Intersection Pairing on Hyperbolic 4-Manifolds

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1. INTRODUCTION

1.1. Begin with the following elementary question: how one can estimate the algebraic intersection number $<\alpha,\beta>$ between two closed curves α, β on an orientable Riemannian surface M ?

Suppose that the metric on M is hyperbolic (has curvature (-1)). Then the answer can be given in the terms of lengths of α,β. For example:

$$|<\alpha,\beta>| \leq K(l(\alpha),l(\beta)) = 2(\pi/2 + l(\alpha))\cdot(e^{l(\beta)}+1)$$

In particular, if $|<\alpha,\beta>|\geq 1$ then

$$1 \;\leq\; \sinh(l(\beta)/2)\;\sinh(l(\alpha)/2)$$

The defect of this answer is that the right side depends on the metric on M while the left side is purely topological.

1.2. Now let M be an arbitrary complete oriented hyperbolic 4-manifold (which isn't necessarily closed), $\sigma_j : \Sigma_j \to M$ $(j = 1,2)$ be two cycles in $Z_2(M,\mathbb{Z})$, where Σ_j are closed oriented connected surfaces. The main aim of this paper is to estimate from above the absolute value of the intersection pairing $|<[\sigma_1],[\sigma_2]>|$ so that the estimate depends just on the Euler characteristics of Σ_j. This will be done under certain condition on σ_j.

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DEFINITION 1. The maps σ_j are *incompressible* if the induced homomorphisms of the fundamental groups σ_{j*} are injective.

THEOREM 1. There exists a function $C(.,.)$ such that for any complete hyperbolic 4-manifold M and for any classes $[\sigma_1], [\sigma_2]$ in $H_2(M, \mathbb{Z})$ which have incompressible representatives, the following inequality holds:

$$| < [\sigma_1], [\sigma_2] > | \leq C(|\chi(\Sigma_1)|, |\chi(\Sigma_2)|) .$$

So, Theorem 1 shows how the intersection pairing together with the simplicial norm on the second homology group provides an obstruction for existence of complete hyperbolic structure on 4-manifolds. The intersection pairing itself isn't too interesting invariant for hyperbolic 4-manifolds. If M^4 is closed then just two symmetric bilinear forms of the given rank can occur as intersection forms. On another hand, any symmetric bilinear form can be realized as the intersection form of a compact convex hyperbolic 4-manifold (cf. [GLT], [Ka 1]). There are no homological obstructions for hyperbolicity in the dimensions 2 and 3 because the length of a curve (unlike the area of a surface) isn't a topological invariant.

1.3. Consider the particular case: suppose $M = M(e, g)$ is homeomorphic to an $\mathbb{R}^2$- bundle over a closed orientable surface F of genus $g > 1$. Such bundles are characterized by their "Euler numbers". In this case it is just the intersection number $e =< [F], [F] >$, where we identified F with the zero section of the bundle.

COROLLARY 1. The condition $|e| \leq C(2g-2, 2g-2)$ is necessary for existence of complete hyperbolic structures on M.

REMARK 1. As it follows from the calculations in Section 3, instead of $C(2g-2, 2g-2)$ one can actually take

exp(Const · exp ~~$\exp(8000g/\mu)$~~

where μ is the Margulis constant for the 4-dimensional hyperbolic space.

Denote by $\mathcal{S}(e, g)$ an orientable 3-manifold which is a circle bundle over the closed oriented surface of genus g such that the Euler number of the fibration is e.

COROLLARY 2. If $\mathcal{S}(e, g)$ has a flat conformal structure with non-surjective development map, then $|e| \leq C(2g-2, 2g-2)$.

1.4. The examples of hyperbolic manifolds $M = M(e, g)$ with $e = 0$ are easy to construct and they exist for arbitrary genus $g > 0$. If the surface F

of genus g is uniformized by the group $\Gamma \subset PSL(2,\mathbb{C})$ then the extension of Γ in $\mathbb{H}^4$ is the holonomy group for a complete hyperbolic structure on $M(0,g)$. The Bieberbach theorem implies that for $e > 0$ the manifolds $M(e,1)$ can't have complete hyperbolic structures. The case $g > 1$, $e > 0$ is less trivial, first examples of complete hyperbolic structures were obtained independently in [GLT] & [Ku] and [Ka 1]:

THEOREM 2. The manifold $M = M(e,g)$ admits a complete hyperbolic structure under the conditions:

[Ku 1]:

$$0 < e \leq (2g-2)/3$$

[Ka 1]:

$$0 < e \leq (2g-2)/22 \ .$$

After [GLT] M.Anderson [A] proved

THEOREM 3. Let $E \to B$ be an arbitrary vector bundle with the compact base B of negative sectional curvature. Then E admits a Riemannian metric with negatively pinched sectional curvature :

$$0 > a_E > K_E > -1$$

for some constant a_E depending on the bundle.

CONJECTURE 1 [GLT]. The (Milnor-Wood) inequality

$$0 \leq |e| \leq 2g-2$$

is the necessary condition for existence of complete hyperbolic structures on the manifold $M = M(e,g)$.

REMARK 2. It is important here that M is fibered. N.Kuiper [Ku 2] constructed a sequence of complete hyperbolic manifolds M_g^4 which are homotopy equivalent to closed surfaces F_g of genus g such that:

$$lim_{g\to\infty} < [F_g],[F_g] > /(2g-2) = 2/\sqrt{3} > 1$$

REMARK 3 (N.Kuiper [Ku 3], V.Marenich [Mar]). Suppose that M is hyperbolic and $\Sigma \subset M$ is an *imbedded* minimal surface of genus g. Then the Milnor-Wood inequality

$$| < [\Sigma],[\Sigma] > | \leq (2g-2)$$

holds.

1.5. Probably it's possible to prove Corollary 1 for convex compact hyperbolic 4-manifolds by comparing two η-invariants for flat conformal structures [Ka 2, 3]. More realistic idea was suggested to the author by M.Gromov who proposed to compactify the moduli space of all hyperbolic structures on the given fiber bundle. Formally speaking this idea doesn't work, since arbitrary large number of points of selfintersections of a zero section can be pinched to a point in the limit. However, what we are using in this paper are some "pre-limit" considerations based on Mamford's compactness theorem and the existence of the Margulis constant.

The idea of the proof is quite simple. Suppose that we realized the classes $[\sigma_1], [\sigma_2]$ by p-l surfaces Σ_1, Σ_2 in M such that :

the number and diameters of simplices in $\Sigma_1 \cup \Sigma_2$ are bounded from above. Then the existence of the universal Margulis constant and the fact

that 2 geodesic planes intersect transversally by not more than one point

immediately imply the assertion of Theorem 1. Certainly it's impossible in general to estimate from above the diameters of simplices since the diameters of the surfaces Σ_1, Σ_2 can be unbounded. However the "long" pieces of Σ_1, Σ_2 are contained in the "thin" part of the manifold M which have very simple topological structure. Then the detailed analysis of geometry in the "thin" part of M (section 2) and a correct choice of the surfaces representing $[\sigma_j]$ (sections 3, 4) give the desired result.

1.6. So Theorem 1 is a "0-th order approximation" to Conjecture 1. The simplest examples of negatively pinched closed manifolds of dimension 4 which do not admit hyperbolic structures are given by complex-hyperbolic manifolds. More sophisticated examples were constructed by Mostow and Siu [MS]; there are no (real) hyperbolic structures on any compact Kahler manifold of (real) dimension > 2 due to theorem of J.Carlson and D.Toledo [CT]. Another series of examples was constructed by Gromov and Thurston in [GT]. Theorem 1 combined with the theorem of Anderson presents the first examples of negatively curved open manifolds of dimension 4 that do not have complete hyperbolic structure, but homptopy-equivalent to hyperbolic manifolds. It's interesting to remark that these manifolds have finite-sheeted branched coverings which are hyperbolic (as well as some examples of Gromov and Thurston).

Certainly there is a gap between Theorems 1 and 3 and Conjecture 1.

CONJECTURE 2. There exists a function $D(.,.,.)$ such that for any

Riemannian 4-manifold M whose curvature is pinched as $0 > a \geq K_M \geq -1$ and for any classes $[Q]$, $[P]$ in $H_2(M, \mathbb{Z})$ we have:

$$| < [Q], [P] > | \leq D(\|[Q]\|, \|[P]\|, a)$$

where $\|W\| = min\{|\chi(W)|; w : W \to M$ is a surface representing the class $[W]\}$.

REMARK 4 (W.Goldman). Conjecture 2 isn't true for *orbifolds*. The example is given by complex-hyperbolic orbifolds covered by a nontrivial $\mathbb{R}^2$-bundle over surface. Namely, let $\Gamma \subset SU(1,1) \subset SU(2,1)$ be a co-compact torsion-free lattice in $SU(1,1)$. Then the manifold $M(\Gamma) = \mathbb{H}^2_{\mathbb{C}}/\Gamma$ is complex- hyperbolic and it admits an isometric $U(1)$-action. On other hand, $M(\Gamma)$ is diffeomorphic to the total space of a nontrivial $\mathbb{R}^2$-bundle over $S = \mathbb{H}^2_{\mathbb{R}}/\Gamma$ (2-sheeted ramified covering over the tangent bundle of S). The group $U(1)$ has cyclic subgroups Z_n of arbitrarily large order n. Then the sequence of orbifolds $M(\Gamma)/Z_n$ has the desired properties.

In particular we have:

CONJECTURE 2'. In Theorem 1 one can drop the condition for the cycles to be incompressible.

1.7. Being true Conjecture 2' would have several applications for flat conformal structures on 3-manifolds.

Suppose that we are given compact oriented geometric 3-manifolds N_1 and N_2 such that:

(1) the interiors of N_i have no Euclidean structures;

(2) $\partial N_i = T_i$ are incompressible tori $(i = 1, 2)$.

Denote by $N_1 \cup_f N_2$ the manifold obtained by gluing of N_1, N_2 via the homeomorphism $f : T_1 \to T_2$.

Fix N_j and consider various manifolds of the type $N_1 \cup_f N_2$.

COROLLARY 3. (i) If N_i are both hyperbolic then not more than finitely many manifolds $N_1 \cup_f N_2$ can be realized as incompressible ideal boundary components of a complete hyperbolic 4-manifolds. (This follows from Morgan's compactness theorem [Mor]).

If Conjecture 2' holds, then we have the conclusions (ii) and (iii):

(ii) If both manifolds are Seifert then not more than finitely many manifolds $N_1 \cup_f N_2$ can be realized as ideal boundary components of complete hyperbolic 4-manifolds.

(iii) If (say) N_1 is hyperbolic, N_2 is Seifert then infinitely many manifolds $N_1 \cup_f N_2$ can be realized as ideal boundary components of complete hyperbolic 4-manifolds (cf. [Ka 1, 4]).

However, if we fix the image of the regular fiber of N_2 then again there are only finitely many manifolds $N_1 \cup_f N_2$ which can be ideal boundary components of complete hyperbolic 4-manifolds.

On other hand it follows from [Ka 1, 4] that, unless the canonical decomposition of a Haken manifold N includes gluings of the type (i), the manifold N always has a finite-sheeted covering N_0 which is an ideal boundary component of a complete hyperbolic 4-manifold.

1.8. We split the proof of Theorem 1 in two cases.

Case 1 (Section 3). We shall suppose that the cycles σ_j satisfy certain condition of maximality.

CONDITION "MAX". If for some $h \in \pi_1(M)$ we have $1 \neq g \in h^{-1}\sigma_{1*}(\pi_1(\Sigma_1))h \cap \sigma_{2*}(\pi_1(\Sigma_2))$ then the maximal almost abelian subgroups of

$$\pi_1(M), h^{-1}\sigma_{1*}(\pi_1(\Sigma_1))h\ , \sigma_{2*}(\pi_1(\Sigma_2))$$

containing g are equal. For example, in Theorem 2 the condition "MAX" is fulfilled. (Cf. the definition of "doubly incompressible map" in [T 2]).

Case 2 (Section 4). This is the general case.

In the Case 1 the analysis of behavior of the surfaces $\sigma_j(\Sigma_j)$ in the "thin part" of the manifold M is more simple, that is why we decided to single out this case.

In the section 2 we discuss the geometry of components of the "thin" parts of hyperbolic 4-manifolds ("Margulis' tubes and cusps"). The following is the reason of difference between the dimensions 3 and 4. Let $< g >$ be an infinite cyclic discrete group of (orientation preserving) isometries of $\mathbb{H}^n$. Consider the set of points $\mathcal{K}(< g >, \mu) = \{x \in \mathbb{H}^n : d(x, g^k(x)) \leq \mu$ for some $k \neq 0\}$; define $q(x) =$ minimal $k > 0$ such that $d(x, g^k(x)) \leq \mu$. Then, for $n \leq 3$ the function q is constant on $\mathcal{K}(< g >, \mu)$. However it's not longer true for $n \geq 4$. In particular, if the element g is parabolic, then $q(x)$ can have infinitely many different values. If g is loxodromic, then the image of q is still finite, but it depends on the element g. Something similar occurs even for $n \leq 3$ if g doesn't preserve the orientation; however, in this case q can have not more than 2 different values.

1.9. There are several other results and conjectures that seems to be similar to Theorem 1 and Conjecture 1.

1.9.1. If C is a smooth curve of genus g in a complex surface X and K is a canonical class of X then (see [F]):

$$2g-2=< C,C > +K\cdot C$$

R.Kirby conjectured [Ki] that if a smooth embedded 2-manifold Σ is in the same homology class as C then the genus of Σ isn't less than g.

