec_{\tau_{\mathrm{mod}}}$ is a partial order.*

Proof Transitivity holds since the composition of folding maps is again a folding map.

To verify reflexivity, pick points $\xi_{\mathrm{mod}} \in \mathrm{int}(\tau_{\mathrm{mod}})$ and $\eta_{\mathrm{mod}} \in \mathrm{int}(\sigma_{\mathrm{mod}})$.

Let $\bar{\sigma} = w\sigma_{\mathrm{mod}} \subset a_{\mathrm{mod}}$ be a chamber and $f \colon a_{\mathrm{mod}} \to a_{\mathrm{mod}}$ a folding map fixing τ_{mod}. Let $\bar{\eta} = w\eta_{\mathrm{mod}}$. If the f–image of the segment $\xi_{\mathrm{mod}}\bar{\eta}$ is again an unbroken geodesic segment, then the two geodesic segments are congruent by an element of $W_{\tau_{\mathrm{mod}}}$, because their initial directions at ξ_{mod} are. On the other hand, if the f–image of $\xi_{\mathrm{mod}}\bar{\eta}$ is a broken geodesic segment, then the distance of its endpoints is strictly smaller than its length, and consequently $f\bar{\sigma} \not\succeq \bar{\sigma}$. This shows that

$$W_{\tau_{\mathrm{mod}}}\bar{\sigma}_1 \preceq_{\tau_{\mathrm{mod}}} W_{\tau_{\mathrm{mod}}}\bar{\sigma}_2 \preceq_{\tau_{\mathrm{mod}}} W_{\tau_{\mathrm{mod}}}\bar{\sigma}_1 \implies W_{\tau_{\mathrm{mod}}}\bar{\sigma}_1 = W_{\tau_{\mathrm{mod}}}\bar{\sigma}_2$$

and hence reflexivity. $\square$

The relative position function

$$\mathrm{pos} \colon \mathrm{Flag}_{\sigma_{\mathrm{mod}}} \times \mathrm{Flag}_{\tau_{\mathrm{mod}}} \to W_{\tau_{\mathrm{mod}}} \backslash W$$

is *lower semicontinuous*; compare the discussion of closures of Schubert cycles above.

It follows from the geometric description of the folding orders in terms of folding maps that for face types $\nu_{\mathrm{mod}} \subset \tau_{\mathrm{mod}}$ the order $\preceq_{\nu_{\mathrm{mod}}}$ refines the order $\preceq_{\tau_{\mathrm{mod}}}$, because a folding map fixing τ_{mod} fixes in particular its face ν_{mod}. Thus, for chambers $\sigma_1, \sigma_2 \subset \partial_\infty X$ and simplices $\nu \subset \tau \subset \partial_\infty X$ of types $\nu_{\mathrm{mod}} = \theta(\nu) \subset \tau_{\mathrm{mod}} = \theta(\tau)$ it holds that

(8.2) $$\mathrm{pos}(\sigma_1, \tau) \preceq_{\tau_{\mathrm{mod}}} \mathrm{pos}(\sigma_2, \tau) \implies \mathrm{pos}(\sigma_1, \nu) \preceq_{\nu_{\mathrm{mod}}} \mathrm{pos}(\sigma_2, \nu).$$

We now describe the action of the longest element $w_0 \in W$ on relative positions. Note that $w_0 W_{\tau_{\mathrm{mod}}} w_0^{-1} = W_{w_0 \tau_{\mathrm{mod}}} = W_{\iota \tau_{\mathrm{mod}}}$ and w_0 maps $W_{\tau_{\mathrm{mod}}}$–orbits to $W_{\iota \tau_{\mathrm{mod}}}$–orbits. The action of w_0 therefore induces a natural map

$$W_{\tau_{\mathrm{mod}}} \backslash W \xrightarrow{w_0} W_{\iota \tau_{\mathrm{mod}}} \backslash W, \quad W_{\tau_{\mathrm{mod}}} w \mapsto w_0 W_{\tau_{\mathrm{mod}}} w = W_{\iota \tau_{\mathrm{mod}}} w_0 w.$$

This map is order-reversing:

$$(8.3) \qquad W_{\tau_{\mathrm{mod}}} w \prec_{\tau_{\mathrm{mod}}} W_{\tau_{\mathrm{mod}}} w' \iff W_{\iota\tau_{\mathrm{mod}}} w_0 w \succ_{\iota\tau_{\mathrm{mod}}} W_{\iota\tau_{\mathrm{mod}}} w_0 w';$$

see [34, Section 3.2].

Definition 8.4 (complementary position) We define the *complementary position* by

$$\text{c-pos} := w_0 \, \text{pos} \,.$$

This terminology is justified by:

Lemma 8.5 [34, Lemma 3.16] *Let $\tau, \hat{\tau}, \sigma \subset \partial_\infty X$ be two simplices and a chamber contained in an apartment a, and suppose that τ and $\hat{\tau}$ are antipodal. Then $\text{pos}(\sigma, \hat{\tau}) = \text{c-pos}(\sigma, \tau)$.*

The relation of "complementarity" is clearly symmetric: c-c-pos = pos. Passing to complementary relative position reverses the partial order (see (8.3)):

$$(8.6) \qquad \text{pos}(\sigma_1, \tau) \prec_{\theta(\tau)} \text{pos}(\sigma_2, \tau) \iff \text{c-pos}(\sigma_1, \tau) \succ_{\iota\theta(\tau)} \text{c-pos}(\sigma_2, \tau).$$

8.2 Further properties of the folding order

This is a technical section whose main result (Proposition 8.12) will be used in the proof of Proposition 9.12, which is the key to proving proper discontinuity of actions of τ_{mod}–regular subgroups.

We begin with a result useful for comparing relative positions.

Lemma 8.7 (i) *Let $\sigma_0, \sigma_1, \sigma_2 \subset \partial_\infty X$ be chambers, and suppose there exists a segment $\xi_0 \xi_2$ with $\xi_0 \in \text{int}(\sigma_0)$ and $\xi_2 \in \text{int}(\sigma_2)$ containing a point $\xi_1 \in \text{int}(\sigma_1)$. Then*

$$\text{pos}(\sigma_1, \sigma_0) \preceq \text{pos}(\sigma_2, \sigma_0)$$

with equality if and only if $\sigma_1 = \sigma_2$.

(ii) *More generally, let $\sigma_1, \sigma_2 \subset \partial_\infty X$ be chambers and let $\tau_0 \subset \partial_\infty X$ be a simplex of type τ_{mod}. Suppose that there exists a segment $\xi_0 \xi_2$ with $\xi_0 \in \text{int}(\tau_0)$ and $\xi_2 \in \text{int}(\sigma_2)$ containing a point $\xi_1 \in \text{int}(\sigma_1)$. Then*

$$\text{pos}(\sigma_1, \tau_0) \preceq_{\tau_{\mathrm{mod}}} \text{pos}(\sigma_2, \tau_0)$$

with equality if and only if $\sigma_1 = \sigma_2$.

Proof We prove the more general assertion (ii). After perturbing ξ_2, we can arrange that the subsegment $\xi_1\xi_2$ avoids codimension-two faces. Along this subsegment we find a gallery of chambers connecting σ_1 to σ_2. We may therefore proceed by induction and assume that the chambers σ_1 and σ_2 are adjacent, ie share a panel π which is intersected transversally by $\xi_1\xi_2$. Working in an apartment containing $\tau_0, \sigma_1, \sigma_2$, the wall through π does not contain τ_0 and separates $\mathrm{st}(\tau_0) \cup \sigma_1$ from σ_2. Folding at this wall yields the desired inequality. $\qquad\square$

We next study the values of the relative position function on stars of simplices.

We fix a reference chamber $\sigma_0 \subset \partial_\infty X$. Let $v \subset \partial_\infty X$ be a simplex. For any interior points $\eta \in \mathrm{int}(v)$ and $\xi_0 \in \mathrm{int}(\sigma_0)$, the segment $\eta\xi_0$ enters the *interior* of a chamber $\sigma_- \supset v$, ie

$$\eta\xi_0 \cap \mathrm{int}(\sigma_-) \neq \varnothing.$$

Note that the chamber σ_- does not depend on the interior points η and ξ_0. Moreover, it is contained in any apartment containing σ_0 and v. We call σ_- the chamber in $\mathrm{st}(v)$ *pointing towards* σ_0.

Similarly, if $\xi_0\xi_+ \supsetneq \xi_0\eta$ is an extension of the segment $\xi_0\eta$ beyond η, then there exists a chamber $\sigma_+ \supset v$ such that $\eta\xi_+ \cap \mathrm{int}(\sigma_+) \neq \varnothing$, and we call σ_+ a chamber in $\mathrm{st}(v)$ *pointing away from* σ_0.

Let $a \subset \partial_\infty X$ be an apartment containing σ_0 and v. Then $\sigma_- \subset a$. Moreover, since geodesic segments inside a extend uniquely, there exists a *unique* chamber $\sigma_+ \subset \mathrm{st}(v) \cap a$ pointing away from σ_0. The chambers $\sigma_\pm \subset a$ can be characterized as follows in terms of separation from σ_0 by walls:

Lemma 8.8 *Let $\sigma \subset \mathrm{st}(v) \cap a$ be a chamber. Then*

(i)   *$\sigma = \sigma_+$ if and only if σ is separated from σ_0 by every wall $s \subset a$ containing v,*

(ii)   *$\sigma = \sigma_-$ if and only if σ is not separated from σ_0 by any wall $s \subset a$ containing v.*

Proof (i)   Clearly, σ_+ is separated from σ_0 by every wall $s \supset v$ because, using the above notation, $\xi_0\xi_+ \cap s = \eta$. Conversely, if σ is separated from σ_0 by all such walls s, then σ and σ_+ lie in the same hemispheres bounded by the walls $s \supset v$ in a, and therefore must coincide.

(ii)   Similarly, σ_- is not separated from σ_0 by any wall $s \supset v$ because $\xi_0\eta \cap s = \eta$, and conversely, if σ is not separated from σ_0 by any wall $s \supset v$, then σ and σ_- lie in the same hemispheres bounded by the walls $s \supset v$ in a, and therefore must coincide. $\quad\square$

Remark 8.9 The assertion of the lemma remains valid if one only admits the walls $s \subset a$ such that $s \cap \sigma$ is a panel containing v.

The chambers pointing towards and away from σ_0 in $\partial_\infty X$ can also be characterized in terms of the folding order:

Lemma 8.10 *The restriction of the function* $\mathrm{pos}(\cdot, \sigma_0)$ *to the set of chambers contained in* $\mathrm{st}(v)$ *attains a unique minimal value in* σ_- *and a unique maximal value[17] precisely in the chambers pointing away from* σ_0.

Proof Let $\sigma \supset v$ be a chamber and let $a \subset \partial_\infty X$ be an apartment containing σ_0 and σ. Then $\sigma_- \subset a$. Let $\sigma_+ \subset \mathrm{st}(v) \cap a$ be the unique chamber pointing away from σ_0.

Still using the above notation, let $\xi_0 \xi_+ \supset \xi_0 \eta$ be an extension of the segment $\xi_0 \eta$ with endpoint $\xi_+ \in \mathrm{int}(\sigma_+)$. Let $\xi_- \in \xi_0 \eta \cap \mathrm{int}(\sigma_-)$. The points ξ_- and η appear in this order on the (oriented) segment $\xi_0 \xi_+$.

We now perturb the segment $\xi_0 \xi_+$ to a segment $\xi_0 \xi'_+$ which intersects $\mathrm{int}(\sigma)$ in a point η' close to η and $\mathrm{int}(\sigma_-)$ in a point ξ'_- close to ξ_-. The perturbation is possible because $\sigma \supset v$. Again, the points ξ'_- and η' appear in this order on the perturbed segment $\xi_0 \xi'_+$. Lemma 8.7 implies that

$$\mathrm{pos}(\sigma_-, \sigma_0) \preceq \mathrm{pos}(\sigma, \sigma_0) \preceq \mathrm{pos}(\sigma_+, \sigma_0)$$

with equality in the first (second) inequality if and only if $\sigma = \sigma_-$ ($\sigma = \sigma_+$). The assertion of the lemma follows because $\mathrm{pos}(\sigma_+, \sigma_0)$ does not depend on the choice of a. $\square$

We now extend the lemma to the case of relative position with respect to a simplex τ_0.

We pick a chamber $\sigma_0 \supset \tau_0$. For a simplex v, the chamber $\sigma_- \subset \mathrm{st}(v)$ pointing towards σ_0 is defined as before.

Corollary 8.11 *The restriction of the function* $\mathrm{pos}(\cdot, \tau_0)$ *to the set of chambers in* $\mathrm{st}(v)$ *attains a unique minimal value in* σ_- *and a unique maximal value precisely in the chambers pointing away from* σ_0.

Proof This is an immediate consequence of Lemma 8.10 and (8.2). $\square$

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[17]By this we mean that it is larger than all other values.

Combing the corollary with the discussion in Section 3.2.3, we obtain:

Proposition 8.12 *Let τ_0, ν be simplices. Then there exists a dense open subset of chambers in $\mathrm{st}_{\mathrm{Fu}}(\nu)$ where the function $\mathrm{pos}(\,\cdot\,, \tau_0)$ attains its unique maximal value.*

8.3 Thickenings

A *thickening* (of the neutral element) in W is a subset

$$\mathrm{Th} \subset W$$

which is a union of sublevels for the folding order, ie which contains with every element w also every element w' satisfying $w' \prec w$; see [34, Section 3.4.1]. In the theory of posets, such subsets are called *ideals*.

Note that

$$\mathrm{Th}^c := w_0(W - \mathrm{Th})$$

is again a thickening. Here, $w_0 \in W$ denotes the longest element of the Weyl group, that is, the element of order two mapping σ_{mod} to the opposite chamber in a_{mod}. It holds that

$$W = \mathrm{Th} \sqcup w_0\mathrm{Th}^c,$$

and we call Th^c the thickening *complementary* to Th.

A thickening $\mathrm{Th} \subset W$ is called *fat* if $\mathrm{Th} \cup w_0\mathrm{Th} = W$, equivalently, $\mathrm{Th} \supseteq \mathrm{Th}^c$. It is called *slim* if $\mathrm{Th} \cap w_0\mathrm{Th} = \varnothing$, equivalently, $\mathrm{Th} \subseteq \mathrm{Th}^c$. It is called *balanced* if it is both fat and slim, equivalently, $\mathrm{Th} = \mathrm{Th}^c$; see [34, Definition 3.25].

For types $\widetilde{\vartheta}_0, \widetilde{\vartheta} \in \sigma_{\mathrm{mod}}$ and a radius $r \in [0, \pi]$ we define the *metric thickening*

$$\mathrm{Th}_{\widetilde{\vartheta}_0, \widetilde{\vartheta}, r} := \{w \in W : d(w\widetilde{\vartheta}, \widetilde{\vartheta}_0) \leq r\}$$

using the natural W–invariant spherical metric d on a_{mod}; see [34, Section 3.4.1, formula (3.26)].

For a face type $\tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$, we denote by $W_{\tau_{\mathrm{mod}}}$ its stabilizer in W. Furthermore, $\iota = -w_0 \colon \sigma_{\mathrm{mod}} \to \sigma_{\mathrm{mod}}$ denotes the canonical involution of the model spherical Weyl chamber.

Lemma 8.13   (i) *If $\widetilde{\vartheta}_0 \in \tau_{\mathrm{mod}}$, then $W_{\tau_{\mathrm{mod}}}\mathrm{Th}_{\widetilde{\vartheta}_0, \widetilde{\vartheta}, r} = \mathrm{Th}_{\widetilde{\vartheta}_0, \widetilde{\vartheta}, r}$.*

  (ii) *If $\iota\widetilde{\vartheta}_0 = \widetilde{\vartheta}_0$, then $\mathrm{Th}_{\widetilde{\vartheta}_0, \widetilde{\vartheta}, r}$ is fat for $r \geq \frac{\pi}{2}$ and slim for $r < \frac{\pi}{2}$.*

Proof (i) For $w' \in W_{\tau_{\mathrm{mod}}}$, we have that $w'\tilde{\vartheta}_0 = \tilde{\vartheta}_0$ and hence

$$d(w'w\tilde{\vartheta}, \tilde{\vartheta}_0) = d(w\tilde{\vartheta}, \underbrace{w'^{-1}\tilde{\vartheta}_0}_{\tilde{\vartheta}_0}).$$

(ii) Since $w_0\tilde{\vartheta}_0 = -\iota\tilde{\vartheta}_0 = -\tilde{\vartheta}_0$, we have

$$d(w_0 w\tilde{\vartheta}, -\tilde{\vartheta}_0) = d(w\tilde{\vartheta}, \underbrace{-w_0\tilde{\vartheta}_0}_{\tilde{\vartheta}_0}),$$

whence the assertion. $\qquad\square$

Corollary 8.14 (existence of balanced thickenings) *If the face type τ_{mod} is ι–invariant, $\iota\tau_{\mathrm{mod}} = \tau_{\mathrm{mod}}$, then there exists a $W_{\tau_{\mathrm{mod}}}$–invariant balanced thickening $\mathrm{Th} \subset W$.*

Proof Since $\iota\tau_{\mathrm{mod}} = \tau_{\mathrm{mod}}$, there exists $\tilde{\vartheta}_0 \in \tau_{\mathrm{mod}}$ such that $\iota\tilde{\vartheta}_0 = \tilde{\vartheta}_0$. Moreover, there exists $\tilde{\vartheta} \in \sigma_{\mathrm{mod}}$ such that $d(\cdot\,\tilde{\vartheta}, \tilde{\vartheta}_0) \neq \frac{\pi}{2}$ on W. (This holds for an open dense subset of types $\tilde{\vartheta} \in \sigma_{\mathrm{mod}}$.) According to the lemma, the metric thickening $\mathrm{Th}_{\tilde{\vartheta}_0, \tilde{\vartheta}, \frac{\pi}{2}}$ is balanced and $W_{\tau_{\mathrm{mod}}}$–invariant. $\qquad\square$

Given a thickening $\mathrm{Th} \subset W$, we obtain *thickenings at infinity* as follows.

First, we define the thickening in $\partial_{\mathrm{Fu}} X$ of a chamber $\sigma \in \partial_{\mathrm{Fu}} X$ as

$$\mathrm{Th}_{\mathrm{Fu}}(\sigma) := \{\mathrm{pos}(\cdot\,, \sigma) \in \mathrm{Th}\} \subset \partial_{\mathrm{Fu}} X.$$

It is a finite union of Schubert cycles relative σ. We then define the thickening of σ inside the Finsler ideal boundary as the "suspension" of its thickening inside the Furstenberg boundary:

$$\mathrm{Th}^{\mathrm{Fins}}(\sigma) := \{[b_{\nu,p}^{\overline{\theta}}] : \mathrm{st}_{\mathrm{Fu}}(\nu) \subset \mathrm{Th}_{\mathrm{Fu}}(\sigma)\} = \bigcup\{X_\nu : \mathrm{st}_{\mathrm{Fu}}(\nu) \subset \mathrm{Th}_{\mathrm{Fu}}(\sigma)\} \subset \partial_\infty^{\mathrm{Fins}} X,$$

where $\mathrm{st}_{\mathrm{Fu}}(\nu)$ denotes the set of chambers containing ν as a face; see Section 3.2.3. Note that $\mathrm{Th}_{\mathrm{Fu}}(\sigma) = \mathrm{Th}^{\mathrm{Fins}}(\sigma) \cap \partial_{\mathrm{Fu}} X$.

Lemma 8.15 $\mathrm{Th}^{\mathrm{Fins}}(\sigma)$ *is compact.*

Proof Consider a sequence of points $[b_{\nu_n, p_n}^{\overline{\theta}}] \in \mathrm{Th}^{\mathrm{Fins}}(\sigma)$, and suppose that it converges in $\partial_\infty^{\mathrm{Fins}} X$:

$$[b_{\nu_n, p_n}^{\overline{\theta}}] \to [b_{\mu, q}^{\overline{\theta}}].$$

We must show that also $[b_{\mu, q}^{\overline{\theta}}] \in \mathrm{Th}^{\mathrm{Fins}}(\sigma)$.

After extraction, we may assume that all simplices ν_n have the same type $\theta(\nu_n) = \nu_{\mathrm{mod}}$. According to Lemma 5.31, $\nu_n \to \nu \subseteq \mu$. By assumption, $\mathrm{st}(\nu_n) \subset \mathrm{Th}_{\mathrm{Fu}}(\sigma)$, and we must show that $\mathrm{st}(\mu) \subset \mathrm{Th}_{\mathrm{Fu}}(\sigma)$. Since $\mathrm{st}(\nu) \supseteq \mathrm{st}(\mu)$, this will follow from $\mathrm{st}(\nu) \subset \mathrm{Th}_{\mathrm{Fu}}(\sigma)$. The latter follows from the closedness of $\mathrm{Th}_{\mathrm{Fu}}(\sigma)$ in $\partial_{\mathrm{Fu}} X$, because every chamber $\sigma' \subset \mathrm{st}(\nu)$ is a limit of a sequence of chambers $\sigma'_n \subset \mathrm{st}(\nu_n)$. $\qquad\square$

Remark 8.16 One can show that $\mathrm{Th}^{\mathrm{Fins}}(\sigma) \subset \partial^{\mathrm{Fins}}_\infty X$ is a contractible CW complex. In the second version of this paper on arXiv (see Theorem 8.21 there), we proved that it is Čech acyclic.

Example 8.17 Suppose that the Weyl group W of X is of type A_2, ie is isomorphic to the permutation group on three letters. Let $s_1, s_2 \in W$ denote the generators which are the reflections in the walls of the positive chamber σ_{mod}. There is the unique balanced thickening $\mathrm{Th} = \{e, s_1, s_2\} \subset W$. The thickening $\mathrm{Th}^{\mathrm{Fins}}(\sigma) \subset \partial^{\mathrm{Fins}}_\infty X$ is the wedge of two closed disks connected at the point σ: These disks are the visual compactifications $\bar{X}_{\tau_1}, \bar{X}_{\tau_2}$ of two rank-1 symmetric spaces X_{τ_i}. Here τ_1, τ_2 are the two vertices of the edge σ.

More generally, we define the thickening in $\partial^{\mathrm{Fins}}_\infty X$ of a set of chambers $A \subset \partial_{\mathrm{Fu}} X$ as

$$\mathrm{Th}^{\mathrm{Fins}}(A) := \bigcup_{\sigma \in A} \mathrm{Th}^{\mathrm{Fins}}(\sigma) \subset \partial^{\mathrm{Fins}}_\infty X.$$

Lemma 8.18 *If A is compact, then $\mathrm{Th}^{\mathrm{Fins}}(A)$ is compact.*

Proof Since $\partial_{\mathrm{Fu}} X$ is a homogeneous space for the maximal compact subgroup K, there exists a chamber $\sigma_0 \in A$ and a compact subset $C \subset K$ such that $A = C\sigma_0$. Then

$$\mathrm{Th}^{\mathrm{Fins}}(A) = C \cdot \mathrm{Th}^{\mathrm{Fins}}(\sigma_0)$$

and is hence compact as a consequence of the previous lemma. $\qquad\square$

If the thickening $\mathrm{Th} \subset W$ is $W_{\tau_{\mathrm{mod}}}$–invariant, then we can define the thickening in $\partial^{\mathrm{Fins}}_\infty X$ of a simplex $\tau \subset \partial_\infty X$ of type τ_{mod} as

$$\mathrm{Th}^{\mathrm{Fins}}(\tau) := \mathrm{Th}^{\mathrm{Fins}}(\sigma) \subset \partial^{\mathrm{Fins}}_\infty X$$

for a chamber $\sigma \supseteq \tau$. It does not depend on σ. For a set $A \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ of simplices of type τ_{mod}, we define its thickening in $\partial^{\mathrm{Fins}}_\infty X$ as

$$\mathrm{Th}^{\mathrm{Fins}}(A) := \bigcup_{\tau \in A} \mathrm{Th}^{\mathrm{Fins}}(\tau) \subset \partial^{\mathrm{Fins}}_\infty X.$$

Again, $\mathrm{Th}^{\mathrm{Fins}}(A)$ is compact if A is.

Lemma 8.19 (fibration of thickenings) *Let $A \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ be compact, and suppose that the thickenings $\mathrm{Th}^{\mathrm{Fins}}(\tau)$ of the simplices $\tau \in A$ are pairwise disjoint. Then the natural map*

$$\mathrm{Th}^{\mathrm{Fins}}(A) \xrightarrow{\pi} A$$

is a fiber bundle.

Proof Regarding continuity of π, suppose that $\xi_n \to \xi$ in $\mathrm{Th}^{\mathrm{Fins}}(A)$ and $\tau_n \to \tau$ in A with $\xi_n \in \mathrm{Th}^{\mathrm{Fins}}(\tau_n)$. Then $\xi \in \mathrm{Th}^{\mathrm{Fins}}(\tau)$ by semicontinuity of relative position, and hence $\pi(\xi) = \tau$.

In order to show that π is a fibration, we need to construct local trivializations. Fix $\tau \in A$ and an opposite simplex $\hat{\tau}$. The unipotent horocyclic subgroup $H_{\hat{\tau}}$ acts simply transitively on an open neighborhood of τ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$. Now, let $S \subset H_{\hat{\tau}}$ denote the closed subset consisting of all $h \in H_{\hat{\tau}}$ which send τ to elements of A. Then $S\tau$ is a neighborhood of τ in A. Restricting the action of $H_{\hat{\tau}}$ to the subset S, we obtain a topological embedding

$$S \times \mathrm{Th}^{\mathrm{Fins}}(\tau) \to \mathrm{Th}^{\mathrm{Fins}}(A)$$

and a local trivialization of π over a neighborhood of τ in A. $\square$

9 Proper discontinuity

Our aim is to construct domains of proper discontinuity for the action

$$\Gamma \curvearrowright \overline{X}^{\mathrm{Fins}}$$

of discrete subgroups $\Gamma < G$ on the Finsler compactification $\overline{X}^{\mathrm{Fins}}$ of X. The proper discontinuity of an action can be rephrased as the absence of dynamical relations, and our construction of domains results from studying the dynamical relations between points in $\overline{X}^{\mathrm{Fins}}$ with respect to the action $G \curvearrowright \overline{X}^{\mathrm{Fins}}$ and determining necessary conditions.

The Furstenberg boundary is naturally embedded in the Finsler boundary as the closed stratum at infinity,

$$\partial_{\mathrm{Fu}} X \subset \partial_{\infty}^{\mathrm{Fins}} X;$$

see Corollary 5.25. The G–action on $\overline{X}^{\mathrm{Fins}}$ is determined by (fills in) the G–action on $\partial_{\mathrm{Fu}} X \cong \mathrm{Flag}_{\sigma_{\mathrm{mod}}}$, and our approach is based on the study of the dynamics of the G–action on its associated flag manifolds in [34]. We first recall from there a combinatorial

inequality for dynamical relations in $\partial_{\mathrm{Fu}} X$ and provide an auxiliary result regarding the dynamics of pure sequences on $\partial_{\mathrm{Fu}} X$; see Section 9.1. We then show in Section 9.2 how dynamical relations in $\partial_\infty^{\mathrm{Fins}} X$ imply dynamical relations in $\partial_{\mathrm{Fu}} X$ and use this to extend the combinatorial inequality for dynamical relations from $\partial_{\mathrm{Fu}} X$ to $\overline{X}^{\mathrm{Fins}}$. With this inequality at hand, one readily obtains domains of proper discontinuity by removing suitable thickenings of limit sets (see Section 8.3), ie the points which have "sufficiently special" position relative to some limit point.

9.1 Dynamics on the Furstenberg boundary

We consider the action

$$G \curvearrowright \partial_{\mathrm{Fu}} X$$

on the Furstenberg boundary $\partial_{\mathrm{Fu}} X \cong \mathrm{Flag}_{\sigma_{\mathrm{mod}}}$. Specifically, we are interested in the dynamics of diverging sequences in G. Let (g_n) be a τ_{mod}–contracting sequence in G with

(9.1) $$g_n|_{C(\tau_-)} \to \tau_+$$

uniformly on compacts, where $\tau_\pm \in \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$.

9.1.1 Dynamical relations We recall from [34] the following necessary condition for dynamical relations between points in $\partial_{\mathrm{Fu}} X$ with respect to the action of (g_n):[18]

Proposition 9.2 (dynamical relation inequality in $\partial_{\mathrm{Fu}} X$; compare [34, Proposition 6.5]) *Let (g_n) be a τ_{mod}–contracting sequence in G satisfying (9.1). Suppose that there is a dynamical relation*

$$\sigma \overset{(g_n)}{\sim} \sigma'$$

between points $\sigma, \sigma' \in \partial_{\mathrm{Fu}} X$. Then

(9.3) $$\mathrm{pos}(\sigma', \tau_+) \prec \mathrm{c\text{-}pos}(\sigma, \tau_-).$$

Intuitively, this means that it cannot happen that σ is far from τ_- *and* σ' is far from τ_+, where "far" is to be understood as having "generic" relative position.

Proof The Furstenberg boundary $\partial_{\mathrm{Fu}} X$ is naturally identified with the regular G–orbits $G\xi$ in the visual boundary $\partial_\infty X$. The assertion is therefore equivalent to the implication (i) $\Longrightarrow$ (ii) of [34, Proposition 6.5] in the special case of regular G–orbits. $\square$

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[18]Proposition 6.5 of [34] applies to τ_{mod}–contracting sequences on arbitrary flag manifolds $\mathrm{Flag}_{\nu_{\mathrm{mod}}}$ for arbitrary face types $\tau_{\mathrm{mod}}, \nu_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$.

9.1.2 Pure sequences The action of the τ_{mod}–contracting sequence (g_n) in G satisfying (9.1) preserves the natural fibration of flag manifolds

$$(9.4) \qquad \pi_{\tau_{\mathrm{mod}}} : \partial_{\mathrm{Fu}} X \cong \mathrm{Flag}_{\sigma_{\mathrm{mod}}} \to \mathrm{Flag}_{\tau_{\mathrm{mod}}} .$$

For a simplex $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$,

$$\mathrm{st}_{\mathrm{Fu}}(\tau) = \pi_{\tau_{\mathrm{mod}}}^{-1}(\tau) \subset \partial_{\mathrm{Fu}} X$$

is the set of chambers $\sigma \supset \tau$; see Section 3.2.3. For $\tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$, we denote by

$$C_{\mathrm{Fu}}(\tau_-) := \pi_{\tau_{\mathrm{mod}}}^{-1}(C(\tau_-)) \subset \partial_{\mathrm{Fu}} X$$

the set of chambers over $C(\tau_-)$, and by $\partial C_{\mathrm{Fu}}(\tau_-) := \pi_{\tau_{\mathrm{mod}}}^{-1}(\partial C(\tau_-))$ its complement in $\partial_{\mathrm{Fu}} X$.

The contraction property (9.1) for the action of (g_n) on the base $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ translates into the property for the dynamics on $\partial_{\mathrm{Fu}} X$ that the (g_n)–orbits in $C_{\mathrm{Fu}}(\tau_-)$ accumulate at $\mathrm{st}_{\mathrm{Fu}}(\tau)$ locally uniformly (in view of the fibration (9.4)), ie that for every compact subset $A \subset C_{\mathrm{Fu}}(\tau_-)$ the sequence of subsets $g_n A$ accumulates at (a subset of) $\mathrm{st}_{\mathrm{Fu}}(\tau)$. In the terminology of [34, Definition 5.8] this means that the sequence (g_n) is $(\partial C_{\mathrm{Fu}}(\tau_-), \mathrm{st}_{\mathrm{Fu}}(\tau))$–*accumulating* on $\partial_{\mathrm{Fu}} X$.

As a τ_{mod}–contracting sequence, (g_n) is in particular τ_{mod}–regular; see Section 4.3. If we make the stronger additional assumption that (g_n) is τ_{mod}–*pure* (see Definition 4.8), then its accumulation dynamics on $\partial_{\mathrm{Fu}} X$ can be described more precisely. The next result expresses that there is only bounded distortion in the direction of the $\pi_{\tau_{\mathrm{mod}}}$–fibers:

Proposition 9.5 *Suppose that the sequence (g_n) in G is τ_{mod}–contracting as in (9.1) and τ_{mod}–pure. Then, after extraction,*

$$g_n|_{C_{\mathrm{Fu}}(\tau_-)} \to \phi$$

uniformly on compacts, where the limit map

$$\phi : C_{\mathrm{Fu}}(\tau_-) \to \mathrm{st}_{\mathrm{Fu}}(\tau_+)$$

is an open (in the classical topology) algebraic map. Moreover, for every $\hat{\tau}_- \in C(\tau_-)$, the restriction

$$\phi|_{\mathrm{st}_{\mathrm{Fu}}(\hat{\tau}_-)} : \mathrm{st}_{\mathrm{Fu}}(\hat{\tau}_-) \to \mathrm{st}_{\mathrm{Fu}}(\tau_+)$$

is given by the restriction of an element in G, and hence is an (algebraic) homeomorphism.

Proof We fix a basepoint $o \in X$.