Conjecture of J.Morgan is that for any smooth oriented 4-manifold for which Donaldson's polynomials are defined and non-zero, and any smoothly embedded oriented surface $\Sigma\subset M$ with positive self-intersection one have the inequality

$$2g-2\geq\ <\Sigma,\Sigma>$$

I learned this information from the paper of P.Kronheimer [Kr] where the reader can find further information on this subject.

1.9.2. Suppose that M is a complex hyperbolic surface and $f: S_g\longrightarrow M$ is a homotopy-equivalence. Let ω_M denote the Kähler form on M. Then Domingo Toledo has proved [To 1] that the number

$$c=\frac{1}{2\pi}\int_{S_g}f^*\omega_M$$

is an integer independent of f which satisfies

$$2-2g\leq c\leq 2g-2$$

Furthermore Toledo [To 2] proved that M is a quotient by a cocompact lattice in $\mathrm{U}(1,1)$ if and only if $|c|=2g-2$.

In [GK] we proved that, subject to Toledo's necessary conditions, every value of c is realized by a complex hyperbolic surface $N(c,g)$ homeomerphic to $M(e=e(g,c),g)$ (see 1.3). In all these examples the selfintersection number $e=e(g,c)$ of the generator of $H_2(N(c,g),\mathbb{Z})$ varies in the closed interval $[1-g,2(1-g)]$. So, in particular, some of these manfolds are homeomorphic, but the actions of their fundamental groups on $\mathbb{H}^2_{\mathbb{C}}$ can't be deformed one to another inside the group $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$. On other hand, W.Goldman showed that in the examples [Ka 1] (see Theorem 1) all representations of $\pi_1(S_g)$ in $SO(4,1)$ are in the component of the trivial representation in $\mathrm{Hom}(\pi_1(S_g),SO(4,1))$.

1.9.3. The condition $2g - 2 \geq |\chi(E)|$ is necessary and sufficient for existence of a smooth foliation transversal to fibers of a smooth $\mathbb{S}^1$ bundle E over a surface of genus g (J.Wood [W]).

1.9.4. See also the paper of N.Mok [Mo].

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2. GEOMETRY OF MARGULIS TUBES

Many results of this section are well known in some form.

2.1. **NOTATIONS.** By $d(x, Y) = \inf \{d(x, y) : y \in Y\}$ we shall denote the (low) hyperbolic distance between the 1-point set $\{x\}$ and the set $Y \subset \mathbb{H}^4$. The ball with the center at x and the radius r will be denoted by $B(x, r)$. If $h \in \mathrm{Isom}(\mathbb{H}^4)$ then $l(h) = inf\{d(h(x), x) : x \in \mathbb{H}^4\}$ is the "length" of h. For each pair of points $a, b \in \mathbb{H}^4$ we shall denote by $[a, b]$ the geodesic segment connecting them. We denote by $[a, b, c]$ the totally geodesic triangle with the vertices $a, b, c \in \mathbb{H}^4$. If $G \subset \mathrm{Isom}(\mathbb{H}^4)$, $x \in \mathbb{H}^4$ then $Ir_G(x) = d(x, Gx)/2$ is the injectivity radius $InjRad([x])$ at the projection $[x]$ of x in $\mathbb{H}^4/G$. We shall assume that all groups below are torsion-free.

If h is a loxodromic or parabolic transformation in $\mathbb{H}^4$ then we denote by Π the canonical fibration of $\mathbb{H}^4$ by totally geodesic hyperplanes orthogonal to either axis of h (if h is loxodromic) or its 1-dimensional invariant horocycle (if h is parabolic). The projection of Π to $\mathbb{H}^4/ < h >$ will be called *canonical foliation* associated with $< h >$.

For the almost abelian group $H \subset \mathrm{Isom}(\mathbb{H}^4)$ define

$$\mathcal{K}(H, \mu) = \{z \in \mathbb{H}^4 : Ir_H(z) \leq \mu\} \tag{1}$$

to be the *Margulis cone.* The projection

$$\mathcal{T}(H, \mu) = \mathcal{K}(H, \mu)/H \tag{2}$$

is the *Margulis' tube* in $\mathbb{H}^4/H$ (we assume that Margulis tubes can be non-compact; in particular cusps are also considered as Margulis' tubes).

Suppose that the hyperbolic 4-space is realized as the "upper half-space", and the loxodromic element h is a Euclidean similarity. The main problem concerned with Margulis tubes in the dimension 4 (and higher) is that even for the cyclic loxodromic group $< h >= H \subset \mathrm{Isom}(\mathbb{H}^4)$ the boundary of the Margulis cone $\mathcal{K}(H, \mu)$ is very far from been a "round" cone (like in dimensions 2 and 3), but rather looks as a cone over an ellipsoid, where the ratio of the largest and smallest axes can be arbitrary large.

2.2. **LEMMA 1.** Let $x \in \mathbb{H}^4$ be a point such that: for some subgroup $H \subset \mathrm{Isom}(\mathbb{H}^4)$ we have: $Ir_H(x) \geq \nu$ for some positive ν. Then the ball $B(x, r)$ contains not more than

$$(exp^3(r + \nu))/\nu^3 \tag{3}$$

points from the orbit Hx.

LEMMA 2. Under the conditions of Lemma 1 the number of elements $h \in H \subset \mathrm{Isom}(\mathbb{H}^4)$ such that the intersection $h(B(x, r)) \cap B(x, r)$ is not empty is not more than

$$(exp^3(2r + \nu))/\nu^3 \tag{4}$$

LEMMA 3. Let H be a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ and $\nu/2 = Ir_H(x)$. Suppose that $d(x, y) < r$. Then $Ir_H(y) > C_1(r, \nu)/2$. Here

$$C_1(r, \nu) = 2r/n(r, \nu)\ ,\ n(r, \nu) = [\frac{\exp(18r + 2\nu)}{\nu^3}] + 1 \tag{5}$$

PROOF.

Let h be an arbitrary nontrivial element of the group H. Let n_0 be such that $d(x,\ h^{n_0}(x)) \geq 3r$. Then $d(y,\ h^{n_0}(y)) \geq r$ and $d(y,\ h(y)) \geq r/n_0$. So, our aim is to estimate n_0 from above. Notice that for $n = n(r, \nu)$ among the elements

$$\{1,\ h,\ \dots,\ h^n\}$$

there is h^k such that $d(x,\ h^k(x)) \geq 3r$ (by Lemma 1). Then we can take $n_0 \leq n$ and $d(y, h(y)) \geq r/n$ for every $h \in H - \{1\}$.

Lemma 3 is proved.

2.3. In Lemmas 4 and 5 below we shall prove the KEY PROPERTY of INDEX (Corollary 5) which will be crucial in our paper.

KEY PROPERTY of INDEX. Let h be a parabolic or loxodromic isometry of $\mathbb{H}^4$, $x \in \mathbb{H}^4$. The index $ind_h(x)$ of x is just $d(x, hx)$. Suppose that x is such that: $Ir_{<h>}(x) > \mu$ and there is a geodesic segment $L = [a, b]$ such that $d(x, L) < R$ for some R and $max\{ind_h(x),\ ind_h(a),\ ind_h(b)\} < R$. Then we shall prove that either

$$min\{d(x,a), d(x,b)\} < C_+(R,\mu)$$

(parabolic alternative) or

$$l(h) > C_-(R,\mu) > 0$$

(hyperbolic alternative). See Figure 1, where $A = A_h$ is the axis of the loxodromic transformation h.

REMARK 5. The Key Property is valid also if instead of $\mathbb{H}^4$ we consider arbitrary simply connected complete space X of the sectional curvature $K_X < -a^2 < 0$.

We shall prove the KEY PROPERTY just for the loxodromic h, the parabolic case easily follows.

Below we assume that the hyperbolic 4-space $\mathbb{H}^4$ is realized as the upper half space $\mathbb{R}^4_+$; $|X - Y|$ denotes the Euclidean distance between points X, Y; $\partial_\infty \mathbb{H}^4 = \overline{\mathbb{R}^3} = \mathbb{R}^3 \cup \infty$.

2.4. Suppose that $g = \theta \circ \lambda$ is a similarity in $\mathbb{E}^4$ preserving $\mathbb{H}^4$, $g(0) = 0$, A is the axis of g, $d(z, A) > 2$. Here λ is the homothety $\lambda : x \to \lambda x$ and θ is the rotation on the angle θ. Let L be the geodesic containing the points ∞, z; let $w \in L$ be a point such that z lies between w and ∞. (See Figure 2).

LEMMA 4. Suppose that the under conditions above:

$$\nu \leq d(g(z), z) \leq R;\ \ d(g(w), w) \leq R$$

Then $d(z, w) \leq R + \frac{1}{\nu}$.

PROOF. *Step 1.* Put $g(z) = z'$ $, g(w) = w'$. Denote by $\alpha(u)$ the angle between the horosphere P with center at ∞ containing the point z and the Euclidean line passing through the points z, u ; $\alpha(u) \leq \pi/2$. Then the condition $d(z, A) > 2$ guarantees that $\alpha(\lambda z) \leq \pi/3$. However $|z - \theta\lambda z| \geq |z - \lambda z|$, then

$$\beta = \alpha(gz) \leq \alpha(\lambda z) \leq \pi/3$$

Step 2. Due to the Step 1 it's sufficient for us to consider the case: $z,\ z' = \lambda z,\ w,\ w' = \lambda w \in \mathbb{H}^2 \subset \mathbb{C}$, $\arg(z) = \beta \leq \pi/3$; $d(z,w) = d(z',w')$, $Re(z) = Re(w)$. (Figure 3). Without loss of generality we can suppose that $Im(z) = 1, y = Im(z')$.

Now we have: $\rho = |z - z'|, y = \rho\sin(\beta) + 1$,

$q = 2\sinh\frac{d(z,z')}{2} = \frac{\rho}{\sqrt{y}} = \frac{\rho}{\sqrt{1+\rho\sin(\beta)}}$.

So $q^2 + \rho q^2\sin(\beta) - \rho^2 = 0$,

$$\rho = (q^2\sin(\beta) + \sqrt{q^4\sin^2(\beta) + 4q^2})/2 \geq q \geq 2\sinh\frac{\nu}{2} \tag{6}$$

On other hand we have: $\sin\beta \leq \sqrt{3}/2$, so $\rho \leq 2q^2 + 2 \leq 8\sinh^2(R/2) + 2$,

$$\sinh\nu/2 \leq \sinh d(z,z")/2 = |z - z"|/2 = \rho(\cos\beta)/2$$

$$\leq (8\sinh^2(R/2) + 2)\cos\beta \leq 8\sinh^2(R/2) + 2$$

and $d(w,w') \geq d(w,w") - d(w',w") \geq d(w',w") - R$.

Let $s = Im(w)$, then $d(z,w) = \log(1/s)$ and

$$\sinh(d(w,w")/2) = |z - z"|/(2s) \geq \rho/(4s) \geq \sinh(\nu/2)/(2s) \tag{7}$$

$$s \geq \frac{1}{2}\sinh(\nu/2)\sinh^{-1}\frac{d(w,w")}{2} \geq \frac{1}{2}\sinh(\nu/2)\sinh^{-1}(R) \tag{8}$$

since $d(w,w") \leq R + d(w,w')$ and $d(w,w') \leq R$. Now

$$d(z,w) = \log(1/s) \leq \log(2\sinh^{-1}(\nu/2)\sinh R) \tag{9}$$

However $\log\sinh a = (a^2 - 1)/2a$ and $2\sinh b \leq e^b$. Therefore: $d(z,w) \leq R + \frac{1}{\nu}$.

Lemma 4 is proved.

Now suppose that $a, b, z \in \mathbb{H}^4$ be points such that: $d(z,[a,b]) \leq R$

Denote by L_a, L_b the geodesic rays connecting the points a, b and ∞.

PROPOSITION 4. Under the conditions above we have:

$$\min\{d(z,L_a)\ ,d(z,L_b)\} \leq 2 + R \tag{10}$$

PROOF. Denote by c the point of $[a,b]$ such that $d(c,z) \leq R$. Let L'_x be the geodesic containing L_x. Then:

$$\cosh d(c, L'_a) = \sin^{-1}\alpha, \ \cosh d(c, L'_b) = \sin^{-1}\beta \tag{11}$$

(see Figure 4) and $\alpha + \beta \geq \pi/2$ so

$$\sin^2\alpha + \sin^2\beta \geq 1 \tag{12}$$

Now there are two opportunities (up to change $\beta \to \alpha$):
(i) $b_4 \geq w_4$ for every $w \in [a,b]$
(ii) else.
Consider (ii). Then $\phi < \pi/2$, $\psi < \pi/2$ where ϕ, ψ are nonzero angles of the triangle formed by L_a, L_b, $[a,b]$. Therefore:

$$d(c, L'_a) = d(c, L_a) \ , \ d(c, L'_b) = d(c, L_b) \tag{13}$$

Now if $\sin^2\alpha \leq 1/4$ then $\sin^{-1}\beta \leq 2$.
This means that

$$\min\{\cosh d(c, L_a), \cosh d(c, L_b)\} \leq 2 \tag{14}$$

so $\min\{d(z,L_a), d(z,L_b)\} \leq 2 + R$ in the case (ii).
Case (i). Then $\phi < \pi/2$, $\psi > \pi/2$ however $\alpha \geq \pi/4$ (since the arc of the geodesic passing through a, b is greater then the quarter of circle).
Then $1/\sin\alpha \leq \sqrt{2}$ and $e^x/2 \leq \cosh x = \cosh(d(c, L'_a)) \leq \sqrt{2}$; $x \leq \log(3) < 2$.
However $\phi < \pi/2$, then $d(c, L'_a) = d(c, L_a)$ that means $d(c, L_a) < 2$.
Therefore

$$\min\{d(z, L_a), d(z, L_b)\} \leq 2 + d(c,z) \leq 2 + R \tag{15}$$

QED.

2.5. **LEMMA 5.** Suppose that g is a loxodromic element with the axis A as in Lemma 4; $a, b, z \in \mathbb{H}^4$ be points such that: $d(z,[a,b]) \leq R$, $d(z,A) \geq 2 + R$, $\nu < d(g(z),z) < R$, $d(a,g(a)) < R$, $d(b,g(b)) < R$, so the points z, a, b have bounded index with respect to $< g >$.

Then: $\min\{d(z,a),d(z,b)\} \le 4R+6+1/k$,
where $k(R,\nu) = k = 2(2+R)\nu^3/\exp(18(2+R)+2\nu)$.

PROOF. According to Proposition 4 we can assume that: $d(z,L_a) \le 2+R$. Let $u \in L_a$ be a point such that $d(z,u) < 2+R$. Then we have: $d(u,A) > 2$, $k < d(g(u),u) < 3R+4$ (the last is by Lemma 3). Now we can apply Lemma 4 to the a,u to obtain: $d(a,u) < 3R+4 = 1/k$ and $d(z,a) < 4R+6+1/k$.

QED.

COROLLARY 5 (KEY PROPERTY OF INDEX). Let h be a parabolic or loxodromic isometry of $\mathbb{H}^4$, $x \in \mathbb{H}^4$ is such that: $Ir_{<h>}(x) > \mu$ and there is a geodesic segment $L=[a,b]$ such that $d(x,L) < R$ for some R and

$$max\{ind(x),\ ind(a),\ ind(b)\} < R$$

Then either

$$min\{d(x,a),d(x,b)\} < C_+(R,\mu) = 4R+6+1/k$$

where $k = 2(2+R)\mu^3/\exp(18(2+R)+2\mu)$ (parabolic alternative)

or $l(h) > C_-(R,\mu) = C_1(R+2,\mu) > 0$ where the function C_1 is defined by Lemma 3 (hyperbolic alternative).

PROOF. Combine Lemma 5 and Lemma 3. QED.

Denote by $C(t,A) = \{w \in \mathbb{H}^2 : d(w,A) = t\}$ the "hypercycle" whose axis is the geodesic A.

PROPOSITION 6. Let z_1, z belong to a connected component of $C(t,A)$. Then

$$d_{C(t,A)}(z_1,z) \le 2\sinh(d(z_1,z)/2) \tag{16}$$

where d_C is the metric on $C = C(t,A)$ induced from the hyperbolic plane.

PROOF. We can suppose that $|z_1| = 1, |z| = r$, $\log r$ is just the distance between projections of z_1, z on the geodesic A. Then

$$2\sinh(d(z_1,z)/2) = \frac{r-1}{\sin(\theta)\sqrt{r}} \tag{17}$$

for $\cosh(t)\sin\theta = 1$. Here $\pi - 2\theta$ is the Euclidean angle at the vertex of $C(t,A)$. Moreover, $a = d_C(z_1,z) = \log(r)/\sin\theta$, $a\sin\theta = \log r$. Our aim is to show that:

$$log(r) \le \frac{r-1}{\sqrt{r}} = \sqrt{r} - \frac{1}{\sqrt{r}} \tag{18}$$

if $r \geq 1$. Let $x = \sqrt{r}$, then $2\log x \leq x - 1/x$ since for $x = 1$ we have the equality and derivative of the left side is $\leq$ of derivative of the right side. **QED.**

REMARK 6. In the situation above we have also:

$$\cosh t \leq 2\frac{1}{\log r} \ \sinh \frac{d(z_1, z)}{2}$$

2.6. Now until 2.13 let $q \in Isom(\mathbb{H}^4)$ be a nonelliptic isometry. If q is loxodromic then we shall suppose that q fixes $\{0, \infty\}$ and 0 is the repulsive point. If q is parabolic then we shall assume that q fixes the point ∞ (i.e. q is an isometry of the Euclidean space).

Remind that $p : \mathbb{H}^4 \to \mathbb{H}^4 / < q >$ is the covering map;

$$\mathcal{K}(< q >, \nu) = \{x \in \mathbb{H}^4 : \inf\{d(x, g(x)) : g \in < q > -\{1\}\} \leq \nu\} ,$$

$\mathcal{T}(< q >, \nu) = p(\mathcal{K}(< q >, \nu))$.

For the element $T \in < q >$ we denote by T_θ its rotational component and $T_\lambda = T_\theta^{-1}T$. Also put $0 \leq \theta_T \leq \pi$ be the angle of rotation of T_θ; denote by λ_T either the Euclidean distance $|X - T_\lambda(X)|$ (in the case of parabolic T) or the coefficient of similarity (in the loxodromic case). If the rotational component of q isn't trivial, put R_x to be the Euclidean distance from the point $x \in \mathbb{H}^4$ to the Euclidean plane of rotation L_q. For loxodromic q define A_q to be the geodesic axis of q. If $\theta_q = 0$ then we put $L_q = \emptyset, R_x = 0$. Direct calculations show that:

$$|T(x) - x|^2 = 2R_x^2(1 - \cos\theta_T) + \lambda_T^2 \tag{19}$$

(in the parabolic case)

$$|T(x) - x|^2 = (2R_x^2(1 - \cos\theta_T) + \lambda_T^2)|x|^2 \tag{20}$$

(in the loxodromic case). Therefore

$$2\sinh^2(d(x, T(x)/2) = (2R_x^2(1 - \cos\theta_T) + \lambda_T^2)/x_4^2 \tag{21}$$

(in the parabolic case)

$$2\sinh^2(d(x, T(x)/2) = (2R_x^2(1 - \cos\theta_T) + (\lambda_T - 1)^2/\lambda_T)|x|^2/x_4^2 \tag{22}$$

(in the loxodromic case).

We shall denote $\sqrt{2}\sinh(d(T(x),x)/2)$ by x_T; put

$$x_T^\theta = |x|R_x\sqrt{2(1-\cos\theta_T)}/x_4$$

and $x_T^\lambda = \sqrt{x_T^2 - (x_T^\theta)^2}$.

The formulas (21), (22) imply that the domain $\mathcal{K}(< q >, \nu)$ is convex near its smooth boundary points, where the boundary from the Euclidean point of view is a piece of a cone (in the loxodromic case) or a cylinder (in the parabolic case) over an ellipsoid of revolution or 2-sheeted hyperboloid of revolution. In smooth points the boundary $\partial\mathcal{K}(< q >, \nu)$ is given by the equation

$$x_{q^{n(x)}} = \sqrt{2}\sinh(\nu/2) \tag{23}$$

However in nonsmooth boundary points the domain $\mathcal{K}(< q >, \nu)$ is not locally convex. Nevertheless the following remark will be important for us.

REMARK 7. Take an arbitrary fiber Π_t of the canonical fibration and let $\mathbb{H}^2 \subset \Pi_t$ be any hyperbolic plane which contains the axis of rotation $L_q \cap \Pi_t$. Then the curve

$$\partial\mathcal{K}(< q >, \nu) \cap \mathbb{H}^2 \cap \Pi_t \tag{24}$$

is given by the equation $x_4 = x_4(R_x)$ which is an increasing function on R_x. In the both parabolic and loxodromic cases this equation is the equation of hyperbola (in smooth points); the difference between parabolic and loxodromic cases is that in the last case the domain of the function is bounded by (say) $R_x \leq 1$. (See Figure 5.)

Suppose $H \subset Isom(\mathbb{H}^4)$ is an almost abelian discrete subgroup; $H(\infty) = \infty$. If H contains a loxodromic element q then denote by $\mathbb{H}^4_*$ the complement in $\mathbb{H}^4$ to the axis $A = A_q$ of q and we put $\mathbb{H}^4_* = \mathbb{H}^4$ in the parabolic case.

Let $\varphi = \varphi_{H,\nu} : a \in \mathbb{H}^4_* \to \partial\mathcal{K}(H,\nu)$ be the projection $\varphi(a) =$ the point of intersection of $\partial\mathcal{K}(H,\nu)$ with the geodesic containing a , ∞ (in the parabolic case), $\varphi(a) =$ the point of intersection of $\partial\mathcal{K}(H,\nu)$ with the semigeodesic containing a and orthogonal to A_q (in the loxodromic case).

2.7. **LEMMA 6.** (See also [B], [SY], [HI]). The map φ is defined correctly.

PROOF. Firstly the formulas (21-22) imply that the set $\mathcal{K}(H,\nu)$ is star-like with respect to the point ∞. If H contains loxodromic element q

and $z \in \partial\mathcal{K}(H,\nu)$ then we can use the fact that the whole Euclidean ray $K_z = \{c \cdot z : c \in \mathbb{R}_+\}$ is also contained in $\partial\mathcal{K}(H,\nu)$. Therefore the domain in the Euclidean plane between the rays A_q and K_z is contained in $\mathcal{K}(H,\nu)$. Hence the point of intersection defining φ in the loxodromic case is unique. Now it follows from (3-4) that the intersection of $\mathcal{K}(H,\nu)$ with any geodesic ending at the point ∞ isn't empty. **QED**.

PROPOSITION 7. Suppose that $Ind_g(a) \leq R$; $\nu \leq R$. (i) Then either $d(a, \varphi(a) \in \mathcal{K}(< g >, \nu)) \leq C_+(R,\nu)$ or $l(g) \geq C_-(R,\nu)$ and $\cosh d(a, A_g) \leq \frac{2\sinh R/2}{C_-(R,\nu)}$. (ii) If g is parabolic then $d(a, \varphi(a) \in \mathcal{K}(< g > , \nu)) \leq 1 + R/2 - \nu/2$.

PROOF. (i) Follows from Lemma 5 and Corollary 5. (ii) Suppose that $d(b, g(b)) \geq \nu$. Then $\frac{\sinh^2(R/2)}{\sinh^2(\nu/2)} \geq b_4^2/a_4^2$ where $b = \varphi(a)$. Now the statement (ii) follows from direct calculations. QED.

2.8. Let $x, z \in \mathbb{H}^4$ be a pair of distinct points, $T \in < q > -\{1\}$. Define the film $S = S_{Txz}$ in $\mathbb{H}^4$ connecting points $x, z, T(z), T(x)$ as follows.

Let $T = T_\theta \circ T_\lambda; T_\theta = \exp(\xi), T_\lambda = \exp(\zeta)$; $\xi = \xi_T, \zeta = \zeta_T \in so(4,1)$, where $\exp : t \in [0,1] \cdot \xi \to SO(4,1)$ is injective.

Then we put

$$S^\lambda = \bigcup_{t \in [0,1]} exp(t\zeta)([x,z]) \tag{25}$$

$$S^\theta = \bigcup_{t \in [0,1]} exp(t\xi)(g_\lambda[x,z]) \tag{26}$$

$$S = S^\lambda \cup S^\theta$$

$$\partial_x S = \bigcup_{t \in [0,1]} exp(t\zeta)(x) \cup \bigcup_{t \in [0,1]} exp(t\xi)(T_\lambda x) \tag{27}$$

$$\partial_z S = \bigcup_{t \in [0,1]} exp(t\zeta)(z) \cup \bigcup_{t \in [0,1]} exp(t\xi)(T_\lambda z) \tag{28}$$

$$\delta S = \partial_x S \cup \partial_z S \tag{29}$$

The films $S = S_{Txz}$ constructed above will be called "ruled films". It's easy to see that number of points of transversal intersection of ruled film with another ruled film (or geodesic plane in $\mathbb{H}^4$) isn't greater than 8.

REMARK 8. The film S_{Txz} is contained in $\mathcal{K}(< q >, \nu)$.

The ruled film S_{Txz} is said to be in the general position if:

(i) $[x, z] \subset \mathbb{H}^4_* - L_q$;

(ii) In the loxodromic case let $\mathbb{H}^3_{[x,z]}$ be the geodesic hyperplane of $\mathbb{H}^4$ which contains $[x, z] \cup A_q$. Then $\mathbb{H}^3_{[x,z]}$ isn't orthogonal to L_q.

2.9. Suppose that we are given a ruled film S_{gxz}, then $\partial_x S_{gxz}$ consists of two arcs:

$$\delta_x^\lambda = \bigcup_{t \in [0,1]} exp(t\zeta)(x)$$

$$\delta_x^\theta = \bigcup_{t \in [0,1]} exp(t\xi)(T_\lambda x).$$

Then take totally geodesic regions $D(x, g_\lambda(x))$ and $D(g_\lambda(x), g(x))$ bounded by $\delta_x^\lambda \cup [x, g_\lambda x]$ and $\delta_x^\theta \cup [g_\lambda x \ , gx]$ respectively.

Define the totally geodesic regions $D(z, g_\lambda(z))$ and $D(g_\lambda(z), g(z))$ in the same way.

DEFINITION 2. The film

$$\tilde{Q}_{gxz} = S_{gxz} \cup D(x, g_\lambda(x)) \cup D(g_\lambda(x), g(x)) \cup$$

$$D(z, g_\lambda(z)) \cup D(g_\lambda(z), g(z)) \cup [x, g_\lambda x \ , gx] \cup [z, g_\lambda z \ , gz]$$

is called the *extended ruled film* . If $g \in G \subset Isom(\mathbb{H}^4)$ then the projection Q_{gxz} of $\tilde{Q}_{gxz}$ to $\mathbb{H}^4/G$ is called the *extended ruled annulus.* Define

$$\partial_x Q_{gxz}, \partial_z Q_{gxz}$$

to be the projections of $[x, gx], [z, gz]$ in $\mathbb{H}^4/G$. Then

$$\partial_x Q_{gxz} \cup \partial_z Q_{gxz} = \partial Q_{gxz}$$

(see Figure 6).