We first note that we can replace the sequence (g_n) by a sequence of transvections. Indeed, let $b_n \to b$ and $b'_n \to b'$ be converging sequences in G, and put $\widetilde{g}_n := b_n g_n b'_n$. Then the τ_{mod}–pureness of (g_n) is equivalent to the τ_{mod}–pureness of $(\widetilde{g}_n)$, and (9.1) to locally uniform convergence $\widetilde{g}_n|_{C(b'^{-1}\tau_-)} \to b\tau_+$. Furthermore, the locally uniform convergence $g_n|_{C_{\mathrm{Fu}}(\tau_-)} \to \phi$ translates into $\widetilde{g}_n|_{C_{\mathrm{Fu}}(b'^{-1}\tau_-)} \to b\phi b'$ with limit map $b\phi b' \colon C_{\mathrm{Fu}}(b'^{-1}\tau_-) \to \mathrm{st}_{\mathrm{Fu}}(b\tau_+)$. Thus, the assertion holds for (g_n) if and only if it holds for $(\widetilde{g}_n)$. Since (g_n) is τ_{mod}–pure and we may pass to a subsequence, we can therefore replace (g_n), using a *KAK*–decomposition of G, by a sequence of transvections t_n with axes through o. Moreover, we may assume that the sequence $(t_n o)$ is contained in the Weyl sector $V(o, \tau_+)$ and τ_{mod}–regular, ie drifts away from the boundary.

In this special situation, things become explicit: The simplex τ_- is o–opposite to τ_+. The unipotent subgroup H_{τ_-} acts simply transitively on $C(\tau_-) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ and, accordingly, the natural map

$$\alpha \colon H_{\tau_-} \times \mathrm{st}_{\mathrm{Fu}}(\tau_+) \to C_{\mathrm{Fu}}(\tau_-)$$

is an algebraic isomorphism. The transvections t_n normalize H_{τ_-}, and it holds that[19]

$$c_{t_n} \to e$$

uniformly on compacts in H_{τ_-}, where c_g denotes conjugation by g. Since t_n acts trivially on $\mathrm{st}_{\mathrm{Fu}}(\tau_+)$, we obtain for $h \in H_{\tau_-}$ and $\sigma \in \mathrm{st}_{\mathrm{Fu}}(\tau_+)$ that

$$t_n(h\sigma) = \underbrace{(t_n h t_n^{-1})}_{\to e} \underbrace{(t_n \sigma)}_{=\sigma} \to \sigma.$$

Hence,

$$t_n \to \phi$$

uniformly on compacts in $C_{\mathrm{Fu}}(\tau_-)$, where $\phi \colon C_{\mathrm{Fu}}(\tau_-) \to \mathrm{st}_{\mathrm{Fu}}(\tau_+)$ is the open continuous limit map given by $\phi(h\sigma) = \sigma$. Moreover, for $\widehat{\tau}_- \in C(\tau_-)$, the restriction $\phi|_{\mathrm{st}_{\mathrm{Fu}}(\widehat{\tau}_-)} \colon \mathrm{st}_{\mathrm{Fu}}(\widehat{\tau}_-) \to \mathrm{st}_{\mathrm{Fu}}(\tau_+)$ coincides with the restriction of the unique element in H_{τ_-} which maps $\widehat{\tau}_-$ to τ_+. In order to see that ϕ is algebraic, we observe that it equals the composition of the algebraic map α^{-1} followed by the projection to the second factor, $H_{\tau_-} \times \mathrm{st}_{\mathrm{Fu}}(\tau_+) \to \mathrm{st}_{\mathrm{Fu}}(\tau_+)$. $\qquad\square$

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[19]See eg [17], Section 2.17].

In general, verifying (uniform) regularity of a subgroup is not an easy task. See eg Theorem 5.53 of [33] for a result of this kind. For Zariski dense subgroups the verification of regularity becomes easier. The next result, which is an interesting application of Proposition 9.5, provides a sufficient condition for τ_{mod}–regularity:

Theorem 9.6 Let $\rho\colon \Gamma \to G$ be a representation whose image is Zariski dense in G. Suppose that Z is a compact metrizable space, $\Gamma \curvearrowright Z$ is a discrete convergence group action (with finite kernel), and $f\colon Z \to \mathrm{Flag}_{\nu_{mod}}$ is a ρ–equivariant topological embedding. Then ρ has finite kernel and $\rho(\Gamma)$ is ν_{mod}–regular.

Proof In view of the Zariski density of $\rho(\Gamma)$, also $f(Z)$ is Zariski dense in $\partial_{Fu}X$. Consequently, the assumption that Γ acts on Z with finite kernel implies that ρ has finite kernel.

We assume that $\rho(\Gamma)$ is not regular.

Suppose that for some sequence $\gamma_i \to \infty$ in Γ, the sequence $g_i = \rho(\gamma_i) \in G$ is not ν_{mod}–regular. Hence, after extraction, since Δ has only finitely many faces, the sequence (g_i) is τ_{mod}–*pure* for some face τ_{mod} of σ_{mod} not containing ν_{mod}. Therefore, according to Proposition 9.5, after further extraction, there exists a pair of simplices $\tau_+ \in \mathrm{Flag}_{\tau_{mod}}$ and $\tau_- \in \mathrm{Flag}_{\iota\tau_{mod}}$ such that the sequence (g_i) converges on the Zariski open and dense subset $C_{Fu}(\tau_-) \subset \partial_{Fu}X$ to a surjective algebraic map $\phi\colon C_{Fu}(\tau_-) \to \mathrm{st}_{Fu}(\tau_+)$. At the same time, by the convergence property on Z, after extraction, (g_i) converges to a constant map $\equiv \nu_+$ on $A := f(Z) - \{\nu_-\}$, for some (exceptional) point $\nu_- \in f(Z)$. Therefore, regarding the g_i as maps $\mathrm{Flag}_{\sigma_{mod}} \to \mathrm{Flag}_{\sigma_{mod}}$, for every point $\sigma \in \pi_{\nu_{mod}}^{-1}(A)$ the accumulation set of $(g_i(\sigma))$ is contained in $\pi_{\nu_{mod}}^{-1}(\nu_+)$. Hence, the image of the intersection

$$\tilde{A} := \pi_{\nu_{mod}}^{-1}(A) \cap C_{Fu}(\tau_-)$$

under the map ϕ is contained in $\pi_{\nu_{mod}}^{-1}(\nu_+)$. Since $\tilde{A}$ is Zariski dense and $\phi\colon C_{Fu}(\tau_-) \to \mathrm{st}_{Fu}(\tau_+)$ is surjective, the image $\phi(\tilde{A})$ is Zariski dense in $\mathrm{st}_{Fu}(\tau_+)$. It follows that $\mathrm{st}_{Fu}(\tau_+) \subset \mathrm{st}_{Fu}(\nu_+)$. However, as $\nu_{mod} \not\subset \tau_{mod}$, this contradicts the property that

$$\mathrm{st}(\tau_+) \not\subset \mathrm{st}(\nu_+). \qquad \square$$

9.2 Dynamics on the Finsler compactification

We now consider the action

$$G \curvearrowright \overline{X}^{Fins}$$

on the Finsler compactification. Our goal is to construct in $\bar{X}^{\text{Fins}}$ domains of proper discontinuity for actions of discrete subgroups $\Gamma < G$. We refer the reader to Section 11.4 for an explicit example illustrating our general construction.

9.2.1 From Finsler to Furstenberg dynamical relations We show first that dynamical relations in the Finsler boundary (with respect to the action on the entire compactification) imply intrinsic dynamical relations in the Furstenberg boundary:

Lemma 9.7 *Let $g_n \to \infty$ be a sequence in G. Suppose there is a dynamical relation*

$$\xi \overset{(g_n)}{\sim} \xi'$$

in $\bar{X}^{\text{Fins}}$ between boundary points $\xi \in X_\nu$ with $\nu \in \text{Flag}_{\nu_{\text{mod}}}$ and $\xi' \in X_{\nu'}$ with $\nu' \in \text{Flag}_{\nu'_{\text{mod}}}$. Then, after extraction, there exist $\nu_- \in \text{Flag}_{\iota\nu_{\text{mod}}}$, $\nu'_- \in \text{Flag}_{\iota\nu'_{\text{mod}}}$ and open continuous maps

$$C_{\text{Fu}}(\nu_-) \overset{\phi}{\to} \text{st}_{\text{Fu}}(\nu) \quad \text{and} \quad C_{\text{Fu}}(\nu'_-) \overset{\phi'}{\to} \text{st}_{\text{Fu}}(\nu')$$

such that for every $\tilde{\sigma} \in C_{\text{Fu}}(\nu_-) \cap C_{\text{Fu}}(\nu'_-)$ there is the dynamical relation in $\partial_{\text{Fu}} X$:

(9.8) $$\phi\tilde{\sigma} \overset{(g_n)}{\sim} \phi'\tilde{\sigma}.$$

Remark 9.9 (1) $C_{\text{Fu}}(\nu_-)$ and $C_{\text{Fu}}(\nu'_-)$ are open dense in $\partial_{\text{Fu}} X$, and hence also their intersection.

(2) The dynamical relation (9.8) is meant to hold intrinsically *inside* $\partial_{\text{Fu}} X$, ie there exists a sequence (σ_n) in $\partial_{\text{Fu}} X$, not just in $\bar{X}^{\text{Fins}}$, such that $\sigma_n \to \phi\tilde{\sigma}$ and $g_n\sigma_n \to \phi'\tilde{\sigma}$.

Proof By assumption, and since X is dense in $\bar{X}^{\text{Fins}}$, there exists a sequence (x_n) in X such that

$$x_n \to \xi \quad \text{and} \quad g_n x_n \to \xi'.$$

We fix a basepoint $o \in X$ and write

$$x_n = a_n o \quad \text{and} \quad g_n x_n = b_n o$$

such that $g_n = b_n a_n^{-1}$. Then the sequences (a_n) and (b_n) in G are ν_{mod}–pure and ν'_{mod}–pure, respectively, and we have flag convergence

$$a_n \to \nu \quad \text{and} \quad b_n \to \nu';$$

see Propositions 5.42 and 5.43. After extraction, we obtain that also the inverse sequences flag converge:

$$a_n^{-1} \to v_- \quad \text{and} \quad b_n^{-1} \to v_-'$$

with $v_- \in \mathrm{Flag}_{\iota v_{\mathrm{mod}}}$ and $v_-' \in \mathrm{Flag}_{\iota v_{\mathrm{mod}}'}$; see Lemma 4.4. The sequences (a_n) and (b_n) are then contracting on the appropriate flag manifolds:

$$a_n|_{C(v_-)} \to v \quad \text{and} \quad b_n|_{C(v_-')} \to v'$$

uniformly on compacts.

Due to pureness, we get more precise information about the accumulation dynamics of these sequences on $\partial_{\mathrm{Fu}} X$. Proposition 9.5 yields that

$$a_n|_{C_{\mathrm{Fu}}(v_-)} \to \phi \quad \text{and} \quad b_n|_{C_{\mathrm{Fu}}(v_-')} \to \phi'$$

uniformly on compacts with open continuous limit maps

$$\phi \colon C_{\mathrm{Fu}}(v_-) \to \mathrm{st}_{\mathrm{Fu}}(v) \quad \text{and} \quad \phi' \colon C_{\mathrm{Fu}}(v_-') \to \mathrm{st}_{\mathrm{Fu}}(v').$$

Then for $\tilde{\sigma} \in C_{\mathrm{Fu}}(v_-) \cap C_{\mathrm{Fu}}(v_-')$, it holds that

$$a_n\tilde{\sigma} \to \phi\tilde{\sigma} \quad \text{and} \quad g_n a_n \tilde{\sigma} = b_n \tilde{\sigma} \to \phi'\tilde{\sigma},$$

ie we obtain the dynamical relation

$$\phi\tilde{\sigma} \overset{(g_n)}{\sim} \phi'\tilde{\sigma}$$

inside $\partial_{\mathrm{Fu}} X$.   □

Remark 9.10 (extrinsic versus intrinsic dynamical relations in $\partial_{\mathrm{Fu}} X$)   In the special case $v_{\mathrm{mod}} = v_{\mathrm{mod}}' = \sigma_{\mathrm{mod}}$ this result says that all *extrinsic* dynamical relations in $\partial_{\mathrm{Fu}} X$, as a subset of $\overline{X}^{\mathrm{Fins}}$, are already *intrinsic*. More precisely, if $\sigma, \sigma' \in \partial_{\mathrm{Fu}} X$ and (x_n) is a sequence in X such that $x_n \to \sigma$ and $g_n x_n \to \sigma'$ in $\overline{X}^{\mathrm{Fins}}$, then there also exists a sequence (σ_n) in $\partial_{\mathrm{Fu}} X$ such that $\sigma_n \to \sigma$ and $g_n \sigma_n \to \sigma'$.

We deduce the following consequence from the technical statement in the last lemma:

Corollary 9.11 *After extraction, the sequence (g_n) satisfies: if $O \subseteq \mathrm{st}_{\mathrm{Fu}}(v)$ and $O' \subseteq \mathrm{st}_{\mathrm{Fu}}(v')$ are dense open subsets, then there exist $\sigma \in O$ and $\sigma' \in O'$ which are intrinsically dynamically related in $\partial_{\mathrm{Fu}} X$ with respect to (g_n).*

Proof Since ϕ is open and continuous, the subset $\phi^{-1}(O)$ is dense open in $C_{\mathrm{Fu}}(\nu_-)$, and hence also in $\partial_{\mathrm{Fu}}X$. Similarly, $\phi'^{-1}(O')$ is dense open in $\partial_{\mathrm{Fu}}X$. Consequently, their intersection is nonempty and contains some $\tilde{\sigma}$. We put $\sigma = \phi\tilde{\sigma}$ and $\sigma' = \phi'\tilde{\sigma}$, and use the lemma. $\qquad\square$

9.2.2 Dynamical relations We can now extend the combinatorial inequality for intrinsic dynamical relations in the Furstenberg boundary (Proposition 9.2) to the Finsler boundary:

Proposition 9.12 (dynamical relation inequality in $\partial_\infty^{\mathrm{Fins}}X$) *Let* (g_n) *be a* $\tau_{\mathrm{mod}}-$*contracting sequence in* G *with* (9.1). *Suppose that there is a dynamical relation*

$$\xi \overset{(g_n)}{\sim} \xi'$$

in $\bar{X}^{\mathrm{Fins}}$ *between boundary points* $\xi \in X_\nu$ *with* $\nu \in \mathrm{Flag}_{\nu_{\mathrm{mod}}}$ *and* $\xi' \in X_{\nu'}$ *with* $\nu' \in \mathrm{Flag}_{\nu'_{\mathrm{mod}}}$. *Then*

(9.13) $$\mathrm{pos}(\sigma', \tau_+) \prec \text{c-pos}(\sigma, \tau_-)$$

for all $\sigma \in \mathrm{st}_{\mathrm{Fu}}(\nu)$ *and* $\sigma' \in \mathrm{st}_{\mathrm{Fu}}(\nu')$.

Proof For any two simplices $\tau, \nu \subset \partial_\infty X$, the relative position $\mathrm{pos}(\cdot, \tau)$ has a unique maximal value on $\mathrm{st}_{\mathrm{Fu}}(\nu)$, ie all other values are smaller, and it attains this maximal value on an dense open subset; see Proposition 8.12. Let $O \subset \mathrm{st}_{\mathrm{Fu}}(\nu)$ denote the open dense subset where $\mathrm{pos}(\cdot, \tau_-)$ is maximal and $O' \subset \mathrm{st}_{\mathrm{Fu}}(\nu')$ the subset where $\mathrm{pos}(\cdot, \tau_+)$ is maximal. By Corollary 9.11, after extraction, there exist $\sigma \in O$ and $\sigma' \in O'$ which are intrinsically dynamically related in $\partial_{\mathrm{Fu}}X$:

$$\sigma \overset{(g_n)}{\sim} \sigma'.$$

Applying Proposition 9.2, we obtain that these σ and σ' satisfy inequality (9.13). The inequality for arbitrary $\sigma \in \mathrm{st}_{\mathrm{Fu}}(\tau)$ and $\sigma' \in \mathrm{st}_{\mathrm{Fu}}(\tau')$ follows.[20] $\qquad\square$

There are no dynamical relations in $\bar{X}^{\mathrm{Fins}}$ between points in X. This leaves the case of dynamical relations between points in X and points in $\partial_\infty^{\mathrm{Fins}}X$, which is easy to deal with:

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[20]Here we use the fact that taking complementary position reverses the folding order; see (8.6).

Lemma 9.14 *Suppose that the sequence (g_n) in G is τ_{mod}–contracting with (9.1). If there is a dynamical relation*

$$x \stackrel{(g_n)}{\sim} \xi'$$

in $\bar{X}^{\mathrm{Fins}}$ between a point $x \in X$ and a boundary point $\xi' \in \partial_\infty^{\mathrm{Fins}} X$, then $\xi' \in \bar{X}_{\tau_+}$.