2.10. **LEMMA 7.** Suppose that the film S_{gxz} is in the general position. Then $p(\partial_x S_{gxz})$, $p(\partial_z S_{gxz})$ are homologous in

$$\partial \mathcal{T}(< q >, \nu) - p(L_q \cap \partial \mathcal{K}(< q >, \nu)) .$$

Moreover, the intersection $p(S_{gxz}) \cap p(L_q \cap \partial \mathcal{K}(< q >, \nu))$ is empty.

PROOF. Due to the property (i) of a film in the general position we have: $[x, z] \cup L_q = \emptyset$. However S_{gxz} results via moving the segment $[x, z]$ by elements contained in $Z_{<q>}$ which leave L_q invariant. **QED**.

2.11. Suppose that $x, y, z, w \in \partial \mathcal{K}(< q >, \nu)) \cap \Pi_t$ - one and the same fiber of the canonical fibration associated with q; $g, h \in< q > -\{1\}$,

$$max\{d(g(x), x), d(g(y), y), d(g(z), z), d(g(w), w)\} \leq C \ , \qquad (30)$$

S_{gxz}, S_{hyw} are in general position, $p(\partial S_{gxz}) \cap p(\partial S_{hyw}) = \emptyset$ and the intersection $p(\partial S_{gxz}) \cap p(\partial S_{hyw})$ is transversal. Our purpose is to estimate the algebraic intersection number between the annuli

$$p(S_{gxz}, \partial S_{gxz}), p(S_{hyw}, \partial S_{hyw}) \subset (\mathcal{T}(< q >, \nu), \partial \mathcal{T}(< q >, \nu))$$

or, equivalently, the algebraic intersection number between S_{gxz} and $< q >$ (S_{hyw}). Notice that this number depends only on

$$p(\partial S_{gxz}), p(\partial S_{hyw}) \subset \partial \mathcal{T}(< q >, \nu)$$

and doesn't depend on the relative cycles in $\mathcal{T}(< q >, \nu)$ with the boundaries $p(\partial S_{gxz})$, $p(\partial S_{hyw})$.

THEOREM 4. Under the conditions above we have the following estimate for the algebraic intersection number:

$$| < p(S_{gxz}), p(S_{hyw}) > | \leq N(C, \nu) \qquad (31)$$

where

$$N(C, \nu) = (\exp^3(4C + 4))/\nu^3 \qquad (32)$$

PROOF. First consider the most interesting case $\theta_q \neq 0$. Lemma 7 implies that if $p(Z_{<q>}(x))$ doesn't divide say $p(Z_{<q>}(y))$ from $p(Z_{<q>}(w))$ then the annulus

$\mathcal{A} = p(S_{hyw})$ can be deformed $rel(\partial \mathcal{A})$ to the new annulus $\mathcal{A}'$ so that $\mathcal{A}' \cap p(\partial_x S_{gxz}) = \emptyset$ and $\#(\mathcal{A}' \cap p(\partial_z S_{gxr})) \leq \#(\mathcal{A} \cap p(\partial_z S_{gxz}))$.

Therefore (up to change of notations) our problem is reduced to the case:

$$R_x \leq R_y \leq R_z \leq R_w \qquad (33)$$

(if q is parabolic) and

$$\min\{R_x, R_z\} \leq R_y \quad , \quad \min\{R_y, R_w\} \leq R_z \tag{34}$$

(if q is loxodromic). The monotonicity of the boundary $\partial\mathcal{K}(<q>, \nu)$ (see Remark 7) implies that:

$$\min\{x_4, z_4\} \leq y_4 \quad , \quad \min\{y_4, w_4\} \leq z_4 \tag{35}$$

If either

$$diam(\varphi(S_{hyw}) \cap (<q> \partial_z S_{gxz})) \leq Const(C, \nu) = 2\sqrt{C+1} \tag{36a}$$

or

$$diam(\varphi(S_{gxz}) \cap (<q> \partial_y S_{hyw})) \leq Const(C, \nu) = 2\sqrt{C+1} \tag{36b}$$

then we can apply Lemma 3 to obtain

$$| < p(S_{gxz}), p(S_{hyw}) > | \leq \exp^3(2Const(C, \nu))/\nu^3 = \exp^3(4C+4)/\nu^3$$

So, our goal is to obtain one of such estimates. The inequalities (35) imply that

$$y_g^\lambda \leq \max\{x_g^\lambda, z_g^\lambda\} \leq \sqrt{2}\sinh(C/2); z_g^\lambda \leq \max\{y_g^\lambda, w_g^\lambda\} \leq \sqrt{2}\sinh(C/2)$$

Therefore, consider

$$z_g^\theta / z_h^\theta = y_g^\theta / y_h^\theta = (\sin\theta_g/2)/(\sin\theta_h/2) = \alpha$$

Now if $\alpha \geq 1$ then $z_g^\theta \geq z_h^\theta$; if $\alpha \leq 1$ then $y_g^\theta \leq y_h^\theta$. So either

$$Ind_h(z) = d(z, h(z)) \leq \sqrt{arcsinh(4\sinh^2 C/2)} = C' \leq \sqrt{C+1} \tag{37}$$

or

$$Ind_g(y) = d(y, g(y)) \leq \sqrt{arcsinh(4\sinh^2 C/2)} = C' \leq \sqrt{C+1} \tag{38}$$

We shall assume that (37) holds.

Notice that the intersection

$$\varphi(S_{hyw}) \cap (< q > \partial_z S_{gxz})$$

is contained in $\varphi(S_{huw}) \cap Z_{<q>}(z)$. Moreover, because our films are in general position (condition (ii)) we have $\varphi[u,w] \cap Z_{<q>}(z) = \{s_1, s_2\}$ where s_1, s_2 can coincide.

Therefore

$$\varphi(S_{huw}) \cap Z_{<q>}(z) =$$

$$\bigcup_{i=1,2} \bigcup_{t\in[0,1]} exp(t\zeta_h)(s_i) \cup \bigcup_{t\in[0,1]} exp(t\xi_h)(h_\lambda s_i). \tag{39}$$

However $Ind_h(s) = Ind_h(z) \leq C'$ due to (37); hence the diameter of the set in (39) is bounded from above by $2C'$ and we are done.

2.12. Now suppose that the rotational component of q is trivial. Then the condition (30) implies that $m, n \leq [C/\nu]$. Take $x, y, z, w \in \Pi_t$. So the films S_{hyw}, S_{gxz} lie in the union of $[C/\nu]$ images of a convex fundamental domain of the group $< q >$. Hence the intersection number isn't greater than $8[C/\nu] \leq N(C,\nu)$. **QED.**

2.13. Consider the case when g, h are parabolic elements which belong to an almost abelian group $\Gamma \subset Isom(\mathbb{H}^4)$ such that Γ isn't a cyclic group. Again suppose that $x, y, z, w \in \partial\mathcal{K}(\Gamma,\nu)$ are points such that the condition (30) is fulfilled. Denote by $p : \mathbb{H}^4 \to \mathbb{H}^4/\Gamma$ the universal covering. Our aim is to estimate the intersection number between pS_{gxz} and pS_{hyw} in terms of C, ν.

THEOREM 5. The intersection number between pS_{gxz} and pS_{hyw} is bounded from above by the constant

$$N'(C,\nu) = (\exp(9\nu + 6))/\nu^3 + 96C/\nu + 120000\exp(72C)/\nu^3 \tag{40}$$

PROOF.

2.14. Let $\Gamma_0 \subset \Gamma$ be a maximal abelian subgroup in Γ; $|\Gamma : \Gamma_0| \leq 12$. Denote by $g_0 = g^{n_g}$, $h_0 = h^{n_h}$ generators of the groups $< g > \cap\Gamma_0$, $< h > \cap\Gamma_0$ respectively. Then n_g and n_h are not greater than 12. Notice also that

$$\partial\mathcal{K}(\Gamma_0,\nu) \subset \mathcal{K}(\Gamma,\nu) - \mathcal{K}(\Gamma,\nu/12)$$

and $\partial\mathcal{K}(\Gamma_0,\nu)$ is a horosphere. For every $u\in\partial\mathcal{K}(\Gamma,\nu)$ by (21) we have

$$d(u,\partial\mathcal{K}(\Gamma,\nu/12))\leq\log(\sinh(\nu/2)/\sinh(\nu/24))\leq\nu/2+1. \tag{41}$$

Therefore for u as above

$$d(u,\partial\mathcal{K}(\Gamma_0,\nu))\leq\nu/2. \tag{42}$$

Denote by $p_0:\mathbb{H}^4\rightarrow\mathbb{H}^4/\Gamma_0$ the universal covering.

2.15. First we reduce the problem to the abelian subgroup Γ_0.

(i) $|<pS_{gxz},pS_{hyw}>|\leq 12|<p_0(<g>S_{gxz}),p_0(<h>S_{hyw})>|$, where HB denotes the orbit of the set B under the group H;

$$p_0(<g>S_{gxz})=p_0(S^0_{gxz});(S^0_{gxz})=S_{gxz}\cup gS_{gxz}\cup g^2S_{gxz}...\cup g_0S_{gxz}$$

(ii) $diam(\{u,Tu,T^2u,...,T_0u\})\leq 12C$ for $u\in\{x,z\},T=g$ and $u\in\{y,w\},T=h$.

Now we enlarge the films S^0_{gxz},S^0_{hyw} in the following way. Let $u\in\{x,y,z,w\}$, $T\in\{g,h\}$ be the corresponding transformation. Then take the union

$$\mathcal{A}^-_u=[u,Tu,T^2u]\cup[u,T^2u,T^3u]\cup...\cup[u,T^{n_u-1}u,T^{n_u}u]$$

$$\mathcal{A}_u=D(u,T_\lambda(u))\cup D(T_\theta(u),T(u))\cup D(T(u),T_\lambda(Tu))\cup...$$
$$\cup D(T^{n_u-1}u),g_\lambda(T^{n_u-1}u))\cup D(g_\lambda(T^{n_u-1}u),T_0(u))\cup D(u,T^{n_u}u)\cup\mathcal{A}^-_u$$

(see 2.9). Then put

$$S^+_{gxz}=S^0_{gxz}\cup\mathcal{A}_x\cup\mathcal{A}_z$$

$$S^+_{hyw}=S^0_{hyw}\cup\mathcal{A}_y\cup\mathcal{A}_w$$

(Figure 7).

All extra pieces that are attached to S^0_{hyw},S^0_{gxz} have diameter $\leq 12C$ and number of them is bounded by 100. Therefore

$$|<p(S_{gxz}),p(S_{hyw})>|\leq 12|<p_0(S^+_{gxz}),p_0(S^+_{hyw})>|+12\cdot 10^4\exp^3(24C)/\nu^3$$

$$|<p_0(S^+_{gxz}),p_0(S^+_{hyw})>|=|<p_0(S_{g_0xz}),p_0(S_{h_0yw})>|$$

since $\partial p_0(S^+_{gxz}) = \partial p_0(S_{g_0xz})$, $\partial p_0(S^+_{hyw}) = \partial p_0(S_{h_0yw})$.

Denote by Γ_1 the maximal subgroup in Γ_0 which has the rank 2 and contains g_0, h_0. We have two possibilities:

(a) Γ_0 has rank 3. (b) Γ_0 has rank 2.

In the case (a) denote by $k_0 \in \Gamma_0$ an element such that $\Gamma_0 =< k_0 > \oplus \Gamma_1$.

Choose a fundamental domain Φ for action of $< k_0 >$ in the horosphere $\partial\mathcal{K}(\Gamma_0 \nu)$ such that Φ is bounded by a pair of Euclidean hyperplanes in $\partial\mathcal{K}(\Gamma_0, \nu)$. In the case (b) let $k_0 = 1$ and $\Phi = \partial\mathcal{K}(\Gamma_0, \nu)$.

Let $\varphi : \mathbb{H}^4 \to \partial\mathcal{K}(\Gamma_0, \nu)$ be the projection defined in 2.6. Then $d(u, u' = \varphi u) \leq 1 + \nu/2$ for every $u \in \partial\mathcal{K}(\Gamma, \nu)$.

For $u \in \{x, z, y, w\}$ we choose $T_u \in < k_0 >$ such as $T_u u' \in \Phi$. Denote $T_u(u')$ by u''. Then above we substitute the films S_{g_0xz}, S_{h_0yw} by $S_{g_0T_x(x')T_z(z')}, S_{h_0T_y(y')T_w(w')}$.

The intersection number between the projections of the ruled films S_j in $\mathbb{H}^4/\Gamma_0$ depends only on projections of their boundaries δS_j. Therefore:

$$| < p_0(S_{g_0xz}), p_0(S_{h_0yw}) > | \leq (\exp(6 + 9\nu))/\nu^3 +$$

$$| < p_0(S_{g_0x''z''}), p_0(S_{h_0y''w''}) > | \, .$$

Next notice that $\Gamma_1(\varphi S_{g_0x''z''}), \Gamma_1(\varphi S_{h_0y''w''}) \subset \Phi$ are Euclidean parallelograms. Hence we can estimate the intersection number

$$| < p_0(S_{g_0x''z''}), p_0(S_{h_0y''w''}) > |$$

in the same way as in 2.11 looking at the intersections of orbit of $\varphi\delta S_{g_0x''z''}$ with $\varphi S_{h_0y''w''}$ under the action of the group Γ_1. However

$$|\Gamma_1 :< g_0 > \oplus < h_0 > | \leq 12C/\nu \tag{43}$$

which implies that

$$| < p_0(S_{g_0x''z''}), p_0(S_{h_0y''w''}) > | \leq 96C/\nu \tag{44}$$

So, the final estimate is

$$< pS_{gxz}, pS_{hyw} > \leq (\exp(6+9\nu))/\nu^3 + 96C/\nu + 120000 \exp(72C)/\nu^3 \tag{45}$$

QED.

3. PROOF of THEOREM 1 UNDER CONDITION OF MAXIMALITY.

3.1. NOTATIONS.

Fix some Margulis constant μ for the 4-dimensional hyperbolic space. So if h_1 , $h_2 \in Isom(\mathbb{H}^4)$ generate a discrete group H and $d(x, h_i(x)) \leq \mu$ $(i = 1, 2)$ for some $x \in \mathbb{H}^4$ then H is an almost abelian group. Recall that the hyperbolic 4-space is realized as the upper half-space $\mathbb{R}^4_+$. By $p : \mathbb{H}^4 \to M$ we shall denote the universal covering of M; its deck-transformation group is G. Let S be a triangulated Riemannian surface so that the edges of triangulation are geodesic arcs. Then a continuous map

$$f : S \to M$$

is called piecewise-geodesic (p-g) if the restriction of f on every simplex is a totally geodesic map. By $M_{(0,\mu]}$ and $M_{[\mu,\infty)}$ we denote μ-thin and μ-thick parts of M respectively. The components of $M_{(0,\mu]}$ are Margulis tubes (see definitions in Section 2). If Δ is a triangle then we shall denote by $\dot{\Delta}$ the set of vertices of Δ. By $l_N(\gamma)$ we denote the length of the curve γ in the metric space N. If a transformation $h \in Isom(\mathbb{H}^4)$ is parabolic then we put $A_h = \emptyset$; if h is loxodromic then A_h is the axis of h (invariant geodesic).

DEFINITION 3. Suppose $H \subset Isom(\mathbb{H}^4)$ is an almost abelian discrete group, $h \in H - \{1\}$; $a, b \in \mathcal{T}(H, \nu)$ belong to one and the same fiber Π_t of the canonical foliation associated with h. Then we define *the p-g annulus* F_{hab} as follows. First connect a, b by the geodesic segments I, J so that $I \subset \Pi_t$, the closed loop $I \cup J$ is homotopic to h. Denote by γ_a, γ_b the shortest loops in $\mathbb{H}^4/H$ which contain x, y and homotopic to h. Then take the pair of geodesic triangles in M whose edges are I, J, γ_a and I, J, γ_b respectively. The union of these triangles is the desired p-g annulus F_{hab}. (See Figure 8 for lift of F_{hab} in $\mathbb{H}^4$).

REMARK 9. In general F_{hab} isn't entirely contained in $\mathcal{T}(H, \nu)$.

3.2. *Step 1.* Let $[\sigma_1], [\sigma_2] \in H_2(M, \mathbb{Z})$ be homology classes, Σ_j has genus g_j $(j = 1, 2)$; we can assume that both Σ_1, Σ_2 are hyperbolic (otherwise the intersection pairing vanishes). Let $\psi_j : \pi_1(\Sigma_j) \to G$ be the representation induced by σ_j.

Now fix j and put $\Sigma = \Sigma_j$ until the step 7 .

3.3. *Step 2.* The group $\psi(\pi_1(\Sigma))$ contains at least one loxodromic element; hence we can construct a pleated map $f^0 : \Sigma \to M$ inducing $\psi : \pi_1(\Sigma) \to G$ (see [T 1], [T 2]).

The pleated locus $\mathcal{L}$ of f^0 is a geodesic lamination on Σ .

Pick a maximal union L_0 of simple closed disjoint geodesics γ on Σ such that

$$0 < l_\Sigma(\gamma) \le \mu .$$

3.4. *Step 3.* For every component $P_j \subset \Sigma - L_0$ and $\mu > 0$ introduce the set

$$W_\mu(P_j) = \{z \in P_j : InjRad(z) \le \mu/2\}$$

Each ideal boundary component $\alpha \subset \partial P_j$ has orientation induced from P_j so we shall distinguish curves $\alpha \subset L_0$ with different orientations but equal underlying sets. Put:

$W_\mu(\alpha, P_j) = \{z \in P_j$: there exists a loop β_z on P_j which is homotopic to α and passes through z, so that $l_\Sigma(\beta_z) \le \mu\}$.

Then

$$W_\mu(P_j) = \bigcup_{\alpha\subset\partial P_j} W_\mu(\alpha, P_j)$$

The Margulis constant μ is $\le$ the Margulis constant for $\mathbb{H}^2$; therefore, for different boundary components we obtain: $W_\mu(\alpha, P_j) \cap W_\mu(\beta, P_j) = \emptyset$. Put

$$\Sigma_\mu = \{z \in \Sigma : InjRad(z) \ge \mu/2\}.$$

Let $L_1 = L_0 - cl(\Sigma_\mu)$ and $P_j^0 = P_j - W_\mu(P)$ for every j. Define

$$diam_0(\Sigma_\mu) = \sum_j diam(P_j^0).$$

LEMMA 8. $diam_0(\Sigma_\mu) \le (2g-2)/\mu$+length of $\partial\Sigma_\mu$.

PROOF. (Cf. [Ab], [Bo], [T 1, 2]). We shall denote by $l(\partial\Sigma_\mu)$ the length of $\partial\Sigma_\mu$. Cover Σ_μ by a maximal set of disjoint discs $D(x, 2\mu)$. Then the number of these discs is $n < Area(\Sigma)/(4\mu^2)$ and every point $z \in \Sigma_\mu$ has the property:

$$d(z, \partial(\Sigma_\mu - \bigcup_{i=1}^{n} D_i(z, 2\mu)) \ < 2\mu$$

Hence, for every $z, w \in P_j^0$ we have: $d(z,w) \leq 4n\mu + l(\partial\Sigma_\mu) \leq (2g - 2)/\mu + l(\partial\Sigma_\mu)$. **QED.**

REMARK 10. If $\alpha^* \subset \partial\Sigma_\mu$ then $l(\alpha^*) \leq 2\sinh\mu$ by Proposition 6. Therefore we obtain:

COROLLARY 8. In Lemma 8 we have:

$$diam_0\Sigma_\mu \leq (2g-2)/\mu + 6(g-1)\sinh\mu = C_2(\mu, g)$$

3.5. *Step 4.*

Let $T_\gamma(\mu) \subset M_{(0,\mu]}$ be the Margulis tube whose fundamental group contains the G-conjugacy class of $\psi(\gamma)$, $\gamma \subset L_1$.

REMARK 11. In general $\pi_1 T_\gamma(\mu) \neq <\psi(\gamma)>$.

For every such geodesic γ we have two (probably equal) components $P_i, P_j \subset \Sigma - L_1$ adjacent to γ. Then $f^0(W_\mu(\gamma, P_j)) \subset T_\gamma(\mu)$ $(k = i, j)$.

Choose points $x_k = x_{\gamma,k} \in \partial W_\mu(\gamma, P_k)$ $(k = i, j)$ such that:

$f^0(x_i), f^0(x_j) \in \Pi_t$ for some fiber of the canonical foliation of $T_\gamma(\mu)$ associated with γ.

Let $\nu = min\{InjRad(f_0(x_k))$, over all points $x_{\gamma,k}$ and all $\gamma \subset L_1\}$.

PROPOSITION 7. $\nu = C_3(\mu, g)$ for some function C_3 which doesn't depend on the manifold M.

PROOF. For every i we have a point $o_i \in P_i - f^{-1}(M_{(0,\mu]})$ since $\psi(\pi_1(P_i))$ isn't almost abelian. Then $d_M(o_i, x_i) \leq C_2(\mu, g)$. Therefore (by Lemma 3)

$$InjRad_M(f\ x_i) \geq C_1(C_2(\mu, g), \mu) = C_3(\mu, g)$$

QED.

REMARK 12. We used the fact that ψ is a monomorphism.

Let $\alpha_k^*, \beta_k^*, ..., \omega_k^*$ be the boundary components of P_k^0. Then we can "triangulate" P_k^0 so that: vertices of this "triangulation" Ω_f are $x_\alpha, x_\beta, ..., x_\omega$; lengths of edges of the triangulation are bounded from above by $(6g - 6)^2 C_2(\mu, g)$ (see Figure 8 for triangulation of the pair of pants). The triangles from this triangulation will be called "short".

3.6. *Step 5.* Now, for each k we map the triangulated surface P_k^0 to a p-g surface in M by the new map $f : P_k^0 \to M$ so that:

for every edge e of the triangulation we have: $f(e) \sim f^0(e)(rel\ \partial e)$.

Hence $l_M(f(e)) \le l_\Sigma(e) \le (16g-16)^2 C_2(\mu, g)$.
Now consider the thin part of Σ.

3.7. *Step 6.* Fix $x_{\gamma,i}$, $x_{\gamma,j}$ lying on the components P_i , P_j adjacent to $\gamma \subset L_1$. Then connect their images $f(x_{\gamma,i})$, $f(x_{\gamma,j})$ by the p-g annulus $F_{\psi(\gamma)f(x_{\gamma,i})f(x_{\gamma,j})}$ (see Definition 3). The boundary of $F = F_{\psi(\gamma)f(x_{\gamma,i})f(x_{\gamma,j})}$ is equal to $f\gamma_i^* \cup f\gamma_j^*$. The annulus F consists of two geodesic triangles. These triangles will be called "long" triangles "sitting in" $T_\gamma(\mu)$. The annulus F itself will be called "long p-g annulus sitting in $T_\gamma(\mu)$", see Remark 9.

So we extended our map from Σ_μ to the p-g map $f : \Sigma \to M$ which is homotopic to σ.

LEMMA 9. Suppose that $d(p(A_{\psi\gamma}), z) \le R+2$ for some $z \in f(\Sigma_\mu)$. Then

$$diam(f\Delta) \le 4\sinh(\mu/2)/C_1(3R+2,\mu)$$

for every $f(\Delta)$ sitting in $T_\gamma(\mu)$.

PROOF. We have $d(z, f\ o_i) \le R, f(o_i) \in M_{(\mu,\infty]} \cap f(P^0)$, $d(f(o_i), p(A_{\psi\gamma})) \le 3R+2$ and so $l(\gamma) \ge C_1(3R+2,\mu)$ by Lemma 3. This implies that

$$d(p(A_{\psi\gamma}), z) \le 2\sinh(\mu/2)/C_1(3R+2,\mu)$$

for every $z \in T_\gamma(\mu)$. Therefore for the triangle $f(\Delta)$ sitting in $T_\gamma(\mu)$ we have:

$$diam(\Delta) \le 4\sinh(\mu/2)/C_1(3R+2,\mu)$$

QED.

Our construction of the map f is sufficiently flexible and given any two classes $\sigma_1, \sigma_2 \in H_2(M,\mathbb{Z})$ we can find transversal maps $f_i : \Sigma_i \to M$ representing these classes.

3.8. *Step 7.* The maps f_i above will be called "nice". Summarize the properties of "nice" maps f_i.

(1) f_i are p-g with respect to some triangulations Ω_{f_i} of the surfaces Σ_i. The number of triangles in Ω_{f_i} is $\le 16(g_i - 1)$.

(2) In the triangulation Ω_{f_i} there are "short" and "long" triangles (with respect to p-l metric induced by f_i). Namely, the internal diameter of a "short" triangle is

$$\le 2C_2(\mu, g_i) = R$$

and their union is $\Sigma_{\mu i}$ so that $\Sigma_i - \Sigma_{\mu i} = W_{\mu i}$ is the union of pairwise disjoint nonhomotopic tubes. All vertices of Ω_{f_i} are contained in $\partial\Sigma_{\mu i}$.

(3) f(0-skeleton of Ω_{f_i}) $\subset M_{[\nu,\infty)}$ for some $\nu = \nu(g_i)$.

(4) Every short triangle $\Delta \subset P^0_{ki} \subset \Sigma_{\mu_i}$ has bounded index with respect to any $\gamma^* \subset \partial P^0_{ki}$. More precisely,

for every $z \in \dot{\Delta}$ and $\gamma^* \subset \partial\Sigma_{\mu i}$ passing through z we have: $l(\gamma^*) \leq R$ (this is just the corollary of (2)).

(5) Denote by $\Omega^1_{f_i}$ the 1-skeleton of Ω_{f_i}. Every component $Q_{ji} \subset W_{\mu i}$ consists of two "long" triangles such that: if for some $h \in \pi_1(M)$ we have $1 \neq h^{-1}\psi_1(\pi_1(Q_{j1}))h \cap \psi_2(\pi_1(Q_{j2})) =< \gamma_j > \subset \pi_1(M)$ then $< \gamma_j >$ is a maximal almost abelian subgroup of $\pi_1(M)$ and the maps

$$f_i : Q_{ji} - (\Omega^1_{f_i}) \to M$$

can be lifted to the fundamental domain $\Phi_j \subset \mathbb{H}^4$ of the group $\pi_1(T_{\gamma_j}(\mu))$. The fundamental domain Φ_j is bounded by a pair of fibers of the canonical fibration of $\mathbb{H}^4$ corresponding to $< \gamma_j >$.

This property follows from the condition "MAX" and Step 6.

(6) For every "long" triangle" Δ we have: $diam(f(\Delta) \cap M_{[\nu,\infty)}) \leq C_4$; where

$$C_4 = 4R + 6 + 1/k(R,\nu) + 4\sinh(\mu/2)/C_1(3R + 2, \mu)\}$$

this follows from Lemma 5 and Lemma 9 .