Proof Let (x_n) be any bounded sequence in X. From $g_n \to \tau_+$ it follows that also $g_n x_n \to \tau_+$. Hence $(g_n x_n)$ accumulates in $\bar{X}^{\mathrm{Fins}}$ at $\bar{X}_{\tau_+}$; see Proposition 5.43. □

Similarly, a dynamical relation $\xi \stackrel{(g_n)}{\sim} x'$ between a boundary point $\xi \in \partial_\infty^{\mathrm{Fins}} X$ and a point $x' \in X$ implies that $\xi \in \bar{X}_{\tau_-}$, as follows by applying the lemma to the inverse sequence (g_n^{-1}).

9.2.3 Accumulation dynamics The dynamical relation inequality for the action of contracting sequences obtained in the previous section can be rephrased in terms of accumulation at pairs of "complementary" thickenings at the attractive and repulsive fixed points in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$; compare the discussion in [34, Sections 5.2, 6.1] for dynamics on flag manifolds.

We refer the reader to Section 8.3 for the definitions of Furstenberg and Finsler thickenings. Lemma 9.14 implies:

Corollary 9.15 *Let (g_n) be a τ_{mod}–contracting sequence in G with (9.1), and let $\varnothing \neq \mathrm{Th} \subsetneq W$ be a $W_{\tau_{\mathrm{mod}}}$–left-invariant thickening. Suppose that there is a dynamical relation*

$$\bar{x} \stackrel{(g_n)}{\sim} \bar{x}'$$

in $\bar{X}^{\mathrm{Fins}}$ between points $\bar{x}, \bar{x}' \in \bar{X}^{\mathrm{Fins}}$. Then $\bar{x} \in (\mathrm{Th}^c)^{\mathrm{Fins}}(\tau_-)$ or $\bar{x}' \in \mathrm{Th}^{\mathrm{Fins}}(\tau_+)$.

Proof By our assumption on Th, we have that $\mathrm{st}_{\mathrm{Fu}}(\tau_-) \subset \mathrm{Th}^c_{\mathrm{Fu}}(\tau_-)$ and $\mathrm{st}_{\mathrm{Fu}}(\tau_+) \subset \mathrm{Th}_{\mathrm{Fu}}(\tau_+)$, equivalently, that $\bar{X}_{\tau_-} \subset (\mathrm{Th}^c)^{\mathrm{Fins}}(\tau_-)$ and $\bar{X}_{\tau_+} \subset \mathrm{Th}^{\mathrm{Fins}}(\tau_+)$. Lemma 9.14 therefore implies the assertion if one of the points $\bar{x}, \bar{x}'$ lies in X.

We may therefore assume that $\bar{x}, \bar{x}' \in \partial_\infty^{\mathrm{Fins}} X$ and apply Proposition 9.12. Let $\bar{x} \in X_\nu$ with $\nu \in \mathrm{Flag}_{\nu_{\mathrm{mod}}}$ and $\bar{x}' \in X_{\nu'}$ with $\nu' \in \mathrm{Flag}_{\nu'_{\mathrm{mod}}}$. Either $\mathrm{pos}(\cdot, \tau_+)$ takes values in Th on $\mathrm{st}_{\mathrm{Fu}}(\nu')$, or the maximal value of $\mathrm{pos}(\cdot, \tau_+)$ on $\mathrm{st}_{\mathrm{Fu}}(\nu')$ is contained in $W - \mathrm{Th} = w_0 \mathrm{Th}^c$. In the latter case, the inequality (9.13) implies that the values of $\mathrm{pos}(\cdot, \tau_-)$ on $\mathrm{st}_{\mathrm{Fu}}(\nu)$ are contained in Th^c.

If $\mathrm{pos}(\,\cdot\,,\tau_+)|_{\mathrm{st}_{\mathrm{Fu}}(\nu')}$ takes values in Th, then $\mathrm{st}_{\mathrm{Fu}}(\nu') \subset \mathrm{Th}_{\mathrm{Fu}}(\tau_+)$, equivalently, $\bar{X}_{\nu'} \subset \mathrm{Th}^{\mathrm{Fins}}(\tau_+)$, and hence $\bar{x}' \in \mathrm{Th}^{\mathrm{Fins}}(\tau_+)$. Similarly, if $\mathrm{pos}(\,\cdot\,,\tau_-)|_{\mathrm{st}_{\mathrm{Fu}}(\nu)}$ takes values in Th^c, then $\bar{x} \in \bar{X}_\nu \subset (\mathrm{Th}^c)^{\mathrm{Fins}}(\tau_-)$. $\qquad\square$

In the language of accumulation dynamics introduced in [34, Section 5.2], this means that the sequence (g_n) is $((\mathrm{Th}^c)^{\mathrm{Fins}}(\tau_-), \mathrm{Th}^{\mathrm{Fins}}(\tau_+))$–accumulating on $\bar{X}^{\mathrm{Fins}}$.

9.2.4 Domains of proper discontinuity We now deduce our main results on proper discontinuity; compare [34, Sections 6.2–6.4].

For a discrete subgroup $\Gamma < G$, we define the *forward/backward τ_{mod}–limit set*

$$\Lambda^{\pm}_{\tau_{\mathrm{mod}}} \subset \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$$

as the set of all simplices $\tau_{\pm}$ as in (4.2) for all τ_{mod}–contracting sequences

$$(g_n =)\gamma_n \to \infty$$

in Γ; see [34, Definition 6.9]. The limit sets $\Lambda^{\pm}_{\tau_{\mathrm{mod}}}$ are Γ–invariant and compact. If τ_{mod} is ι–invariant, then $\Lambda^{+}_{\tau_{\mathrm{mod}}} = \Lambda^{-}_{\tau_{\mathrm{mod}}} =: \Lambda_{\tau_{\mathrm{mod}}}$.

Consider first the case of τ_{mod}–regular, equivalently, τ_{mod}–convergence subgroups. Here we obtain domains by removing suitable thickenings of the τ_{mod}–limit set:

Theorem 9.16 (domains of proper discontinuity for τ_{mod}–convergence subgroups) *Let $\Gamma < G$ be a τ_{mod}–convergence subgroup, and let $\varnothing \neq \mathrm{Th} \subsetneq W$ be a $W_{\tau_{\mathrm{mod}}}$–left-invariant thickening. Then the action*

$$\textbf{(9.17)} \qquad \Gamma \curvearrowright \bar{X}^{\mathrm{Fins}} - \left((\mathrm{Th}^c)^{\mathrm{Fins}}(\Lambda^{-}_{\tau_{\mathrm{mod}}}) \cup \mathrm{Th}^{\mathrm{Fins}}(\Lambda^{+}_{\tau_{\mathrm{mod}}})\right)$$

is properly discontinuous. In particular, if τ_{mod} is ι–invariant and Th is fat, then the action

$$\Gamma \curvearrowright \bar{X}^{\mathrm{Fins}} - \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}})$$

is properly discontinuous.

Proof Suppose that there is a dynamical relation

$$\textbf{(9.18)} \qquad \bar{x} \overset{(\gamma_n)}{\sim} \bar{x}'$$

in $\bar{X}^{\mathrm{Fins}}$ with respect to a sequence $\gamma_n \to \infty$ in Γ. After extraction we may assume that (γ_n) is τ_{mod}–contracting with

$$\gamma_n|_{C(\tau_-)} \to \tau_+$$

uniformly on compacts, where $\tau_\pm \in \Lambda_{\pm\tau_{\mathrm{mod}}}$. Corollary 9.15 implies that

$$\bar{x} \in (\mathrm{Th}^c)^{\mathrm{Fins}}(\tau_-) \subset (\mathrm{Th}^c)^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^-) \quad \text{or} \quad \bar{x}' \in \mathrm{Th}^{\mathrm{Fins}}(\tau_+) \subset \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^+).$$

This yields the first assertion; the second follows because $\mathrm{Th}^c \subseteq \mathrm{Th}$ due to fatness.[21] □

Note that the thickenings of limit sets $\mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^\pm(\Gamma))$ are Γ–invariant and compact. This scheme of constructing domains of proper discontinuity applies equally well to arbitrary discrete subgroups $\Gamma < G$; compare the discussion in [34, Section 6.4]. One then has to take into account the τ_{mod}–limit sets for all face types τ_{mod}. There are several ways to proceed. The most immediate family of possibilities is the following.

Theorem 9.19 (domains of proper discontinuity for discrete subgroups I) *Let $\Gamma < G$ be a discrete subgroup, and let $\varnothing \neq \mathrm{Th}_{\tau_{\mathrm{mod}}} \subsetneq W$ be $W_{\tau_{\mathrm{mod}}}$–left-invariant thickenings for all face types $\tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$. Then the action*

$$(9.20) \qquad \Gamma \curvearrowright \bar{X}^{\mathrm{Fins}} - \bigcup_{\tau_{\mathrm{mod}}} \left((\mathrm{Th}_{\tau_{\mathrm{mod}}}^c)^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^-) \cup \mathrm{Th}_{\tau_{\mathrm{mod}}}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^+) \right)$$

is properly discontinuous.

Proof The proof is the same as for the previous theorem: Suppose that there is a dynamical relation (9.18) in $\bar{X}^{\mathrm{Fins}}$. Then (γ_n) contains for some face type τ_{mod} a τ_{mod}–contracting subsequence and it follows as before that $\bar{x} \in (\mathrm{Th}_{\tau_{\mathrm{mod}}}^c)^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^-)$ or $\bar{x}' \in \mathrm{Th}_{\tau_{\mathrm{mod}}}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^+)$. □

In general, these domains of proper discontinuity can be further enlarged by only removing the thickenings of the limit simplices arising from *pure* sequences in the group: Define the *pure forward/backward τ_{mod}–limit set*

$$\Lambda_{\tau_{\mathrm{mod}}}^{\mathrm{pure},\pm} \subseteq \Lambda_{\tau_{\mathrm{mod}}}^\pm$$

as the closure of the set of all simplices $\tau_\pm$ as in (4.2) for all τ_{mod}–pure τ_{mod}–contracting sequences (γ_n) in Γ. As above, we conclude:

Theorem 9.21 (domains of proper discontinuity for discrete subgroups II) *Let $\Gamma < G$ be a discrete subgroup, and let $\varnothing \neq \mathrm{Th}_{\tau_{\mathrm{mod}}} \subsetneq W$ be $W_{\tau_{\mathrm{mod}}}$–left-invariant thickenings for all face types $\tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$. Then the action*

$$(9.22) \qquad \Gamma \curvearrowright \bar{X}^{\mathrm{Fins}} - \bigcup_{\tau_{\mathrm{mod}}} \left((\mathrm{Th}_{\tau_{\mathrm{mod}}}^c)^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^{\mathrm{pure},-}) \cup \mathrm{Th}_{\tau_{\mathrm{mod}}}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^{\mathrm{pure},+}) \right)$$

is properly discontinuous.

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[21] Here we use that if $\mathrm{Th}_1 \subseteq \mathrm{Th}_2$, then $\mathrm{Th}_1^{\mathrm{Fins}} \subseteq \mathrm{Th}_2^{\mathrm{Fins}}$.

Proof Same argument as before, taking into account that every sequence $\gamma_n \to \infty$ in Γ contains a subsequence which for some face type τ_{mod} is τ_{mod}–contracting and τ_{mod}–pure. $\qquad\square$

Since the domain in (9.22) is larger than the domain in (9.20), one can in general not expect the Γ–action (9.20) to be cocompact.

If Γ is τ_{mod}–regular, then it contains ν_{mod}–pure sequences only for the face types $\nu_{\mathrm{mod}} \supseteq \tau_{\mathrm{mod}}$; hence only these limit sets $\Lambda^{\pm}_{\nu_{\mathrm{mod}}}$ can be nonempty. Since $W_{\nu_{\mathrm{mod}}} \leq W_{\tau_{\mathrm{mod}}}$, we may choose $\mathrm{Th}_{\nu_{\mathrm{mod}}} = \mathrm{Th}_{\tau_{\mathrm{mod}}}$ for these face types, and then the domain in (9.20) coincides with the domain in Theorem 9.16.

9.2.5 Nonemptiness of domains of proper discontinuity at infinity If we assume in addition to the hypotheses of Theorem 9.16 that Γ is τ_{mod}–antipodal, then it is easy to see that, in higher rank, the domains (9.17) strictly enlarge X:

Proposition 9.23 *Let* $\Gamma < G$ *be a* τ_{mod}*–antipodal* τ_{mod}*–convergence subgroup and let* $\mathrm{Th} \subset W$ *be a* $W_{\tau_{\mathrm{mod}}}$*–left-invariant slim thickening. If* $\mathrm{rank}(X) \geq 2$, *then* $\mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}) \subsetneq \partial^{\mathrm{Fins}}_{\infty} X$.

Proof If $\mathrm{Th} = \varnothing$, there is nothing to show. Suppose therefore that $\mathrm{Th} \neq \varnothing$.

We consider the subcomplex C of a_{mod} corresponding to the thickening $\mathrm{Th}_{\mathrm{Fu}}(\tau_{\mathrm{mod}})$ of the model chamber. Since $\mathrm{rank}(X) \geq 2$, we have that a_{mod} is connected. By slimness, C does not contain all chambers of a_{mod}. Therefore there exists a panel $\bar{\pi} \subset C$ such that exactly one of two chambers in a_{mod} adjacent to it belongs to C. In terms of chambers, this means that

$$\mathrm{st}_{\mathrm{Fu}}(\bar{\pi}) \cap \mathrm{Th}_{\mathrm{Fu}}(\tau_{\mathrm{mod}}) \neq \varnothing \quad \text{and} \quad \mathrm{st}_{\mathrm{Fu}}(\bar{\pi}) \not\subset \mathrm{Th}_{\mathrm{Fu}}(\tau_{\mathrm{mod}}).$$

Let $\tau_0 \in \Lambda_{\tau_{\mathrm{mod}}}$, and let $\pi \subset \partial_{\infty} X$ be a panel with $\mathrm{pos}(\pi, \tau_0) = \bar{\pi}$. Then $\mathrm{st}_{\mathrm{Fu}}(\pi)$ intersects $\mathrm{Th}_{\mathrm{Fu}}(\tau_0)$, but is not contained in it. It follows that $\mathrm{st}_{\mathrm{Fu}}(\pi) \not\subset \mathrm{Th}_{\mathrm{Fu}}(\tau)$ for all $\tau \in \Lambda_{\tau_{\mathrm{mod}}}$, because due to our assumptions of antipodality and slimness, the thickenings $\mathrm{Th}_{\mathrm{Fu}}(\tau)$ for $\tau \in \Lambda_{\tau_{\mathrm{mod}}}$ are pairwise disjoint. Consequently, $X_{\pi} \not\subset \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}})$. $\qquad\square$

Remark 9.24 Nonemptiness of domains of proper discontinuity for the Γ–actions on flag manifolds is much harder to prove and requires additional assumptions. For the case of actions on the Furstenberg boundary, see [34, Section 8].

10   General cocompactness results

10.1   General discrete topological group actions

The main result of this section is a cocompactness theorem for a certain class of properly discontinuous group actions $\Gamma \curvearrowright \Omega$, where Ω is an open subset of a compact metrizable space Z and the action extends to a topological action

$$\Gamma \curvearrowright Z.$$

In order to prove cocompactness, we need to impose certain assumptions on both Γ and the action. We assume that there exists a *model action*

$$\Gamma \curvearrowright Y$$

on a contractible simplicial complex which is simplicial, properly discontinuous and cocompact. We further assume that Y admits a Γ–equivariant contractible metrizable compactification $\overline{Y} \supset Y$. The extended action $\Gamma \curvearrowright \overline{Y}$ serves as a model for $\Gamma \curvearrowright Z$. Examples of such model actions abound in geometric group theory. For instance, if Γ is Gromov-hyperbolic, we can take for Y a suitable Rips complex of Γ and for $\overline{Y}$ the Gromov compactification of Y. Other examples are given by isometric properly discontinuous cocompact actions $\Gamma \curvearrowright Y$ on piecewise-Riemannian CAT(0) complexes Y, where $\overline{Y}$ is the visual compactification of Y.

Our next set of hypotheses relates the model action $\Gamma \curvearrowright Y$ to the action $\Gamma \curvearrowright Z$. We assume that there exists a Γ–equivariant continuous map of triads

$$(\textbf{10.1}) \qquad (\overline{Y}, Y, \underbrace{\overline{Y} - Y}_{=:\Lambda_{\mathrm{mod}}}) \xrightarrow{\tilde{f}} (Z, \Omega, \underbrace{Z - \Omega}_{=:\Lambda})$$

such that the restriction $\tilde{f}|_{\Lambda_{\mathrm{mod}}} \colon \Lambda_{\mathrm{mod}} \to \Lambda$ is a $\mathbb{Z}_2$–Čech cohomology equivalence.