(7) Suppose that $d(p(A_{\psi\gamma}), z) \leq R + 2$ for some $z \in f_i(\Sigma_{i\mu})$. Then

$$diam(f_j\Delta) \leq 4\sinh(\mu/2)/C_1(3R + 2, \mu)$$

for every long triangle $f_j(\Delta)$ sitting in $T_\gamma(\mu)$, $j = 1, 2$; see Lemma 9.

3.9. *Step 8.* Now we can count the number of intersections $\#(f_1(\Sigma_1) \cap f_2(\Sigma_2))$.

(i) Consider intersections of "short" triangles. Pick a pair of such triangles $f_1(\Delta_1) \subset f_1(\Sigma_1), f_2(\Delta_2) \subset f_2(\Sigma_2)$, let $\Delta'_j \subset \mathbb{H}^4$ be the geodesic triangles covering them ($j = 1, 2$). Then we are to estimate the number of $h \in G$ such that $h\Delta'_1 \cap \Delta'_2$ isn't empty. Remind that diam$\Delta'_j \leq R/2$ and both Δ'_j contain points o_j such that $Ir_G(o_j) \geq \mu/2$. Therefore we can apply Lemma 2 to obtain:

$\#(f_1(\Delta_1) \cap f_2(\Delta_2)) \leq 8 \exp^3(2R + \mu/2))/\mu^3$.

(ii) First consider the case when Δ_1 is short while Δ_2 is long.

Suppose that $h(z) \in h\Delta_1' \cap \Delta_2'$. Then we can apply Lemma 5 and the property 7 of nice maps to obtain

$$d(z, \dot{\Delta}_2') \leq \max\{4R + 6 + 1/(C_1(R+2, \nu)),\ C_4\} = C_5 \ .$$

Let $\{w_1, w_2, w_3\} = \dot{\Delta}_2'$.

Hence we obtain estimate in the same manner as in the case (i):

$$\#(f_1(\Delta_1) \cap f_2(\Delta_2)) \leq \#\{h \in G : h(B(w_i, 2C_5)) \cap \Delta_1' \neq \emptyset, i = 1, 2, 3\}$$

$$\leq 3 \exp^3(2C_5 + \nu/2))/\nu^3$$

since diam $\Delta_1' \leq 2R < C_5$.

(iii) Assume now that both Δ_1, Δ_2 are long. Denote by $T_{\gamma_j}(\mu) \subset M_{(0,\mu]}$ those Margulis tubes where Δ_j are sitting.

(a) First count the number of intersections that occur in $T_\gamma(\mu)$ if Δ_1, Δ_2 are sitting in the same $T_\gamma(\mu) \subset M_{(0,\mu]}$. The influence of the fundamental group is trivial (property 5) and here we have not more than 1 intersections between the "long" triangles $f_1(\Delta_1), f_2(\Delta_2)$.

(b) If $T_{\gamma_j}(\mu)$ are different then

$$f_1(\Delta_1) \cap T_{\gamma_1}(\mu) \cap f_2(\Delta_2) \cap T_{\gamma_2}(\mu) = \emptyset$$

So consider intersections $(f_1(\Delta_1) - T_{\gamma_1}(\mu)) \cap f_2(\Delta_2)$ outside $T_{\gamma_2}(\mu)$. However $\Delta_1 - f_1^{-1}(T_{\gamma_1}(\mu))$ is the union of 2 subsets each having diameter $\leq C_4 < C_5$ and therefore the number of intersections is not more than

$$6 \exp^3(2C_5 + \nu/2)/\nu^3$$

analogously to the case (ii).

Hence the total estimate in the case (iii) is

$$6(12(g-1))^2 \exp^3(2C_5 + \nu/2)/\nu^3$$

where $g = \max\{g_1, g_2\}$.

Direct calculations now show that the number $| < [\sigma_1], [\sigma_2] > |$ can be estimated as:

$$300(g-1)^2\exp(4000(g-1)/\mu)/\mu^2$$

This finishes the proof of Theorem 1 under condition MAX. **QED.**

4. PROOF OF THEOREM 1 IN GENERAL CASE

4.1. The proof proceeds in the same way as in the section 3 until the step 6.

Step 6'. Fix $x_{\gamma,i}$, $x_{\gamma,j}$ lying on the components P_i , P_j adjacent to $\gamma\subset L_1$. Lift $x_{\gamma,i}$, $x_{\gamma,j}$ to points $\tilde{f}x_{\gamma,i}$, $\tilde{f}x_{\gamma,j}$ in $\mathcal{K}(H,\mu)$ where H is the maximal almost abelian subgroup of G which contains $<\psi(\gamma)>$. Denote by u' the projection $\varphi_{H,\nu}(u)$ of the point $u\in\mathbb{H}^4$ to the $\partial\mathcal{K}(H,\nu)$. For $k\in\{i,j\}$ we have:

(i) $d(\tilde{f}x_{\gamma,k},\psi(\gamma)\tilde{f}x_{\gamma,k})\le\mu$;

(ii) $Ir_G(\tilde{f}x_{\gamma,k})\ge\nu/2$.

Therefore, according to Proposition 7, either

(a) $d(\varphi\tilde{f}x_{\gamma,k},\tilde{f}x_{\gamma,k})\le C_+(\mu,\nu)$ or

(b) $\cosh d(\varphi\tilde{f}x_{\gamma,k},A_{\psi(\gamma)})\le\frac{2\sinh\mu/2}{C_-(\mu,\nu)}$.

On another hand, $f(x_{\gamma,k})\in M_{(\nu,\infty]}$. Therefore, if the possibility (b) holds then (by Lemma 3) we obtain the lower estimates

$$C_1(arccosh(\frac{2\sinh\mu/2}{C_-(\mu,\nu)}),\nu)\le l(\psi\gamma)\le l_\Sigma(\gamma)$$

and moreover:

$$C_1(arccosh(\frac{2\sinh\mu/2}{C_-(\mu,\nu)}),\nu)\le Ir_G(u)$$

for every $u\in A_{\psi(\gamma)}$.

Then we can consider the triangles which constitute the p-g annulus

$$F=F_{\psi(\gamma)f(x_{\gamma,i})f(x_{\gamma,j})}$$

as "short triangles" and exclude $T_\gamma(\nu)$ from the consideration of the thin part of M. So let's suppose that the alternative (b) doesn't hold. Denote by $x''_{\gamma,k}$ the projections $p(\varphi\tilde{f}x_{\gamma,i})$

Then construct the extended ruled film $\tilde{Q} = \tilde{Q}_{\gamma(\varphi\tilde{f}x_{\gamma,i})(\varphi\tilde{f}x_{\gamma,j})}$; and connect the loop $f^0(\gamma_i^*)$ with $\partial_{\varphi\tilde{f}x_{\gamma,i}}p\tilde{Q}$ and $f^0(\gamma_j^*)$ with $\partial_{\varphi\tilde{f}x_{\gamma,j}}p\tilde{Q}$ by the p-g annuli

$$F_{\varphi\tilde{f}x_{\gamma,i},f_0x_{\gamma,i}}\ ,F_{\varphi\tilde{f}x_{\gamma,j},f^0x_{\gamma,j}}$$

(see Figure 9). Now instead of the long p-g annulus F sitting in $T_\gamma(\mu)$ (as in 3.7) take the union:

$$p(\tilde{Q}_{\gamma\varphi\tilde{f}x_{\gamma,i}\varphi\tilde{f}x_{\gamma,j}})) \cup F_{x''_{\gamma,i},f^0x_{\gamma,i}} \cup F_{x''_{\gamma,j},f^0x_{\gamma,j}}.$$

In this way we extended the p-g map from Σ_μ to the new map ϕ defined on Σ.

4.2. *Step 7'.* The maps ϕ_i above will be called "improved nice" maps. Restrictions of ϕ_i to $\Sigma_{i,\mu}$ have all properties of nice maps. However instead of "long p-g annuli sitting in $T_\gamma(\mu)$" we are using extended ruled annuli (see Step 6').

4.3. Now we can count the intersection number $< \phi_1(\Sigma_1), \phi_2(\Sigma_2) >$ in the same way as it was done in 3.9.

(i) Number of intersections between short triangles is estimated exactly as in 3.9(i).

(ii) Every extended ruled film $\tilde{Q}_k$ is contained in $2R$-neighborhood of the geodesic segment

$$[\tilde{\phi}_k(x_{\gamma,i})\ ,\ \tilde{\phi}_k(x_{\gamma,j})]$$

($k = 1, 2$). Number of points of transversal intersection between any extended ruled film with geodesic plane is not more than 8. Therefore again we can use Lemma 5 to obtain

$$\#(p\tilde{Q}_1 \cap f_2\Delta) \leq 24\exp^3(2C_5 + \nu/2))/\nu^3$$

for every short geodesic triangle $f_i\Delta$(see 3.9(ii)).

REMARK 13. If the loxodromic alternative holds for the segment $[\tilde{\phi}_1(x_{\gamma,i})\ ,\ \tilde{\phi}_1(x_{\gamma,j})]$, then it holds for $[\tilde{\phi}_2(x_{\gamma,i})\ ,\ \tilde{\phi}_2(x_{\gamma,j})]$ too.

(iii) The algebraic intersection number between two extended ruled annuli that belong to one and the same Margulis tube was estimated in Theorems 4, 5. It is not more than

$$\max\{N(R,\nu), N'(R,\nu), R/\nu\}$$

This finishes the proof of Theorem 1 in the general case. **QED.**

APPENDIX

LEMMA 11. Let p_1, p_2 be closed geodesics on the hyperbolic surface F. Then:

$$\#(p_1 \cap\ p_2) \leq \exp(l_1 + l_2 + 1)$$

where $\#(.\cap .)$ is the number of points of intersection, l_i is the length of p_i.

PROOF. Denote by $G \subset Isom(\mathbb{H}^2)$ the fundamental group of F. Denote by g_i representative of p_i in the deck-transformation group G; and let q_i be the axis of g_i . Let f_i be a segment of the length l_i on q_i . Without loss of generality we can assume that there is $x \in f_1 \cap f_2$ such that $Ir_G(x) \geq 1$ (since 2 can stand for the Margulis constant in $\mathbb{H}^2$). Then

$$< p_1,\ p_2 > \leq \#\{h \in G : h(B(x, l_1 + l_2)) \cap B(x, l_1 + l_2) \neq \emptyset\}$$

Now we can apply 2-dimensional version of Lemma 2 to obtain that

$$\#\{h \in G : h(B(x, l_1 + l_2)) \cap B(x, l_1 + l_2) \neq \emptyset\} \leq \exp(l_1 + l_2 + 1)$$

Lemma is proved.

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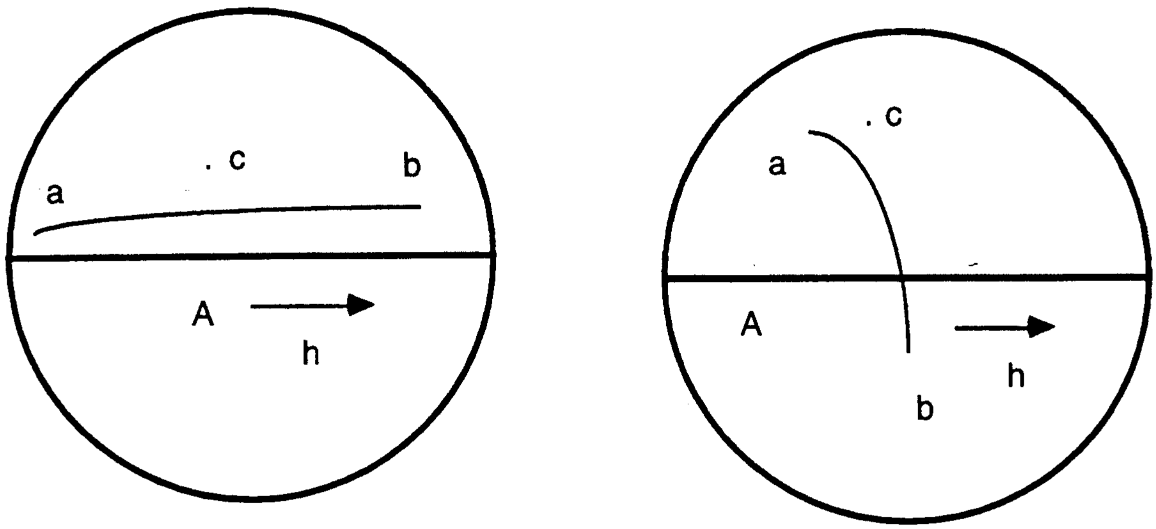


loxodromic alternative   parabolic alternative

Figure 1

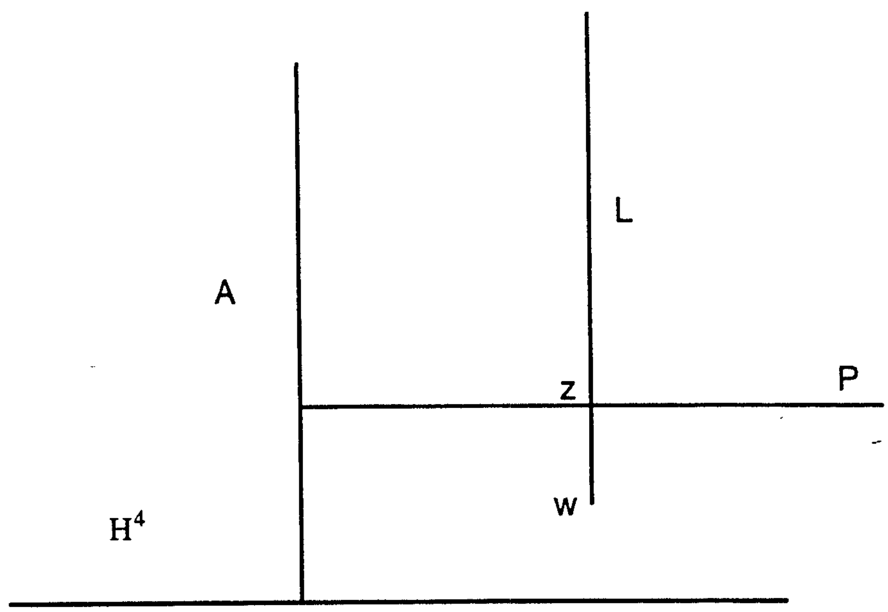


Figure 2

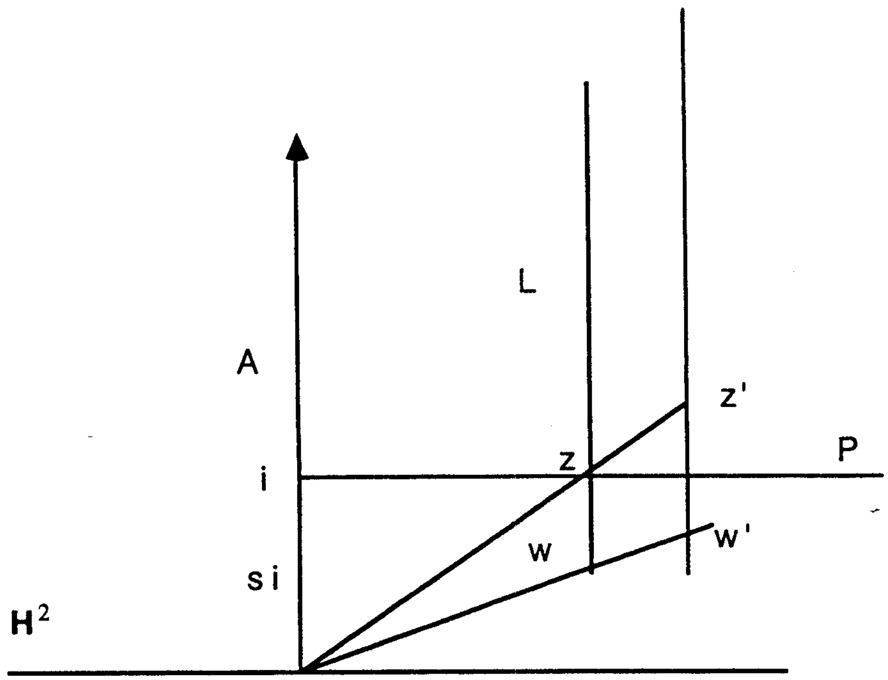


Figure 3

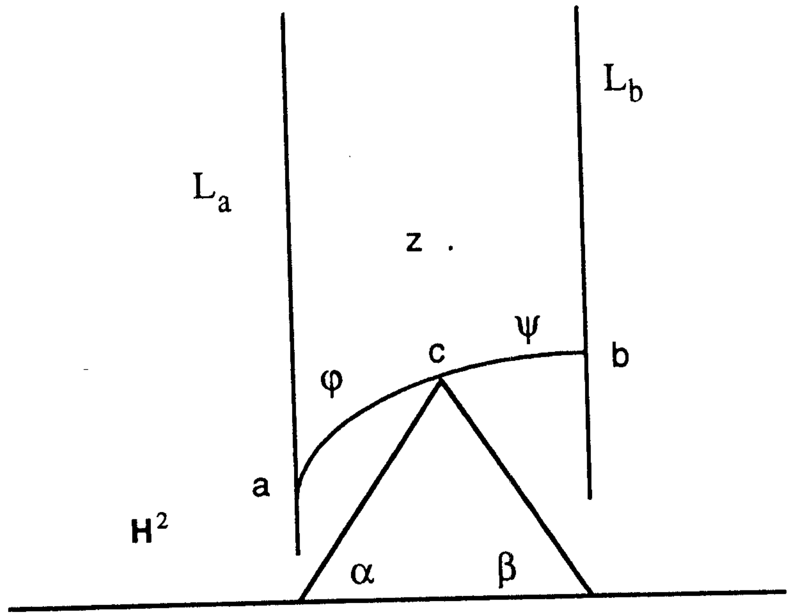
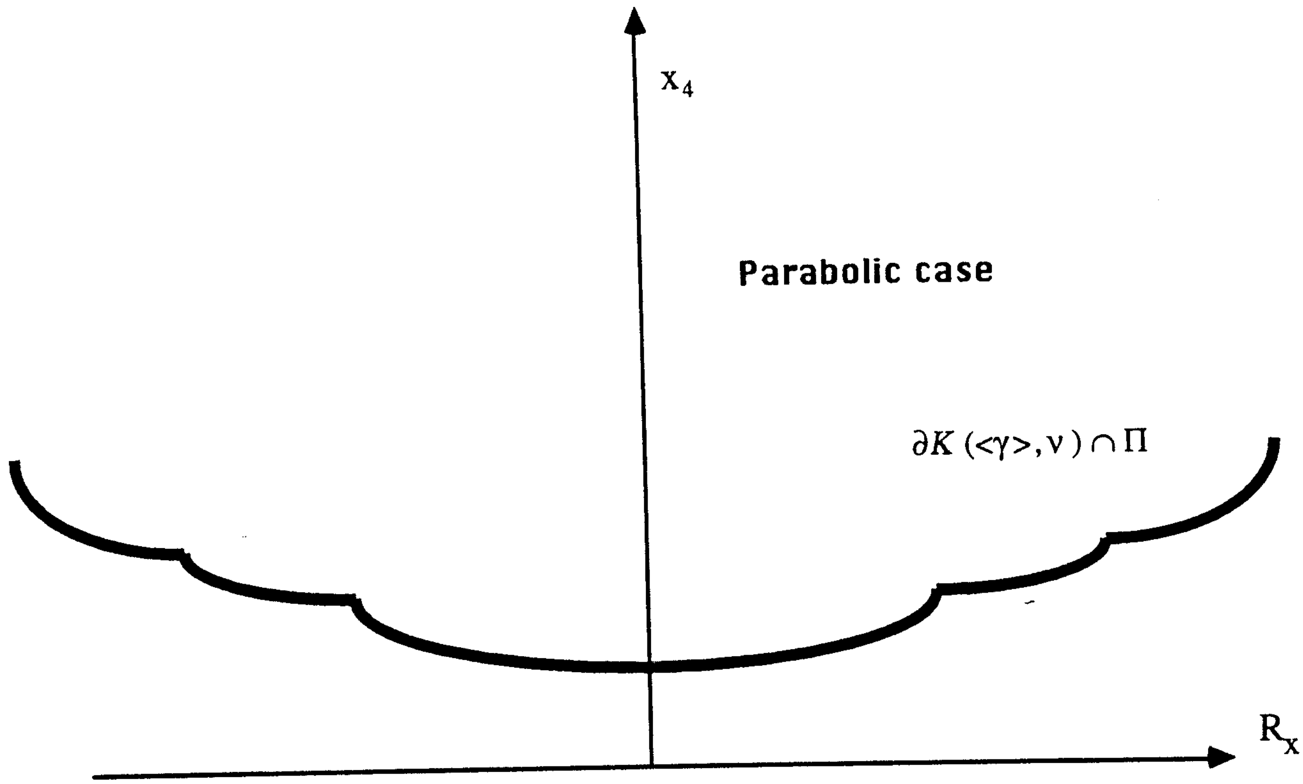


Figure 4


x4
Parabolic case
∂K (<γ>, ν) ∩ Π
Rx

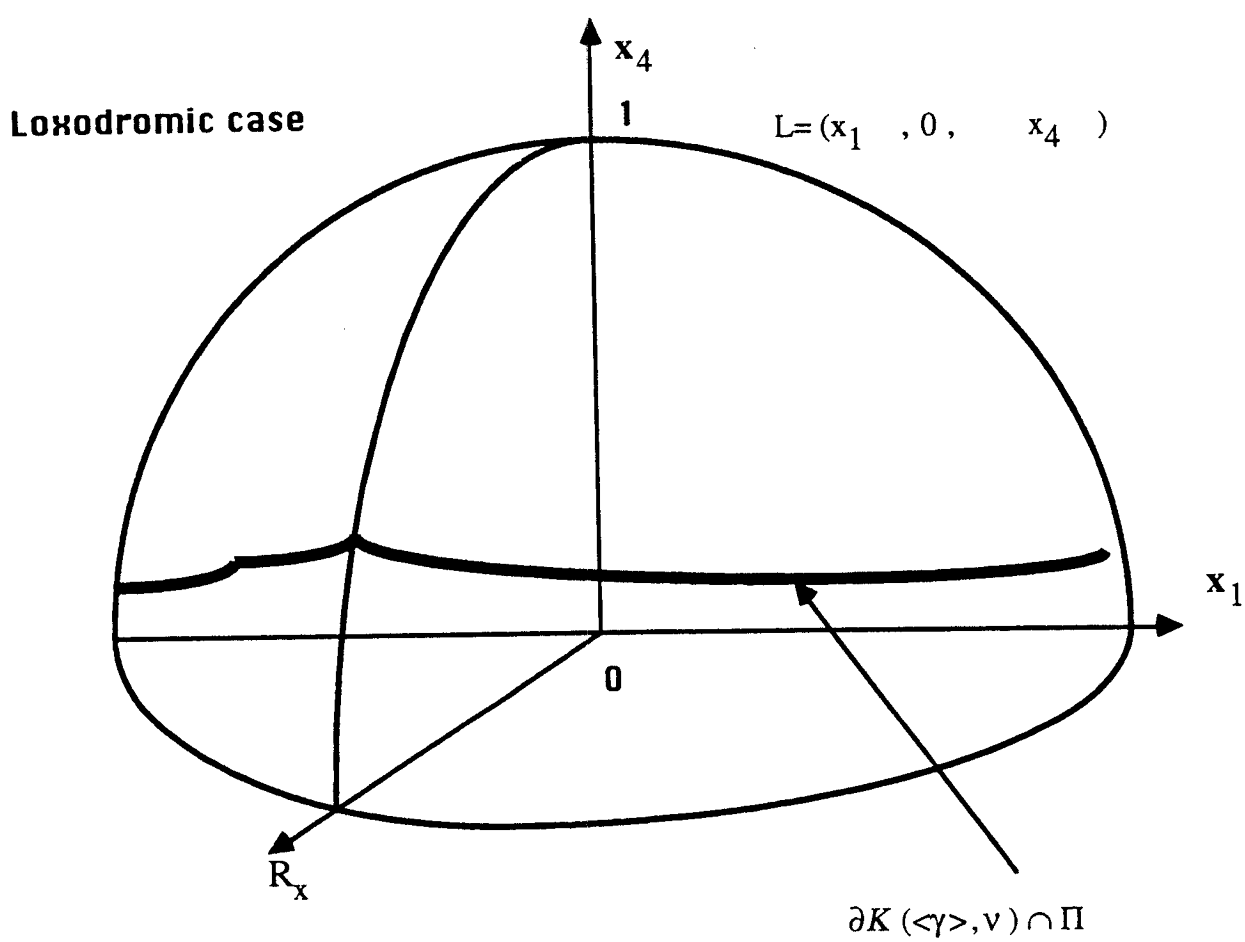

x4
Loxodromic case
1
L= (x1 , 0 , x4)
x1
0
Rx
∂K (<γ>, ν) ∩ Π

Figure 5

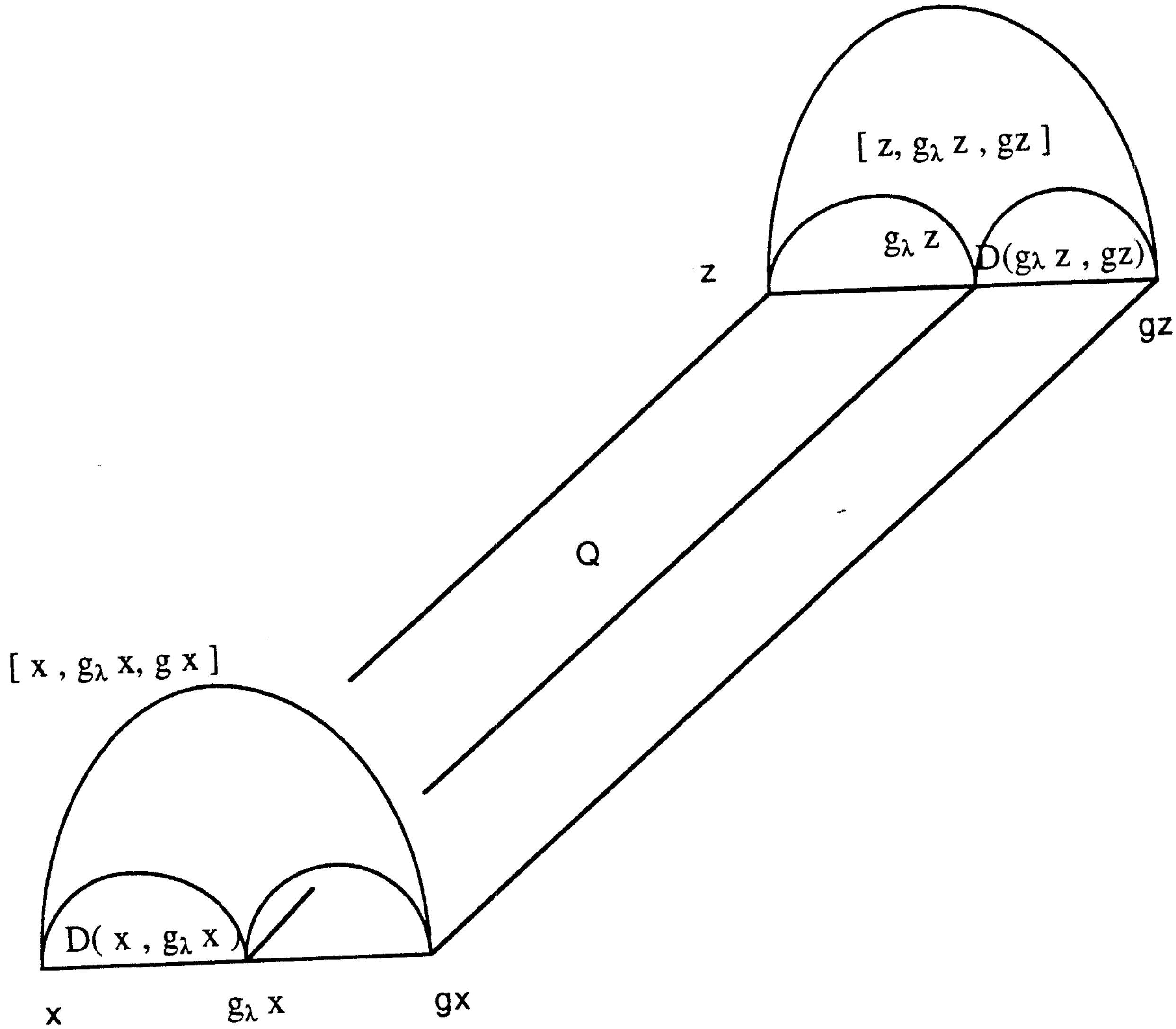

[z, g_λ z , gz]
g_λ z
D(g_λ z , gz)
z
gz
Q
[x , g_λ x, g x]
D(x , g_λ x)
x
g_λ x
gx