Theorem 10.2 *Under the above assumptions, and if Γ is torsion-free and Ω is path connected, then Ω / Γ is compact.*

Proof   We proceed in four steps.

Step 1   (passing to a model action on a manifold with boundary)   We replace the action $\Gamma \curvearrowright Y$ with an action of Γ on a suitable manifold. Since $\pi_1(Y) = 1$ and the action $\Gamma \curvearrowright Y$ is free, the quotient space $R := Y / \Gamma$ satisfies $\pi_1(R) \cong \Gamma$.

We thicken R to a closed manifold without changing the fundamental group. To do so, we first embed R as a subcomplex into the (suitably triangulated) euclidean space E^{2n+1}, where $n = \dim(R)$. We denote by N the regular neighborhood of R in E^{2n+1}, and let $D = \partial N$.

Lemma 10.3   *D is connected and $\pi_1(D) \to \pi_1(N) \cong \pi_1(R)$ is surjective.*

Proof   Let $N' := N - R$. We claim that the map $D \hookrightarrow N'$ is a homotopy equivalence. The proof is the same as the one for the homotopy equivalence $R \to N$: Each simplex $c \subset N$ is the join $c_1 \star c_2$ of a simplex c_1 disjoint from R (and hence contained in D) and a simplex $c_2 \subset R$ (in the extreme cases, c_1 or c_2 could be empty). Now, use the straight line segments given by these join decompositions to homotope each $c - R$ to $c_1 \subset D$.

Since R has codimension ≥ 2 in N, it does not separate N' and each loop in N is homotopic to a loop in N'. Hence, N' is connected and

$$\pi_1(D) \xrightarrow{\cong} \pi_1(N') \to \pi_1(N)$$

is surjective. □

Lemma 10.4   *There exists a connected closed manifold M which admits a map $h\colon R \to M$ inducing an isomorphism of fundamental groups $\pi_1(R) \to \pi_1(M)$.*

Proof   We start with N (the regular neighborhood of $R \subset E^{2n+1}$) as above. As noted in the proof of the previous lemma, the inclusion $R \to N$ is a homotopy equivalence, and N is a compact manifold with boundary. Consider the closed manifold M obtained by doubling N along its boundary D,

$$M = N_1 \cup_D N_2,$$

where N_1, N_2 are two copies of N. We let $i\colon D \to M$ and $i_k\colon N_k \to M$ denote the inclusion maps. Since M is the double of N, we have the canonical retraction $r\colon M \to N_1$ (whose restriction to N_2 is a homeomorphism given by reflecting at D). Define the map $h = i_1 \circ g$,

$$h\colon R \xrightarrow{g} N_1 \xrightarrow{i_1} M,$$

where g corresponds to the inclusion $R \to N$ and hence is a homotopy equivalence. We claim that h induces an isomorphism h_* of fundamental groups.

The existence of the retraction r implies the injectivity of i_{1*} and hence of h_*.

By Lemma 10.3, D is connected. Hence, the Seifert–van Kampen theorem implies that $\pi_1(M)$ is generated by the two subgroups $i_{k*}(\pi_1(N_k)), k = 1, 2$. Since the homomorphisms

$$\pi_1(D) \to \pi_1(N_k)$$

are surjective (Lemma 10.3), we obtain

$$i_{1*}(\pi_1(N_1)) = i_*(\pi_1(D)) = i_{2*}(\pi_1(N_2)).$$

Hence, both homomorphisms $i_{k*}\colon \pi_1(N_k) \to \pi_1(M)$ are surjective. The surjectivity of h_* follows.   □

We let $m = 2n + 1$ denote the dimension of the manifold M and its universal cover $\widetilde{M}$.

Step 2   We let M be a manifold as in Lemma 10.4. We consider the triads (10.1) and the diagonal Γ–action on their products with the universal cover $\widetilde{M}$. Dividing out the action, we obtain bundles over M and $\widetilde{f}$ induces the map of triads of bundles

$$(10.5)\quad (\underbrace{(\bar{Y} \times \widetilde{M})/\Gamma}_{\bar{E}_{\mathrm{mod}}}, \underbrace{(Y \times \widetilde{M})/\Gamma}_{E_{\mathrm{mod}}}, \underbrace{(\Lambda_{\mathrm{mod}} \times \widetilde{M})/\Gamma}_{L_{\mathrm{mod}}})$$

$$\xrightarrow{F} (\underbrace{(Z \times \widetilde{M})/\Gamma}_{\bar{E}}, \underbrace{(\Omega \times \widetilde{M})/\Gamma}_{E}, \underbrace{(\Lambda \times \widetilde{M})/\Gamma}_{L}).$$

Note that E also fibers over Ω/Γ with fiber $\widetilde{M}$.

The map F of triads of bundles satisfies:

  (i) $F|_{E_{\mathrm{mod}}}\colon E_{\mathrm{mod}} \to E$ is proper.

  (ii) $F|_{L_{\mathrm{mod}}}\colon L_{\mathrm{mod}} \to L$ is a cohomology equivalence of bundles.

Lemma 10.6   *Both spaces $\bar{E}$ and $\bar{E}_{\mathrm{mod}}$ are metrizable.*

Proof   These spaces are fiber bundles with compact metrizable bases and fibers. Therefore, $\bar{E}$ and $\bar{E}_{\mathrm{mod}}$ are both compact and Hausdorff. Hence, they are metrizable, for instance, by Smirnov's metrization theorem, because they are paracompact, Hausdorff and locally metrizable.   □

Our approach to proving Theorem 10.2 is based on the following observation.

In a noncompact connected manifold, the point represents the zero class in H_0^{lf}. Similarly, let $\iota\colon F \to E \xrightarrow{\pi} B$ be a fiber bundle over a noncompact space, where $\iota\colon F \to E_b$

is the homeomorphism of F to the fiber $E_b = \pi^{-1}(b)$. If the base B is path-connected, then the induced map

$$\iota_*\colon H_*^{\mathrm{lf}}(F) \to H_*^{\mathrm{lf}}(E)$$

on locally finite homology is independent of the choice of b. In order to show triviality of this map provided that B is noncompact, note that for each class $[\eta] \in Z_c^i(E)$ and each locally finite class $[\xi] \in H_m^{\mathrm{lf}}(F)$, if b is chosen so that E_b is disjoint from the support of η, then $\langle[\eta], [\xi]\rangle = 0$. Here and in the sequel we use (co)homology with $\mathbb{Z}_2$–coefficients. Hence, $\iota_* = 0$.

The compactness of Ω/Γ therefore follows from showing that the fiber of the bundle

$$\widetilde{M}^m \to E \to \Omega/\Gamma$$

represents a nontrivial class in $H_m^{\mathrm{lf}}(E)$, ie that the locally finite fundamental class $[\widetilde{M}] \in H_m^{\mathrm{lf}}(\widetilde{M})$ has a nonzero image under the inclusion induced map $H_m^{\mathrm{lf}}(\widetilde{M}) \to H_m^{\mathrm{lf}}(E)$.

The proper map $F\colon E_{\mathrm{mod}} \to E$ induces the map $F_*\colon H_m^{\mathrm{lf}}(E_{\mathrm{mod}}) \to H_m^{\mathrm{lf}}(E)$ which carries the class represented by the $\widetilde{M}$–fiber in the model E_{mod} to the corresponding class in E. It therefore suffices to show that

$$(10.7) \qquad \underbrace{H_m^{\mathrm{lf}}(\widetilde{M})}_{\cong \mathbb{Z}_2} \xrightarrow{\iota_*} H_m^{\mathrm{lf}}(E_{\mathrm{mod}}) \xrightarrow{F_*} H_m^{\mathrm{lf}}(E)$$

is a composition of injective maps. We will show injectivity of F_* in step 3 and injectivity of ι_* in step 4 below.

Step 3 (injectivity of F_*) We pass to compactly supported cohomology. We recall that locally finite homology (with field coefficients) is dual to compactly supported cohomology in the same degree via Kronecker duality. We therefore must show that the dual map

$$H_c^m(E) \xrightarrow{F^*} H_c^m(E_{\mathrm{mod}})$$

is surjective.

We now switch the fiber direction and regard E and E_{mod} as bundles over M. We use their compactifications $\bar{E}$ and $\bar{E}_{\mathrm{mod}}$ mentioned earlier which allow us to replace compactly supported cohomology by relative cohomology. Since E is compact and metrizable, while L is compact, we have a natural isomorphism of Alexander–Spanier

cohomology groups (see [49, Lemma 11, page 321]):

$$H_c^m(E) \cong H^m(\bar{E}, L).$$

Similarly, we have a natural isomorphism

$$H_c^m(E_{\text{mod}}) \cong H^m(\bar{E}_{\text{mod}}, L_{\text{mod}}).$$

Thus, the surjectivity of the previous map F^* is equivalent to the surjectivity of the map

$$H^m(\bar{E}, L) \xrightarrow{F_{\text{rel}}^*} H^m(\bar{E}_{\text{mod}}, L_{\text{mod}})$$

induced by the map of pairs

(10.8) $(\bar{E}_{\text{mod}}, L_{\text{mod}}) \xrightarrow{F} (\bar{E}, L).$

To verify the surjectivity of F_{rel}^*, we use the long exact cohomology sequence of F:

$$\cdots \; H^{m-1}(\bar{E}) \longrightarrow H^{m-1}(L) \longrightarrow H^m(\bar{E}, L) \longrightarrow H^m(\bar{E}) \longrightarrow H^m(L) \; \cdots$$

$$\cdots H^{m-1}(\bar{E}_{\text{mod}}) \longrightarrow H^{m-1}(L_{\text{mod}}) \longrightarrow H^m(\bar{E}_{\text{mod}}, L_{\text{mod}}) \xrightarrow{j} H^m(\bar{E}_{\text{mod}}) \longrightarrow H^m(L_{\text{mod}}) \cdots$$

A diagram chase (as in the proof of the five lemma) shows that the surjectivity of F_{rel}^* follows from the surjectivity of F_{abs}^*. Indeed, one first checks that $\ker j \subset \operatorname{im} F_{\text{rel}}^*$, and uses this to verify the inclusion

$$j^{-1}(\operatorname{im} F_{\text{abs}}^*) \subset \operatorname{im}(F_{\text{rel}}^*).$$

To see that F_{abs}^* is surjective, we consider the map of bundles:

The fibration $\pi_{\bar{E}_{\text{mod}}}$ is a homotopy equivalence because its fibers $\bar{Y}$ are contractible. Let

$$s \colon M \to \bar{E}_{\text{mod}}$$

denote a section. It follows that $s \circ \pi_{\bar{E}}$ is a left homotopy inverse for F, that is, $s \circ \pi_{\bar{E}} \circ F \simeq \operatorname{id}_{\bar{E}_{\text{mod}}}$. Thus, the induced map on cohomology F_{abs}^* is surjective.

Step 4 (injectivity of ι_*)   We consider the fiber bundle

$$\widetilde{M} \to E_{\mathrm{mod}} \to R.$$

The map $h\colon R \to M$ in Lemma 10.4 yields a section of this bundle. Since the base R of the bundle is a finite CW complex and its fiber $\widetilde{M}$ is a connected m–manifold, Lemma 2.4 implies that the induced map

$$H_m^{\mathrm{lf}}(\widetilde{M}) \xrightarrow{\iota_*} H_m^{\mathrm{lf}}(E_{\mathrm{mod}}),$$

is injective.

This concludes the proof of Theorem 10.2.   □

10.2   Haïssinsky–Tukia conjecture for convergence actions

We now apply our general cocompactness result (Theorem 10.2) from the previous section to the theory of abstract convergence groups.[22] The following natural question is due to P Haïssinsky.[23] An equivalent question was asked by P Tukia in [51, page 77]. We owe the observation of the equivalence of the questions to V Gerasimov.

Let $\Gamma \curvearrowright \Sigma$ be a convergence group action of a hyperbolic group on a metrizable compact space, and suppose that $\Lambda \subset \Sigma$ is an invariant compact subset which is equivariantly homeomorphic to $\partial_\infty \Gamma$. Then the action $\Gamma \curvearrowright \Omega = \Sigma - \Lambda$ is properly discontinuous.

Question 10.9   Is the action $\Gamma \curvearrowright \Omega$ always cocompact?

Remark 10.10   This is true for actions which are *expanding* at the limit set Λ [34].

The main result of this section is the following theorem which provides strong evidence for a positive answer to Question 10.9 in the case of convergence group actions with path-connected discontinuity domains.

Theorem 10.11   *Let $\Gamma \curvearrowright \Sigma$ be a convergence group action of a virtually torsion-free hyperbolic group on a metrizable compact space Σ, and suppose that $\Lambda \subset \Sigma$ is an invariant compact subset which is equivariantly homeomorphic to $\partial_\infty \Gamma$. Then the action*

$$\Gamma \curvearrowright \Sigma - \Lambda$$

is cocompact provided that $\Sigma - \Lambda$ has finitely many path-connected components.

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[22] See eg [13] or [50] for background on convergence groups.

[23] This was posed at the problem session at the Joint Seminar CNRS/JSPS *Aspects of representation theory in low-dimensional topology and* 3*–dimensional invariants*, Carry le Rouet, November 5–9, 2012.

Proof We proceed in three steps.

Step 1 After passing to a finite-index subgroup of Γ preserving each connected component of $\Omega' := \Sigma - \Lambda$, it suffices to consider the case when Ω' is path connected (and nonempty). It also suffices to consider the case when Γ is torsion-free. We let Y be a contractible locally compact simplicial complex on which Γ acts properly discontinuously and cocompactly, eg a suitable Rips complex of Γ. The Gromov compactification $\overline{Y}$ of Y is contractible and metrizable [6], and $\overline{Y} - Y \cong \partial_\infty \Gamma$ equivariantly.

Step 2 (construction of a map of triads) Pick a point $x \in \Omega'$ and define the orbit map

$$f \colon \Gamma \to \Omega', \quad \gamma \mapsto \gamma x.$$

This map is injective since Γ is torsion-free and, hence, acts freely on Ω'. Let $f_\infty \colon \partial_\infty \Gamma \to \Lambda$ be an (the) equivariant homeomorphism. We further let $\overline{\Gamma} = \Gamma \cup \partial_\infty \Gamma$ denote the Gromov compactification of Γ. We define the map

$$\overline{f} \colon \overline{\Gamma} \to \Sigma,$$

whose restriction to Γ is f and to $\partial_\infty \Gamma$ is f_∞.

Lemma 10.12 *The map $\overline{f}$ is an equivariant homeomorphism onto $\Gamma x \cup \Lambda$.*

Proof We first note that the natural action $\Gamma \curvearrowright \overline{\Gamma}$ is a convergence action.

Suppose that (γ_n) is a sequence in Γ converging to $\xi \in \partial_\infty \Gamma$; let $\lambda = f_\infty(\xi)$. We claim that

$$\lim_{n \to \infty} f(\gamma_n) = \lambda.$$

Case 1 (Γ is nonelementary) Without loss of generality (in view of compactness of Σ and the convergence property of the action $\Gamma \curvearrowright \Sigma$), there exists $\lambda_- \in \Lambda$ such that the sequence $\gamma_n|_{\Sigma - \{\lambda_-\}}$ converges to some $\lambda_+ \in \Lambda$ uniformly on compacts. Since f_∞ is a homeomorphism, γ_n converges to $f_\infty^{-1}(\lambda_+)$ uniformly on compacts in $\partial_\infty \Gamma - f_\infty^{-1}(\lambda_-)$. The assumption that Γ is nonelementary implies that $\partial_\infty \Gamma - f_\infty^{-1}(\lambda_-)$ consists of more than one point. Therefore, in view of the convergence property for the action $\Gamma \curvearrowright \overline{\Gamma}$, it follows that γ_n converges to $f_\infty^{-1}(\lambda_+)$ on Γ (here we again pass to a subsequence if necessary). Hence, $\xi = f_\infty^{-1}(\lambda_+)$, $\lambda_+ = \lambda$ and the continuity of $\overline{f}$ follows; see Lemma 2.1.

Case 2 (Γ is elementary, ie $\Gamma \cong \mathbb{Z}$) Then Γ is generated by a single *loxodromic* homeomorphism $\gamma \colon \Sigma \to \Sigma$; ie $\Lambda = \{\lambda_+, \lambda_-\}$. Tukia proved [50, Lemma 2D] that

the sequence (γ^n) converges uniformly on compacts in $\Sigma - \{\lambda_-\}$ to λ_+, while the sequence (γ^{-n}) converges uniformly on compacts in $\Sigma - \{\lambda_+\}$ to λ_-. This implies continuity of the map $\bar{f}$. $\qquad\square$

We now amalgamate the spaces $\bar{Y}$ and Σ using the homeomorphism

$$\bar{f} \colon \bar{\Gamma} = \Gamma \cup \partial_\infty \Gamma \to \overline{\Gamma x} = \Gamma x \cup \Lambda,$$

where we identify Γ with a subset of the vertex set of the complex Y. We denote by Z the result of the amalgamation. This space is metrizable by Urysohn's metrization theorem, since it is Hausdorff, compact and first-countable; compare also Proposition 11.5, which provides a different proof.