Figure 6

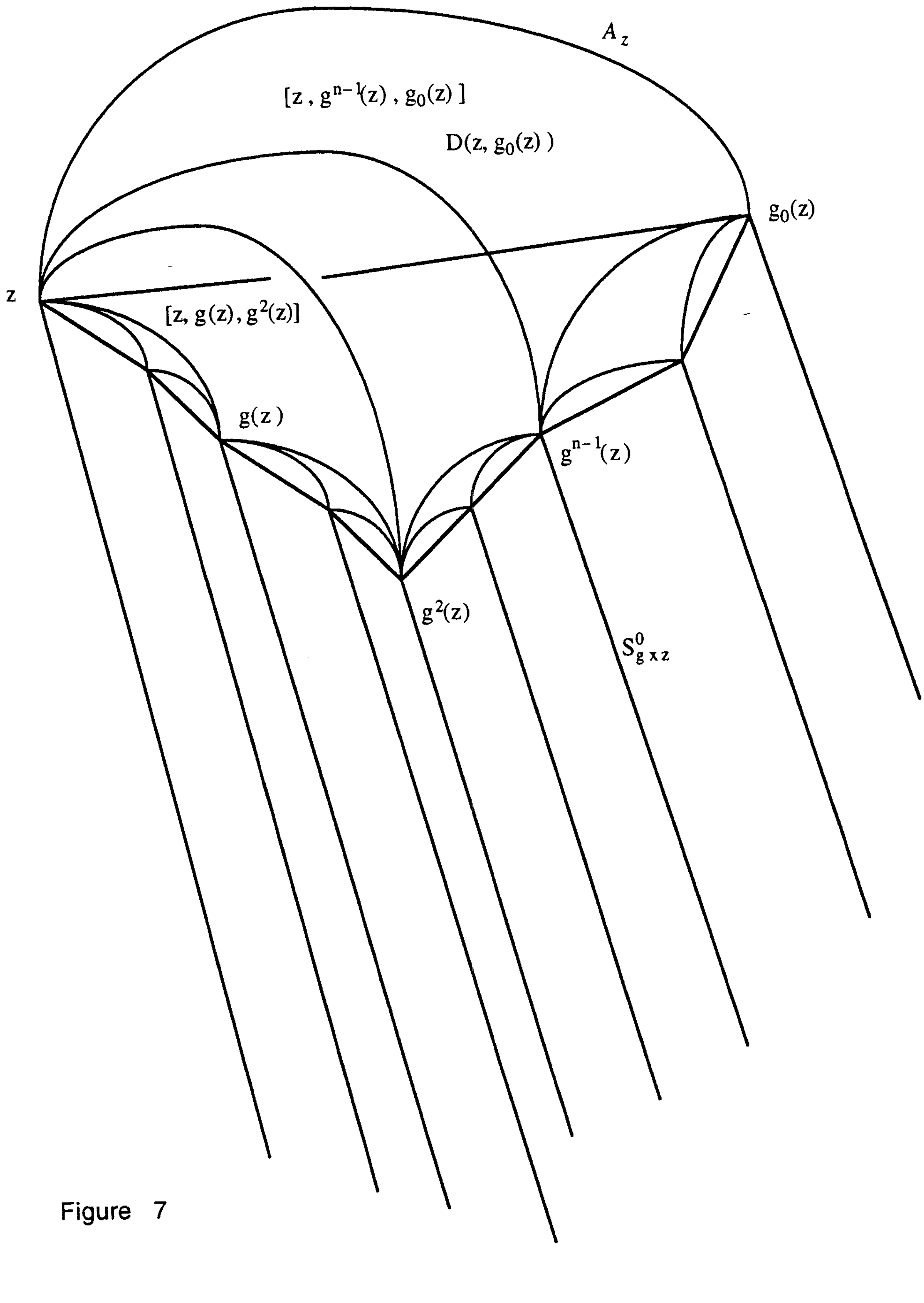
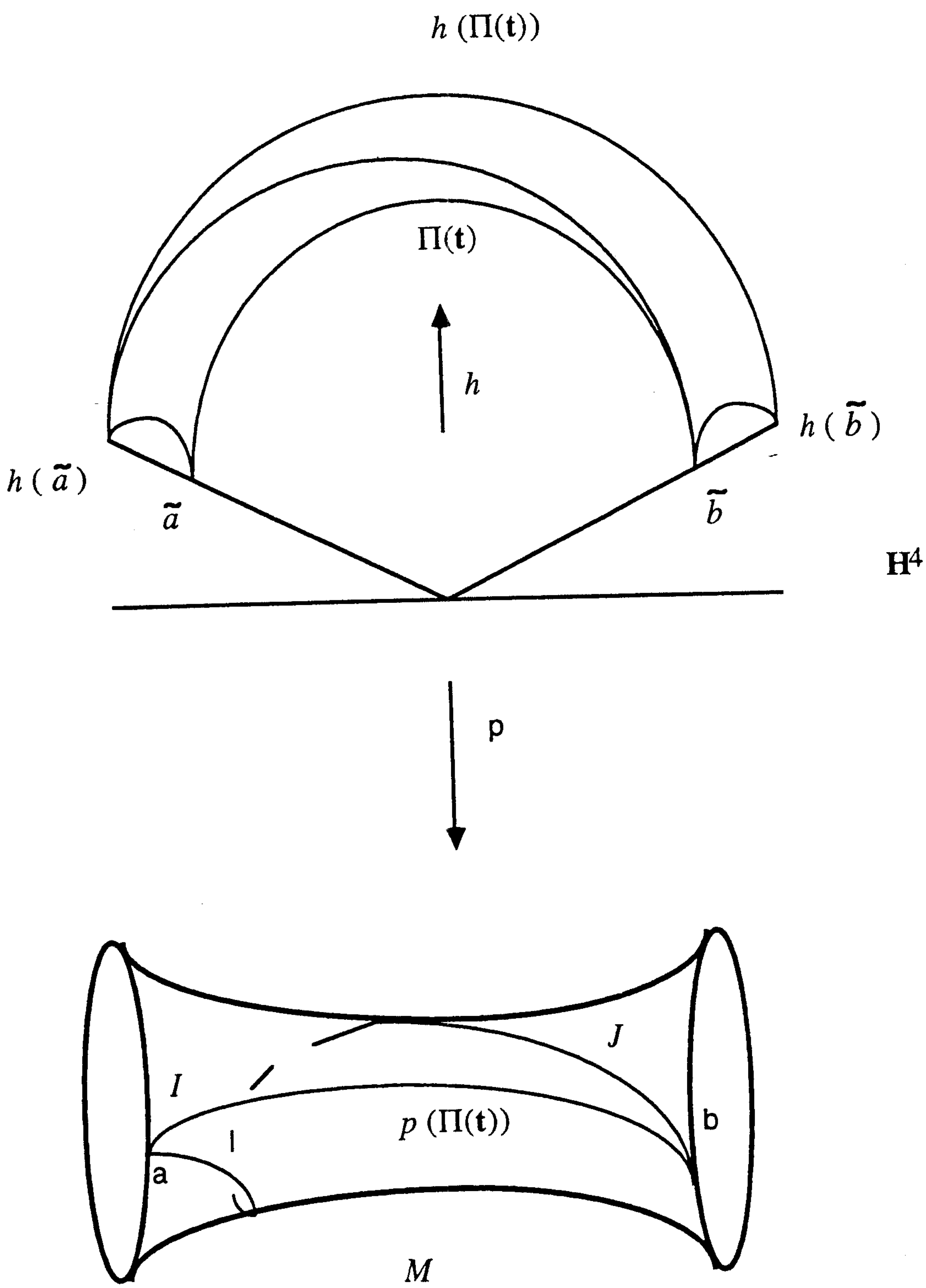


Figure 7



$h\,(\Pi(\mathbf{t}))$
$\Pi(\mathbf{t})$
h
$h\,(\tilde{b})$
$h\,(\tilde{a})$
$\tilde{a}$
$\tilde{b}$
$\mathbf{H}^4$
p
J
I
b
$p\,(\Pi(\mathbf{t}))$
l
a
M

Figure 8

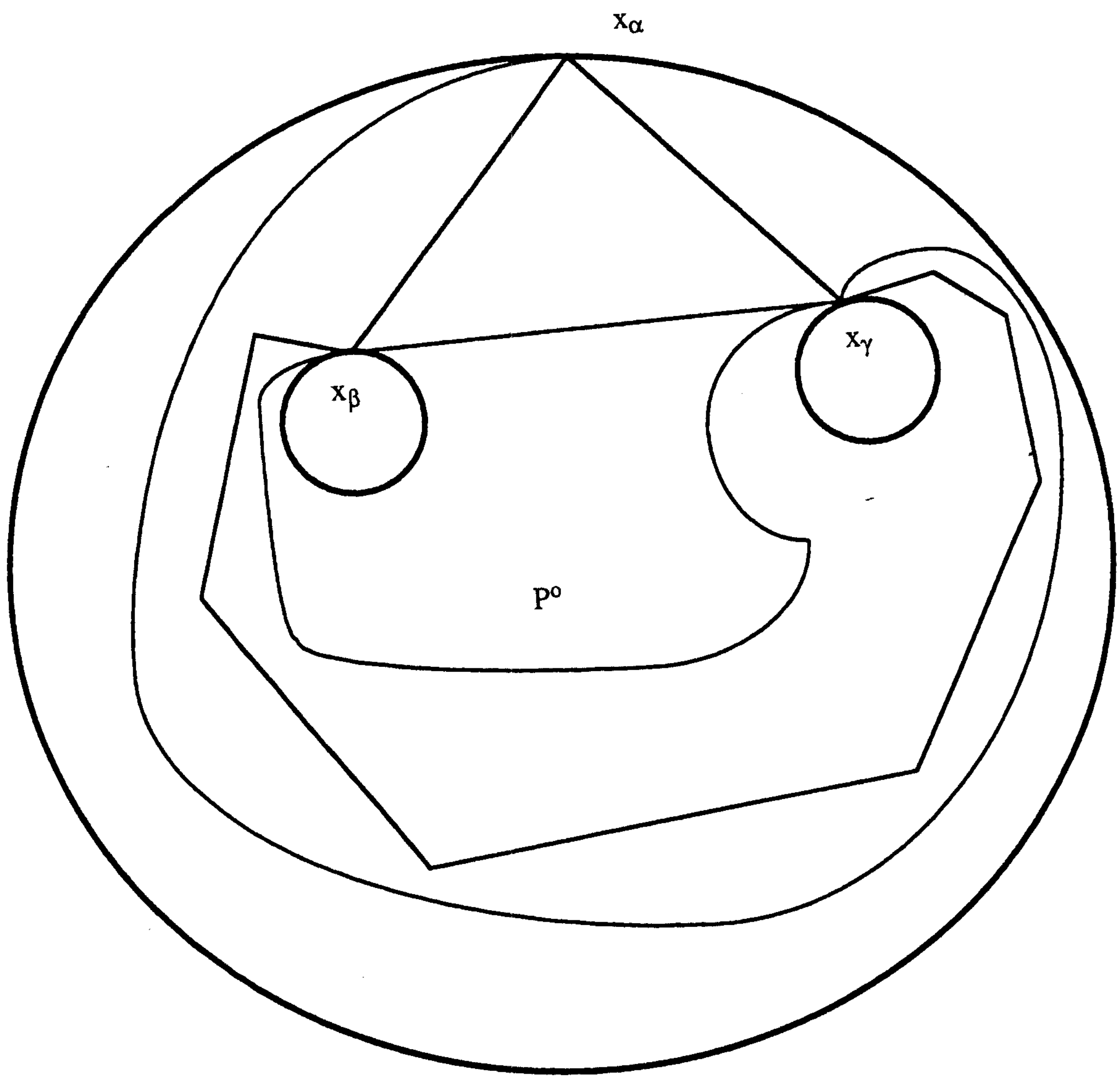


Figure 9