Since $\bar{f}$ is Γ–equivariant, the topological action of Γ on $\bar{Y} \sqcup \Sigma$ descends to a topological action $\Gamma \curvearrowright Z$. This action is properly discontinuous on $\Omega := Y \cup \Omega' \subset Z$ as for each compact $C \subset \Omega$, its intersections with Y and Ω' are both compact and the actions $\Gamma \curvearrowright Y$ and $\Gamma \curvearrowright \Omega'$ are properly discontinuous. Lastly, we note that, in view of the connectivity of Y, since Ω' is path connected, so is Ω. Since the embedding $\Omega' \to \Omega$ is proper, Ω / Γ is compact if and only if Ω' / Γ is compact. We let $\tilde{f} \colon \bar{Y} \to Z$ be the inclusion map.

Step 3 According to Theorem 10.2, Ω / Γ is compact. Therefore, Ω' / Γ is compact as well. $\qquad\square$

Remark 10.13 It is not hard to check that $\Gamma \curvearrowright Z$ is a convergence action; however, this is not needed for our argument.

11 Cocompactness

We return to the discussion of discrete subgroups of Lie groups and their actions on Finsler compactifications. In Section 9, we constructed domains of proper discontinuity. We will now prove the cocompactness of these actions for certain classes of discrete subgroups.

Let τ_{mod} be ι–invariant. In Sections 4.2 and 4.5 we defined τ_{mod}–regular and τ_{mod}–antipodal discrete subgroups $\Gamma < G$ and the τ_{mod}–limit set $\Lambda_{\tau_{\mathrm{mod}}} \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$. In [30] we defined the following class of τ_{mod}–antipodal τ_{mod}–regular subgroups:

Definition 11.1 (asymptotically embedded) We say that a discrete subgroup $\Gamma < G$ is τ_{mod}–*asymptotically embedded* if it is τ_{mod}–regular, τ_{mod}–antipodal and word hyperbolic (as an abstract group), and there is a Γ–equivariant homeomorphism

$$\alpha \colon \partial_\infty \Gamma \xrightarrow{\cong} \Lambda_{\tau_{\mathrm{mod}}} \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

from its Gromov boundary onto its τ_{mod}–limit set.

We proved in [30] that a subgroup $\Gamma < G$ is τ_{mod}–asymptotically embedded if and only if it is τ_{mod}–Anosov.[24]

Suppose now that $\Gamma < G$ is τ_{mod}–regular and τ_{mod}–antipodal, and that $\mathrm{Th} \subset W$ is a $W_{\tau_{\mathrm{mod}}}$–invariant balanced thickening. In this section, we will use the following notation:

$$\widehat{\Sigma} := \bar{X}^{\mathrm{Fins}}, \quad \widehat{\Lambda} := \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}), \quad \widehat{\Omega} := \widehat{\Sigma} - \widehat{\Lambda}.$$

According to Theorem 9.16, the action $\Gamma \curvearrowright \widehat{\Omega}$ is properly discontinuous. We will show that it is also cocompact provided that Γ is τ_{mod}–asymptotically embedded, by replacing the action $\Gamma \curvearrowright \widehat{\Sigma}$ with a convergence action $\Gamma \curvearrowright \Sigma$ on a certain quotient space of $\widehat{\Sigma}$ and then applying our cocompactness result for convergence actions (Theorem 10.11). The collapse takes place only in the thickening $\widehat{\Lambda}$ at infinity, so that the action $\Gamma \curvearrowright \widehat{\Omega}$ is not affected.

11.1 Decompositions and collapses

A *decomposition* $\mathcal{R}$ of a set Z is an equivalence relation on Z. We let $\mathcal{D} = \mathcal{D}_{\mathcal{R}}$ denote the subset of the power set 2^Z consisting of the equivalence classes of $\mathcal{R}$.

A decomposition of a Hausdorff topological space Z is *closed* if the elements of $\mathcal{D}$ are closed subsets of Z; a decomposition is *compact* if its elements are compact subsets. Given a decomposition $\mathcal{R}$ of Z, one defines the quotient space $Z/\mathcal{R}$. Quotient spaces of closed decompositions are T_1 but in general not Hausdorff.

Definition 11.2 A decomposition of Z is *upper semicontinuous* (usc) if it is closed and for each $D \in \mathcal{D}$ and each open subset $U \subset Z$ containing D, there exists another open subset $V \subset Z$ containing D such that every $D' \in \mathcal{D}$ intersecting V nontrivially is already contained in U.

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[24]This means $P_{\tau_{\mathrm{mod}}}$–Anosov in the terminology of [24].

Lemma 11.3 [16, Proposition 1, page 8] *The following are equivalent for a closed decomposition $\mathcal{R}$ of Z:*

(i) *$\mathcal{R}$ is usc.*

(ii) *For every open subset $U \subset Z$, the saturated subset*

$$U^* = \bigcup \{D \in \mathcal{D} : D \subset U\}$$

is open.

(iii) *The quotient projection*

$$Z \xrightarrow{\kappa} Z/\mathcal{R}$$

is closed.

Proof (i) $\Longrightarrow$ (ii) Let $x \in U$ and let $D \in \mathcal{D}$ be the decomposition subset through x. The usc property implies that U^* contains a neighborhood of x.

(ii) $\Longrightarrow$ (i) Take $V = U^*$.

(ii) $\Longrightarrow$ (iii) Let $C \subset Z$ be closed, and let U be the complement. Then $U^* = \kappa^{-1}\kappa(Z - C)$ is open, and it follows that $\kappa(C)$ is closed.

(iii) $\Longrightarrow$ (ii) Let $U \subset Z$ be open. Then $U^* = \kappa^{-1}(Z/\mathcal{R} - \kappa(Z - U))$ is open. $\qquad\square$

Let $Z' \subset Z$ be the union of all elements of $\mathcal{D}$ which are not singletons, and denote by $\mathcal{R}'$ the equivalence relation on Z' induced by $\mathcal{R}$.

Lemma 11.4 *Suppose that Z' is closed. Then $\mathcal{R}$ is usc if and only if $\mathcal{R}'$ is usc.*

Proof Suppose that $\mathcal{R}'$ is usc. Let $D \in \mathcal{D}$. If D is a singleton, then $Z - Z'$ is a saturated open neighborhood of D. On the other hand, if $D \subset Z'$ then D has a saturated open neighborhood V' in Z'. It is an intersection $V' = V \cap Z'$ with an open subset $V \subset Z$ which is necessarily again saturated. This verifies that $\mathcal{R}$ is usc.

Conversely, suppose that $\mathcal{R}$ is usc. Then the intersection of a saturated open subset in Z with Z' is open and saturated in Z'. Hence $\mathcal{R}'$ is usc. $\qquad\square$

We will use the following result:

Proposition 11.5 [16, Proposition 2, page 13] *If Z is metrizable and $\mathcal{R}$ is a compact usc decomposition of Z, then $Z/\mathcal{R}$ is again metrizable.*

We now apply the notion of usc decompositions in the context of the Finsler thickening $\widehat{\Lambda}$ of $\Lambda_{\tau_{\mathrm{mod}}} \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$. Since Γ is τ_{mod}–antipodal and the thickening Th is slim,[25] we obtain a compact decomposition $\mathcal{R}$ of $\widehat{\Sigma}$, whose elements are singletons, namely the points in $\widehat{\Omega}$, and the thickenings $\mathrm{Th}^{\mathrm{Fins}}(\tau)$ of the simplices $\tau \in \Lambda_{\tau_{\mathrm{mod}}}$. (One can show that the latter are contractible; see Remark 8.16.) We let

$$\kappa \colon \widehat{\Sigma} \to \Sigma$$

denote the quotient projection, and

$$\Lambda := \kappa(\widehat{\Lambda}) \cong \Lambda_{\tau_{\mathrm{mod}}}, \quad \Omega := \kappa(\widehat{\Omega}) \cong \widehat{\Omega}.$$

Lemma 11.6 *The decomposition $\mathcal{R}$ of $\widehat{\Sigma}$ is compact usc.*

Proof The restriction $\widehat{\Lambda} \to \Lambda$ of κ is a map of compact Hausdorff spaces and hence closed. Thus the restriction of the decomposition $\mathcal{R}$ to $\widehat{\Lambda}$ is usc; see Lemma 11.3. Hence, by Lemma 11.4, the decomposition $\mathcal{R}$ is usc as well. It is also compact. □

Corollary 11.7 $\Sigma = \widehat{\Sigma}/\mathcal{R}$ *is metrizable.*

This corollary is relevant to us in order to do computations with Čech cohomology.

Remark 11.8 We showed in the second version of this paper on arXiv (see Lemma 10.7 there) that Σ is Čech acyclic; compare Remark 8.16.

11.2 Convergence action

Suppose that $\Gamma < G$ is τ_{mod}–regular and τ_{mod}–antipodal. We continue using the notation from the previous section. The action of Γ on $\widehat{\Sigma}$ descends to a continuous action $\Gamma \curvearrowright \Sigma$.

Lemma 11.9 $\Gamma \curvearrowright \Sigma$ *is a convergence action.*

Proof Let $\gamma_n \to \infty$ be a sequence in Γ. Since the group $\Gamma < G$ is a τ_{mod}–convergence subgroup, we may assume after extraction that (γ_n) is τ_{mod}–contracting: There exist simplices $\tau_\pm \in \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$ such that $\gamma_n \to \tau_+$ uniformly on compacts in $C(\tau_-)$. We claim that γ_n converges uniformly on compacts in $\Sigma - \kappa(\tau_-)$ to $\lambda_+ = \kappa(\tau_+)$. It suffices

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[25] See Section 8.3 for the definition.

to show that for each sequence $z_n \in \Sigma$ converging to $z \neq \lambda_- = \kappa(\tau_-)$, we have after extraction that

$$(\textbf{11.10}) \qquad \lim_{n \to \infty} \gamma_n z_n = \lambda_+.$$

Take $\zeta_n \in \kappa^{-1}(z_n)$. Then, after extraction, $\zeta_n \to \zeta \notin \mathrm{Th}^{\mathrm{Fins}}(\tau_-)$. According to Corollary 9.15, the accumulation set of the sequence $\gamma_n \zeta_n$ is contained in $\mathrm{Th}^{\mathrm{Fins}}(\tau_+)$. This implies (11.10). □

We note that in view of Theorem 9.16 the group Γ acts properly discontinuously on Ω. It is also clear that Λ is the limit set of the action $\Gamma \curvearrowright \Sigma$. Since $\hat{\Omega}$ is path-connected, so is Ω.

11.3 Cocompactness

We now make the stronger assumption that $\Gamma < G$ is τ_{mod}–asymptotically embedded. Continuing the discussion of the previous section, we then also have an equivariant homeomorphism $\partial_\infty \Gamma \to \Lambda_{\tau_{\mathrm{mod}}} \to \Lambda$. Thus, Theorem 10.11 together with Lemma 11.9 imply that the action of $\Gamma \curvearrowright \Omega$ is cocompact. Therefore the action of $\Gamma \curvearrowright \hat{\Omega}$ is cocompact as well. By combining this with Theorems 7.2 and 9.16, we obtain the main result of this paper:

Theorem 11.11 *Let $\Gamma < G$ be a τ_{mod}–asymptotically embedded subgroup, and let $\mathrm{Th} \subset W$ be a $W_{\tau_{\mathrm{mod}}}$–invariant balanced thickening. Then the action*

$$\Gamma \curvearrowright \bar{X}^{\mathrm{Fins}} - \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}})$$

is properly discontinuous and cocompact. The quotient

$$(\bar{X}^{\mathrm{Fins}} - \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}))/\Gamma$$

has a natural structure as a compact real-analytic orbifold[26] with corners.

Remark 11.12 The starting point of our proof of Theorem 11.11, namely the usage of the bundles E and E_{mod} in the proof of Theorem 10.2, is similar to the one in [24, Proposition 8.10]. However, we avoid the use of Poincaré duality and do not need homological assumptions on the space Σ. An essential ingredient in our proof is the map of triads (10.1), ie the existence of a continuous extension of the equivariant proper map $\tilde{f} \colon Y \to \Omega$ to a map of compactifications.

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[26]Note that this quotient space is not, in general, a manifold but only an *orbifold* since the group Γ may contain elements of finite order.

Remark 11.13 (cocompactness on $\partial_{\mathrm{Fu}}X$) Intersecting the domain in the theorem with $\partial_{\mathrm{Fu}}X$ yields that the corresponding actions

$$\Gamma \curvearrowright \partial_{\mathrm{Fu}}X - \mathrm{Th}_{\mathrm{Fu}}(\Lambda_{\tau_{\mathrm{mod}}})$$

are cocompact, thus recovering some of the cocompactness results obtained in [34].

11.4 An example of a nonregular discrete subgroup

We now consider a simple example of a *nonregular* discrete subgroup and show that the action on the domain of proper discontinuity constructed earlier in Theorem 9.19 is cocompact.

Let $G = \mathrm{PSL}(3, \mathbb{R})$, and let $\Gamma \cong \mathbb{Z}^2$ be a discrete subgroup of transvections preserving a maximal flat $F \subset X$. As in Theorem 9.19, we choose a *multithickening* $\mathrm{Th}_\bullet$, ie a collection of $W_{\tau_{\mathrm{mod}}}$–left-invariant thickenings $\varnothing \neq \mathrm{Th}_{\tau_{\mathrm{mod}}} \subsetneq W$ for all face types $\tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$. In addition, we require that

$$\mathrm{Th}_{\iota\tau_{\mathrm{mod}}} = \mathrm{Th}_{\tau_{\mathrm{mod}}}^c$$

for all τ_{mod}. In particular, $\mathrm{Th}_{\sigma_{\mathrm{mod}}}$ is the unique balanced thickening. To simplify notation, we drop the face-type index, and write $\mathrm{Th}(\tau)$ instead of $\mathrm{Th}_{\tau_{\mathrm{mod}}}(\tau)$ for simplices $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$.

There are exactly two distinct multithickenings $\mathrm{Th}_\bullet$ which are swapped by the involution $\iota: a_{\mathrm{mod}} \to a_{\mathrm{mod}}$. We let p_{mod} denote the vertex of σ_{mod} of the type *point* in terms of the projective incidence geometry associated with the group G, and let l_{mod} denote the other vertex of σ_{mod} of the type *line*. We require $\mathrm{Th}(p_{\mathrm{mod}}) \subset a_{\mathrm{mod}}$ to consist of the two chambers containing p_{mod} as a vertex. In other words, for a point $p \in \mathbb{R}P^2$ the thickening $\mathrm{Th}_{\mathrm{Fu}}(p)$ of p in the full flag manifold $\partial_{\mathrm{Fu}}X = \mathrm{Flag}_{\sigma_{\mathrm{mod}}}$ consists of all the flags (p, l). Accordingly, for each line l in the projective plane, its thickening $\mathrm{Th}_{\mathrm{Fu}}(l) \subset \partial_{\mathrm{Fu}}X$ consists of all the flags (p, l') where $p \in \mathbb{R}P^2$ are points incident to the line l. Topologically speaking, $\mathrm{Th}_{\mathrm{Fu}}(p) \cong S^1$ while $\mathrm{Th}_{\mathrm{Fu}}(l)$ is the 2–torus, the trivial circle bundle over l whose fibers are the thickenings $\mathrm{Th}_{\mathrm{Fu}}(p)$, $p \in l$.

We let p_i and l_i for $i = 1, 2, 3$ denote the singular points in $\partial_\infty F$, where $p_i \in \mathbb{R}P^2$ are the fixed points of Γ and $l_i \in (\mathbb{R}P^2)^\vee$ are the fixed lines of Γ, labeled so that l_i is the line through p_{i-1} and p_{i+1} (where i is taken mod 3).

We obtain

$$\Lambda_{p_{\mathrm{mod}}}^+ = \{p_1, p_2, p_3\} = \Lambda_{l_{\mathrm{mod}}}^-, \quad \Lambda_{p_{\mathrm{mod}}}^- = \{l_1, l_2, l_3\} = \Lambda_{l_{\mathrm{mod}}}^+.$$

Furthermore,

$$\Lambda_{\sigma_{\mathrm{mod}}}^{\pm} = \{(p_i, l_j) : i \neq j\}.$$

Thus,

$$\Lambda_{\bullet} = \bigcup_{\tau_{\mathrm{mod}}} \Lambda_{\tau_{\mathrm{mod}}}^{\pm} = \{p_1, p_2, p_3\} \cup \{l_1, l_2, l_3\} \cup \Lambda_{\sigma_{\mathrm{mod}}}^{\pm}.$$

We have

$$\mathrm{Th}_{\mathrm{Fu}}(\Lambda_{p_{\mathrm{mod}}}^{+}) = \mathrm{Th}_{\mathrm{Fu}}^{c}(\Lambda_{l_{\mathrm{mod}}}^{-}), \quad \mathrm{Th}_{\mathrm{Fu}}(\Lambda_{l_{\mathrm{mod}}}^{+}) = \mathrm{Th}_{\mathrm{Fu}}^{c}(\Lambda_{p_{\mathrm{mod}}}^{-}),$$

while

$$\mathrm{Th}_{\mathrm{Fu}}(\Lambda_{\sigma_{\mathrm{mod}}}^{+}) = \mathrm{Th}_{\mathrm{Fu}}^{c}(\Lambda_{\sigma_{\mathrm{mod}}}^{-}).$$

Therefore, the union

$$\mathrm{Th}_{\mathrm{Fu}}(\Lambda_{\bullet}) = \bigcup_{\tau_{\mathrm{mod}}} \left(\mathrm{Th}_{\mathrm{Fu}}^{c}(\Lambda_{\tau_{\mathrm{mod}}}^{-}) \cup \mathrm{Th}_{\mathrm{Fu}}(\Lambda_{\tau_{\mathrm{mod}}}^{+}) \right) \subseteq \mathrm{Th}_{\mathrm{Fu}}(\Lambda_{l_{\mathrm{mod}}}^{+}) = \partial_{\mathrm{Fu}} X$$

is the set of all flags (p, l) such that p is incident to one of the lines l_1, l_2, l_3. Topologically speaking, this set is the union of three trivial circle bundles $\mathrm{Th}_{\mathrm{Fu}}(l_i)$ over the circles l_i, such that

$$\mathrm{Th}_{\mathrm{Fu}}(l_{i-1}) \cap \mathrm{Th}_{\mathrm{Fu}}(l_{i+1}) = \mathrm{Th}_{\mathrm{Fu}}(p_i),$$

where i is taken modulo 3.

The fact that the action

(11.14) $$\Gamma \curvearrowright \Omega_{\mathrm{Fu}} := \partial_{\mathrm{Fu}} X - \mathrm{Th}_{\mathrm{Fu}}(\Lambda_{\bullet})$$

is properly discontinuous can be seen as a special case of [34, Proposition 6.21] in our earlier work. It can be also seen directly by observing that the action of Γ is properly discontinuous on $\Omega_{p_{\mathrm{mod}}} := \mathbb{R}P^2 - (l_1 \cup l_2 \cup l_3)$, since (11.14) is the preimage of $\Omega_{p_{\mathrm{mod}}}$ under the fibration $\partial_{\mathrm{Fu}} X \to \mathbb{R}P^2$. Since Γ acts cocompactly on $\Omega_{p_{\mathrm{mod}}}$ (the quotient is the disjoint union of four 2–tori), the group Γ also acts cocompactly on the domain (11.14). The quotient $\Omega_{\mathrm{Fu}} / \Gamma$ is a circle bundle over $\Omega_{p_{\mathrm{mod}}} / \Gamma$.

We now discuss the corresponding Finsler thickening

$$\mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\bullet}) = \bigcup_{\tau_{\mathrm{mod}}} \left((\mathrm{Th}^{c})^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^{-}) \cup \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}}^{+}) \right);$$

see Theorem 9.19. For each point p_i, the Finsler thickening of p_i is the closed 2–disk, which is the closed stratum $\overline{X}_{p_i}$ naturally isomorphic to a compactified hyperbolic plane whose ideal boundary is the circle $\mathrm{Th}_{\mathrm{Fu}}(p_i)$.

For each l_i, its Finsler thickening $\mathrm{Th}^{\mathrm{Fins}}(l_i)$ is the union of a solid torus, whose boundary is the torus $\mathrm{Th}_{\mathrm{Fu}}(l_i)$, and the closed 2–disk $\bar{X}_{l_i}$, whose boundary circle C_i is the set of flags (p, l_i) for $p \in l_i$. The circle $C_i \subset \mathrm{Th}_{\mathrm{Fu}}(l_i)$ is a section of the circle bundle $\mathrm{Th}_{\mathrm{Fu}}(l_i) \to l_i$. In particular, $\mathrm{Th}^{\mathrm{Fins}}(l_i)$ is contractible.

Furthermore, we have

$$\mathrm{Th}^{\mathrm{Fins}}(l_{i-1}) \cap \mathrm{Th}^{\mathrm{Fins}}(l_{i+1}) = \mathrm{Th}^{\mathrm{Fins}}(p_i),$$

and the triple intersection of the $\mathrm{Th}^{\mathrm{Fins}}(l_i)$ is empty. The thickening $\mathrm{Th}^{\mathrm{Fins}}(\Lambda_\bullet)$ equals the union

$$\bigcup_{i=1}^{3} \mathrm{Th}^{\mathrm{Fins}}(l_i),$$

which is homotopy-equivalent to the circle. The inclusion

$$\partial_\infty^{\mathrm{Fins}}(F) \hookrightarrow \mathrm{Th}^{\mathrm{Fins}}(\Lambda_\bullet)$$

is a homotopy equivalence. This inclusion is the restriction of the natural embedding

$$\bar{F}^{\mathrm{Fins}} \hookrightarrow \bar{X}^{\mathrm{Fins}}.$$

We are now in the position to apply Theorems 9.19 and 10.2, taking $\Gamma \curvearrowright \bar{F}^{\mathrm{Fins}}$ as the compactified model action, and conclude:

Proposition 11.15 *The action*

$$\Gamma \curvearrowright \bar{X}^{\mathrm{Fins}} - \mathrm{Th}^{\mathrm{Fins}}(\Lambda_\bullet)$$

is properly discontinuous and cocompact.

12 Characterizations of Anosov subgroups

In our earlier papers [30; 31] we gave various characterizations of Anosov subgroups in terms of dynamics and coarse extrinsic geometry; see also our surveys [32; 27]. The most relevant characterizations for this paper are *asymptotically embedded* (see Definition 11.1), and *URU*.

We assume from now on that τ_{mod} is ι–invariant. A discrete subgroup $\Gamma < G$ is called τ_{mod}–*URU* if it is uniformly τ_{mod}–regular and undistorted; see [31]. In that work, we proved that the τ_{mod}–URU property is equivalent to τ_{mod}–Anosov. In this section, we will give further characterizations of the Anosov property in terms of dynamics

(S-cocompactness) and coarse extrinsic geometry (Finsler quasiconvexity and existence of retractions).

12.1 Finsler quasiconvexity

One can define *Finsler convex* subsets $C \subset X$ as subspaces such that any two points in C can be connected by *some* Finsler geodesic in C.

In this section we introduce the notion of *Finsler quasiconvex subgroups* of G, which mimics the notion of quasiconvex subgroups of hyperbolic groups. Recall that a subgroup Γ of a word hyperbolic group Γ' is called *quasiconvex* if discrete geodesic segments in Γ' with endpoints in Γ are uniformly close to Γ.

Fix a type $\bar{\theta} \in \text{int}(\tau_{\text{mod}})$. Recall that $d^{\bar{\theta}}$ is in general only a *pseudometric* on X.

Definition 12.1 A discrete subgroup $\Gamma < G$ is τ_{mod}–*Finsler quasiconvex* if for each $x \in X$ there is a constant $R < +\infty$ such that any two points in Γx can be connected by a $d^{\bar{\theta}}$–geodesic segment contained in the R–neighborhood $N_R(\Gamma x)$ of Γx with respect to $d^{\bar{\theta}}$.

Proposition 12.2 *A uniformly τ_{mod}–regular subgroup $\Gamma < G$ is τ_{mod}–Finsler quasiconvex if and only if it is τ_{mod}–URU.*

Proof We first reduce the assertion to the case when the pseudometric $d^{\bar{\theta}}$ is a metric. We recall from the end of Section 5.1.2 that X splits as a product $X_1 \times X_2$ such that $d^{\bar{\theta}}$ is degenerate precisely in the X_2–direction and induces a metric on X_1. In particular, $\bar{\theta}$ points in the X_1–direction, ie the visual boundary points of type $\bar{\theta}$ are contained in $\partial_\infty X_1$. Then $d^{\bar{\theta}}$–balls split off X_2–factors, ie they are products of X_2 with $d^{\bar{\theta}}$–balls in X_1. The same applies to τ_{mod}–Weyl cones and τ_{mod}–diamonds. A map $I \to X$ from an interval is a $d^{\bar{\theta}}$–geodesic if and only if its projection to X_1 is a $d^{\bar{\theta}}$–geodesic. Furthermore, a map into X is a quasiisometric embedding with respect to $d^{\bar{\theta}}$ if and only if its X_1–component is. We can therefore assume that $d^{\bar{\theta}}$ is a metric.

Suppose now that $\Gamma < G$ is τ_{mod}–Finsler quasiconvex. The closed R–neighborhood $N_R(\Gamma x)$ is path-connected and Γ acts cocompactly on it. Therefore, Γ is finitely generated, and the orbit map

$$o_x \colon \Gamma \to N_R(\Gamma x)$$

is a quasiisometric embedding, where we equip $N_R(\Gamma x)$ with a path-metric induced by $d^{\bar{\theta}}$. Since the metrics d^{Riem} and $d^{\bar{\theta}}$ on X are equivalent, the definition of Finsler quasiconvexity implies that the inclusion map

$$N_R(\Gamma x) \to (X, d^{\mathrm{Riem}})$$

is a quasiisometric embedding. Therefore, $\Gamma < G$ is undistorted. Since Γ was assumed to be uniformly τ_{mod}–regular, it is τ_{mod}–URU.

The converse direction follows from our Morse lemma [31, Theorem 1.3] and the description of the geometry of $d^{\bar{\theta}}$–geodesics; see Section 5.1.3. □

12.2 S-cocompactness and retractions

We call an open subset $\Omega \subset \partial_\infty^{\mathrm{Fins}} X$ *saturated* if it is a union of small strata X_ν.

We start with the following simple observation about Finsler convergence at infinity: If (x_n) and (y_n) are sequences in X which are bounded distance apart (ie $d(x_n, y_n)$ is uniformly bounded) and $x_n \to [b]$, $y_n \to [b'] \in \partial_\infty^{\mathrm{Fins}} X$, then the limit points $[b]$ and $[b']$ lie in the same small stratum X_ν; see Lemma 5.28. In particular, for each saturated open subset $\Omega \subset \partial_\infty^{\mathrm{Fins}} X$,

$$[b] \in \omega \iff [b'] \in \Omega.$$

It follows that if $[b] \in \Omega$, then the entire accumulation set of the sequence of balls $B(x_n, R)$,

$$\mathrm{Acc}((B(x_n, R))) \subset \partial_\infty^{\mathrm{Fins}} X,$$

is a compact subset of Ω.

Lemma 12.3 *Let $\Gamma < G$ be a discrete subgroup. Suppose that $\Omega \subset \partial_\infty^{\mathrm{Fins}} X$ is a Γ–invariant saturated open subset such that the action*

$$\Gamma \curvearrowright X \sqcup \Omega$$

is properly discontinuous. Then each compact subset $C \subset X \sqcup \Omega$ satisfies the following **uniform finiteness property**: *there exists a function $f_C(R)$ such that for each ball $B(x, R) \subset X$ it holds that*

$$\mathrm{card}\big(\{\gamma \in \Gamma : \gamma C \cap B(x, R) \neq \varnothing\}\big) \leq f_C(R).$$

Proof Suppose the contrary. Then there is a sequence of balls $B(x_i, R)$ intersecting C and a sequence $\gamma_i \to \infty$ in Γ such that also the balls $B(\gamma_i x_i, R)$ intersect C. We may assume after extraction that $x_i \to \bar{x}$ and $\gamma_i x_i \to \bar{x}'$ in $\bar{X}^{\text{Fins}}$. By the observation preceding the lemma, it holds that $\bar{x}, \bar{x}' \in X \sqcup \Omega$. Since these points are dynamically related with respect to the Γ–action, we obtain a contradiction with proper discontinuity. $\quad\square$

The lemma leads to the following definition.

Definition 12.4 A discrete subgroup $\Gamma < G$ is *S-cocompact* if there exists a Γ–invariant saturated open subset $\Omega \subset \partial_\infty^{\text{Fins}} X$ such that the action

$$\Gamma \curvearrowright X \sqcup \Omega$$

is properly discontinuous and cocompact.

Note that each S-cocompact subgroup is necessarily finitely generated because it acts properly discontinuously and cocompactly on a connected manifold with boundary.

Theorem 12.5 *Each S-cocompact subgroup $\Gamma < G$ admits a Γ–equivariant coarse Lipschitz retraction $r \colon X \to \Gamma$. In particular, Γ is undistorted in G.*

Proof Let $\Omega \subset \partial_\infty^{\text{Fins}} X$ be as in the definition. Let $C \subset X \sqcup \Omega$ be a compact subset whose Γ–orbit covers the entire $X \sqcup \Omega$. We define the coarse retraction r first by sending each point $x \in X$ to the subset

$$r(x) := \{\gamma \in \Gamma : x \in \gamma C\} \subset \Gamma.$$

This subset is clearly finite because of the proper discontinuity of the Γ–action, and the assignment $x \mapsto r(x)$ is equivariant. According to Lemma 12.3, the cardinality of the subset

$$\{\gamma \in \Gamma : \gamma \in r(B(x, 1))\} = \{\gamma \in \Gamma : B(x, 1) \cap \gamma C \neq \varnothing\}$$

is bounded by $f_C(1)$, independently of x. It follows that r is coarse Lipschitz. $\quad\square$

We now apply the previous theorem to the cocompact domains of proper discontinuity obtained earlier by removing Finsler thickenings of limit sets. The next result relates conicality and S-cocompactness:

Theorem 12.6 *Suppose that $\Gamma < G$ is uniformly τ_{mod}–regular and τ_{mod}–antipodal. Then Γ is τ_{mod}–Anosov if and only if it is S-cocompact.*

Proof We use that τ_{mod}–Anosov is equivalent to τ_{mod}–asymptotically embedded. That τ_{mod}–asymptotically embedded implies S-cocompact is our main result, Theorem 11.11. To prove the converse, note that each S-cocompact subgroup is undistorted in G by Theorem 12.5. Hence, Γ is τ_{mod}–URU, and therefore τ_{mod}–Anosov. □

A converse to Theorem 11.11 is a consequence of the following:

Corollary 12.7 *Suppose that $\Gamma < G$ is uniformly τ_{mod}–regular and that $\mathrm{Th} \subset W$ is a $W_{\tau_{\mathrm{mod}}}$–invariant balanced thickening. Then the following are equivalent:*

(i) *The properly discontinuous action (see Theorem 9.19)*

$$\Gamma \curvearrowright \bar{X}^{\mathrm{Fins}} - \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\tau_{\mathrm{mod}}})$$

is cocompact.

(ii) *Γ is S-cocompact.*

(iii) *Γ is τ_{mod}–Anosov.*

Proof The implication (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii) Γ is S-cocompact, hence τ_{mod}–URU by Theorem 12.5, and therefore τ_{mod}–Anosov.

(iii) $\Longrightarrow$ (i) Since τ_{mod}–Anosov is equivalent to τ_{mod}–asymptotically embedded, the implication is the content of Theorem 11.11. □

We are now ready to state the equivalence of a variety of conditions on discrete subgroups, extending the list of equivalent conditions from [30; 31].

Theorem 12.8 *The following are equivalent for uniformly τ_{mod}–regular subgroups $\Gamma < G$:*

(1) *Γ is an equivariant coarse retract.*

(2) *Γ is a coarse retract.*

(3) *Γ is undistorted in G, ie τ_{mod}–URU.*

(4) *Γ is τ_{mod}–Finsler quasiconvex.*

(5) *Γ is τ_{mod}–asymptotically embedded.*

(6) *Γ is S-cocompact.*

(7) *Γ is τ_{mod}–Anosov.*

Proof The implications $(1) \implies (2) \implies (3)$ are immediate. The equivalence $(3) \iff (4)$ is proven in Proposition 12.2. The equivalence $(3) \iff (5)$ is one of the main results of [31]; see Corollary 1.6 of that paper. The equivalence $(5) \iff (7)$ is established in [30]. The implication $(5) \implies (6)$ is the main result, Theorem 11.11, of this paper, while the implication $(6) \implies (1)$ is established in Theorem 12.5. $\square$

We note that this list of equivalences is nearly a perfect match to the list of equivalent definitions of convex cocompact subgroups of rank-1 Lie groups (see [27]), except that convex-cocompactness is (by necessity) missing; see [38].

12.3 Examples

Example 12.9 Consider $X = X_1 \times X_2$, the product of two real hyperbolic spaces, $g = (g_1, g_2)$ an infinite-order isometry of X, where g_1, g_2 are isometries of X_1, X_2. Then the cyclic subgroup $\Gamma = \langle g \rangle$ is σ_{mod}–regular if and only if neither g_1 nor g_2 is elliptic. The subgroup Γ is uniformly σ_{mod}–regular if and only if both g_1, g_2 are hyperbolic isometries of X_1, X_2 or both are parabolic isometries. A cyclic group generated by an element of *mixed type* is *not* uniformly σ_{mod}–regular. The Furstenberg boundary of X is the product $\partial_\infty X_1 \times \partial_\infty X_2$. If Γ, as above, is σ_{mod}–regular and λ_i^+, λ_i^- are the fixed points of g_i in $\partial_\infty X_i$,[27] then $\Lambda_{\sigma_{\mathrm{mod}}}(\Gamma) = \{(\lambda_1^-, \lambda_2^-), (\lambda_1^+, \lambda_2^+)\}$. In particular, in the mixed case if, say, g_1 is hyperbolic and g_2 is parabolic with the unique fixed point $\lambda_2^+ = \lambda_2^- =: \lambda_2$, then $\Lambda_{\mathrm{ch}}(\Gamma) = \{(\lambda_1^-, \lambda_2), (\lambda_1^+, \lambda_2)\}$. Note that if Γ is uniformly σ_{mod}–regular then the limit set $\Lambda_{\sigma_{\mathrm{mod}}}(\Gamma)$ is antipodal, but it is not antipodal if Γ is merely regular. The limit chambers are conical limit points if g is uniformly regular of type hyperbolic-hyperbolic, and otherwise they are not.

The Finsler compactification of X is naturally homeomorphic to $\bar{X}_1 \times \bar{X}_2$. Assume that $g = (g_1, g_2)$, where g_1 is hyperbolic (with the fixed points λ_1^+, λ_1^-) and g_2 is parabolic (with the fixed point λ_2). As we noted above, the group $\Gamma = \langle g \rangle < \mathrm{Isom}(X)$ is σ_{mod}–regular but not uniformly regular. Therefore, it is not σ_{mod}–Anosov. On the other hand, it is S-cocompact. Namely, it acts properly discontinuously and cocompactly on

$$(\bar{X}_1 - \{\lambda_1^-, \lambda_1^+\}) \times \bar{X}_2.$$

In particular, Γ is a coarse retract, and hence undistorted. Thus, uniform regularity cannot be weakened to regularity in Theorem 12.8(1–4, 6), Theorem 12.6 and Corollary 12.7.

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[27]That is, λ_i^+ and λ_i^- are the attractive and repulsive fixed points if g_i is hyperbolic, and $\lambda_i^+ = \lambda_i^-$ is the unique fixed point if g_i is parabolic.
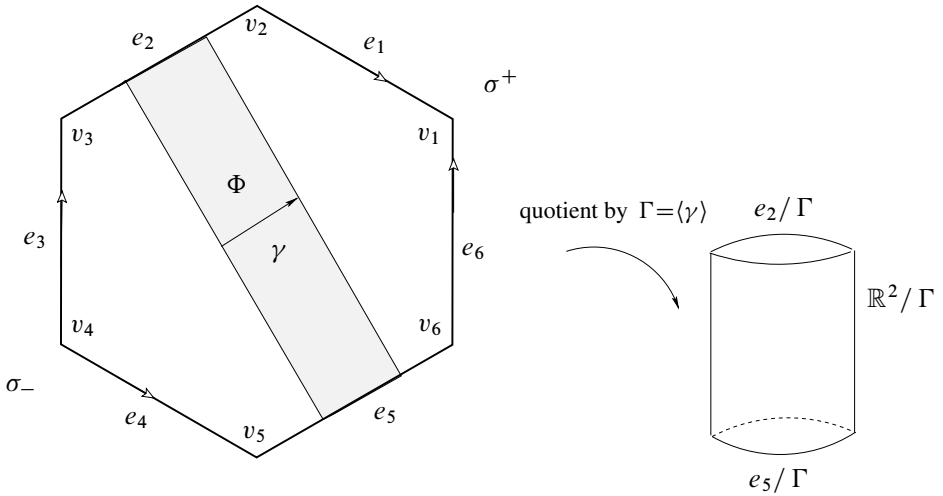
Figure 1: The action of a cyclic subgroup $\Gamma = \langle \gamma \rangle$ on the Finsler compactification $\overline{F}^{\mathrm{Fins}}_{\mathrm{mod}}$ of the model flat and the quotient space of $\overline{F}^{\mathrm{Fins}}_{\mathrm{mod}} - \mathrm{Th}^{\mathrm{Fins}}(\Lambda_{\mathrm{ch}})$ by the Γ–action

Example 12.10 We now work out an example illustrating the equivariant collapsing of $\bar{X}^{\mathrm{Fins}}$ described in Section 11. Consider an infinite cyclic subgroup $\Gamma = \langle \gamma \rangle <$ $\mathrm{PGL}(3, \mathbb{R})$ generated by a regular hyperbolic isometry γ. For simplicity, we only describe the action and the collapse on the Finsler compactification of the unique Γ–invariant maximal flat $F \subset X$. The Finsler compactification $\overline{F}^{\mathrm{Fins}}$ is a hexagon with vertices $v_1, \ldots, v_6$ and edges $e_1, \ldots, e_6$. The vertex set equals the Furstenberg boundary, $\partial_{\mathrm{Fu}} F = \{v_1, \ldots, v_6\}$. We label the vertices so that v_1 and v_4 correspond to the repulsive and attractive chambers $\sigma_-, \sigma_+ \in \partial_{\mathrm{Fu}} F$. The vertices are fixed by γ, but γ has nontrivial dynamics on the edges: the interior points of each edge $e_i = [v_i, v_{i+1}]$ are moved by γ towards one of the two endpoints of e_i, namely to the one which corresponds to the chamber in $\partial_{\mathrm{Fu}} F$ whose position relative to the attractive chamber σ_+ is smaller in the Bruhat order. This is in stark contrast with the action of γ on the visual boundary of $\partial_\infty F$ (with respect to the flat metric), which is fixed pointwise. The σ_{mod}–limit set $\Lambda_{\sigma_{\mathrm{mod}}}(\Gamma) \subset \partial_{\mathrm{Fu}} X$ is the 2–point set $\{\sigma_-, \sigma_+\} = \{v_1, v_4\} \subset \partial_{\mathrm{Fu}} F$. The balanced thickening of $\Lambda_{\sigma_{\mathrm{mod}}}(\Gamma)$ inside $\partial_\infty^{\mathrm{Fins}} F$ is the union (of closed edges)

$$\mathrm{Th}^{\mathrm{Fins}}(\sigma_-) \cup \mathrm{Th}^{\mathrm{Fins}}(\sigma_+) = (e_3 \cup e_4) \cup (e_1 \cup e_6).$$

The intersection

$$\Omega = \Omega(\Gamma) = \Omega_{\mathrm{Th}}^{\mathrm{Fins}}(\Gamma) \cap \partial_\infty^{\mathrm{Fins}} F$$

is the union of the interiors of the edges e_2 and e_5. The rectangle Φ in Figure 1 is a (compact) fundamental domain for the action of Γ on $F \cup \Omega$. The quotient $(F \cup \Omega)/\Gamma$ is homeomorphic to the cylinder $S^1 \times [-1, 1]$. Now, let us collapse each thickening $\mathrm{Th}^{\mathrm{Fins}}(\sigma_-), \mathrm{Th}^{\mathrm{Fins}}(\sigma_+)$ to a point. The result is a convergence action of Γ on the quotient space Q, homeomorphic to the closed 2–disk D^2. Note that collapsing is natural here since, before the collapse, the mapping γ has too many fixed points in $\partial_\infty^{\mathrm{Fins}} F$, namely all vertices $v_1, \ldots, v_6$, while an infinite cyclic group acting as a discrete convergence group can have at most two fixed points [50]. After the collapse only two fixed points are left, namely the projections (still denoted by σ_+, σ_-) of v_1 and v_4. On the quotient space Q we recover the familiar attractive-repulsive dynamics of hyperbolic isometries γ of $\mathbb{H}^2$ acting on the visual compactification of $\mathbb{H}^2$: The point σ_+ is the attractive point and the point σ_- is the repulsive point for the action of γ. That is,

$$\lim_{n \to \infty} \gamma^n = \sigma_+$$

uniformly on compacts in $Q - \{\sigma_-\}$, and

$$\lim_{n \to -\infty} \gamma^{-n} = \sigma_-$$

uniformly on compacts in $Q - \{\sigma_+\}$.

References

[1] **A Ash**, **D Mumford**, **M Rapoport**, **Y Tai**, *Smooth compactification of locally symmetric varieties*, Lie Groups: History, Frontiers and Applications IV, Math. Sci., Brookline, MA (1975) MR

[2] **W L Baily, Jr**, **A Borel**, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. 84 (1966) 442–528 MR

[3] **W Ballmann**, *Lectures on spaces of nonpositive curvature*, DMV Seminar 25, Birkhäuser, Basel (1995) MR

[4] **W Ballmann**, **M Gromov**, **V Schroeder**, *Manifolds of nonpositive curvature*, Progress in Math. 61, Birkhäuser, Boston (1985) MR

[5] **Y Benoist**, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. 7 (1997) 1–47 MR

[6] **M Bestvina**, **G Mess**, *The boundary of negatively curved groups*, J. Amer. Math. Soc. 4 (1991) 469–481 MR

[7] **A Borel**, *Linear algebraic groups*, 2nd edition, Grad. Texts in Math. 126, Springer (1991) MR

[8] **A Borel**, **L Ji**, *Compactifications of symmetric and locally symmetric spaces*, Birkhäuser, Boston (2006) MR

[9] **A Borel**, **J-P Serre**, *Corners and arithmetic groups*, Comment. Math. Helv. 48 (1973) 436–491 MR

[10] **B H Bowditch**, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. 113 (1993) 245–317 MR

[11] **B H Bowditch**, *Geometrical finiteness with variable negative curvature*, Duke Math. J. 77 (1995) 229–274 MR

[12] **B H Bowditch**, *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. 11 (1998) 643–667 MR

[13] **B H Bowditch**, *Convergence groups and configuration spaces*, from "Geometric group theory down under" (J Cossey, W D Neumann, M Shapiro, editors), de Gruyter, Berlin (1999) 23–54 MR

[14] **B Brill**, *Eine Familie von Kompaktifizierungen affiner Gebäude*, dissertation, Goethe University Frankfurt (2006)

[15] **C Chevalley**, *Sur les décompositions cellulaires des espaces G/B*, from "Algebraic groups and their generalizations: classical methods" (W J Haboush, B J Parshall, editors), Proc. Sympos. Pure Math. 56, Amer. Math. Soc., Providence, RI (1994) 1–23 MR

[16] **R J Daverman**, *Decompositions of manifolds*, Pure and Applied Math. 124, Academic, Orlando, FL (1986) MR

[17] **P B Eberlein**, *Geometry of nonpositively curved manifolds*, Univ. of Chicago Press (1996) MR

[18] **C Frances**, *Lorentzian Kleinian groups*, Comment. Math. Helv. 80 (2005) 883–910 MR

[19] **H Furstenberg**, *A Poisson formula for semisimple Lie groups*, Ann. of Math. 77 (1963) 335–386 MR

[20] **F W Gehring**, **G J Martin**, *Discrete convergence groups*, from "Complex analysis, I" (C A Berenstein, editor), Lecture Notes in Math. 1275, Springer (1987) 158–167 MR

[21] **M Gromov**, *Hyperbolic manifolds, groups and actions*, from "Riemann surfaces and related topics" (I Kra, B Maskit, editors), Ann. of Math. Stud. 97, Princeton Univ. Press (1981) 183–213 MR

[22] **B Grünbaum**, *Convex polytopes*, 2nd edition, Grad. Texts in Math. 221, Springer (2003) MR

[23] **O Guichard**, **F Kassel**, **A Wienhard**, *Tameness of Riemannian locally symmetric spaces arising from Anosov representations*, preprint (2015) arXiv

[24] **O Guichard**, **A Wienhard**, *Anosov representations: domains of discontinuity and applications*, Invent. Math. 190 (2012) 357–438 MR

[25] **L Ji**, **A-S Schilling**, *Polyhedral horofunction compactification as polyhedral ball*, preprint (2016) arXiv

[26] **D Joyce**, *D-manifolds, d-orbifolds and derived differential geometry: a detailed summary*, preprint (2012) arXiv

[27] **M Kapovich**, **B Leeb**, *Discrete isometry groups of symmetric spaces*, preprint (2017) arXiv To appear in Handbook of group actions, IV

[28] **M Kapovich**, **B Leeb**, **J Millson**, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, J. Differential Geom. 81 (2009) 297–354 MR

[29] **M Kapovich**, **B Leeb**, **J Porti**, *Dynamics at infinity of regular discrete subgroups of isometries of higher rank symmetric spaces*, preprint (2013) arXiv

[30] **M Kapovich**, **B Leeb**, **J Porti**, *Morse actions of discrete groups on symmetric space*, preprint (2014) arXiv

[31] **M Kapovich**, **B Leeb**, **J Porti**, *A Morse lemma for quasigeodesics in symmetric spaces and euclidean buildings*, preprint (2014) arXiv To appear in Geom. Topol.

[32] **M Kapovich**, **B Leeb**, **J Porti**, *Some recent results on Anosov representations*, Transform. Groups 21 (2016) 1105–1121 MR

[33] **M Kapovich**, **B Leeb**, **J Porti**, *Anosov subgroups: dynamical and geometric characterizations*, Eur. J. Math. 3 (2017) 808–898 MR

[34] **M Kapovich**, **B Leeb**, **J Porti**, *Dynamics on flag manifolds: domains of proper discontinuity and cocompactness*, Geom. Topol. 22 (2018) 157–234 MR

[35] **A Karlsson**, **V Metz**, **G A Noskov**, *Horoballs in simplices and Minkowski spaces*, Int. J. Math. Math. Sci. (2006) art. id. 23656, 1–20 MR

[36] **F I Karpelevič**, *The geometry of geodesics and the eigenfunctions of the Beltrami–Laplace operator on symmetric spaces*, Tr. Mosk. Mat. Obs. 14 (1965) 48–185 MR In Russian; translated in Trans. Moscow Math. Soc. 1965 (1967) 51–199

[37] **B Kleiner**, **B Leeb**, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Inst. Hautes Études Sci. Publ. Math. 86 (1997) 115–197 MR

[38] **B Kleiner**, **B Leeb**, *Rigidity of invariant convex sets in symmetric spaces*, Invent. Math. 163 (2006) 657–676 MR

[39] **F Labourie**, *Anosov flows, surface groups and curves in projective space*, Invent. Math. 165 (2006) 51–114 MR

[40] **A Marden**, *The geometry of finitely generated kleinian groups*, Ann. of Math. 99 (1974) 383–462 MR

[41] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Annals of Math. Studies 76, Princeton Univ. Press (1974) MR

[42] **S A Mitchell**, *Quillen's theorem on buildings and the loops on a symmetric space*, Enseign. Math. 34 (1988) 123–166 MR

[43] **S A Mitchell**, *Parabolic orbits in flag varieties*, preprint (2008)

[44] **C C Moore**, *Compactifications of symmetric spaces, II: The Cartan domains*, Amer. J. Math. 86 (1964) 358–378 MR

[45] **A Parreau**, *La distance vectorielle dans les immeubles affines et les espaces symétriques*, in preparation

[46] **J G Ratcliffe**, *Foundations of hyperbolic manifolds*, Grad. Texts in Math. 149, Springer (1994) MR

[47] **I Satake**, *On compactifications of the quotient spaces for arithmetically defined discontinuous groups*, Ann. of Math. 72 (1960) 555–580 MR

[48] **I Satake**, *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math. 71 (1960) 77–110 MR

[49] **E H Spanier**, *Algebraic topology*, McGraw-Hill, New York (1966) MR

[50] **P Tukia**, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. 23 (1994) 157–187 MR

[51] **P Tukia**, *Conical limit points and uniform convergence groups*, J. Reine Angew. Math. 501 (1998) 71–98 MR

[52] **C Walsh**, *The horofunction boundary of finite-dimensional normed spaces*, Math. Proc. Cambridge Philos. Soc. 142 (2007) 497–507 MR

[53] **C Walsh**, *The horoboundary and isometry group of Thurston's Lipschitz metric*, from "Handbook of Teichmüller theory, IV" (A Papadopoulos, editor), IRMA Lect. Math. Theor. Phys. 19, Eur. Math. Soc., Zürich (2014) 327–353 MR

[54] **G M Ziegler**, *Lectures on polytopes*, Grad. Texts in Math. 152, Springer (1995) MR

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