

On normal subgroups in the fundamental groups of complex surfaces

Michael Kapovich*

August 17, 1998

Abstract

We show that for each aspherical compact complex surface X whose fundamental group π fits into a short exact sequence

$$1 \rightarrow K \rightarrow \pi \rightarrow \pi_1(S) \rightarrow 1$$

where S is a compact hyperbolic Riemann surface and the group K is finitely-presentable, there is a complex structure on S and a nonsingular holomorphic fibration $f : X \rightarrow S$ which induces the above short exact sequence. In particular, the fundamental groups of compact complex-hyperbolic surfaces cannot fit into the above short exact sequence. As an application we give the first example of a non-coherent uniform lattice in $Isom(\mathbb{H}_{\mathbb{C}}^2)$.

1 Introduction

The goal of this paper is threefold:

(a) We will establish a restriction on the fundamental groups of compact aspherical complex surfaces.

(b) We find the first examples of incoherent uniform lattices in $PU(2, 1)$.

(c) We show that the answer to the Question 1 below is negative in the class of uniform lattices in $PU(2, 1)$.

Question 1 *Is there a Gromov-hyperbolic group π which fits into a short exact sequence:*

$$1 \rightarrow K \rightarrow \pi \rightarrow Q \rightarrow 1$$

where K and Q are closed hyperbolic surface groups?

Suppose that X is an aspherical compact complex surface whose fundamental group π fits into a short exact sequence

$$1 \rightarrow K \rightarrow \pi \rightarrow Q = \pi_1(S) \rightarrow 1$$

where S is a compact hyperbolic Riemann surface and the group K is finitely-presentable. The main theorem of this paper is

*Supported by NSF grant DMS-96-26633

Theorem 2 *Under the above assumptions there a complex structure on S and a nonsingular holomorphic fibration $f : X \rightarrow S$ which induces the above short exact sequence.*

Remark 3 *Actually in Theorem 2 it is enough to assume that Q is a torsion-free group with nonzero $\beta_1^{(2)}(Q)$, the 1-rst L_2 -Betti number. On the other hand, in this case we have to assume that X is Kähler. Our proof also works under the assumption that the group K is of the type FP_2 .*

After proving Theorem 2 I have learned that J. Hillman [10] proved the same result under stronger assumption that K is the fundamental group of a compact Riemann surface. Our methods seem to be completely different except application of the result of [1]. Later it turned out that the same result as Hillman's was independently proven by D. Kotschick.

Acknowledgements. I am grateful to Bill Goldman, János Kollár, Mohan Ramachandran and Domingo Toledo for discussions of issues related to this paper and complex-hyperbolic manifolds in general.

2 Milnor fibration

Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a nonconstant holomorphic function, we assume that $0 \in \mathbb{C}^2$ is a critical point of f . Let $S_\epsilon = S_\epsilon(0)$ be a sufficiently small metric sphere in \mathbb{C}^2 centered at the origin. Let $B_\epsilon(0)$ denote the closed ϵ -ball centered at the origin. Let $K := f^{-1}(0) \cap S_\epsilon$, this is a smooth knot (or link) in the 3-sphere. The *Milnor fibration* $\phi : S_\epsilon - K \rightarrow S^1$ associated with f is defined as $\phi(z, w) = f(z, w)/|f(z, w)|$, see [16, §4].

Below we list some properties of ϕ (see [16, §4], [6]):

(a) If ϵ is sufficiently small then ϕ determines a smooth fibration of $S_\epsilon - K$ over S^1 .

(b) Fibers of ϕ are connected provided that the germ of f at zero is reduced, otherwise ϕ will have disconnected fibers.

(b) The knot (link) K is distinct from a single unknot in S^3 unless the germ of f at 0 is isomorphic to $((z, w) \mapsto z^p, (0, 0))$.

(c) If K not an unknot, then each component of $\phi^{-1}(t), t \in S^1$, is not simply-connected.

(d) Let $r > 0$ be sufficiently small. Consider $s \in C_r(0)$, a point on the unit circle in \mathbb{C} centered at zero. Let $\mathcal{F}_{\epsilon,s} := f^{-1}(s) \cap B_\epsilon$. The two surfaces $\mathcal{F}_{\epsilon,s}$ and $F_{\epsilon,s} = \phi^{-1}(s/|s|) - f^{-1}(B_r(0))$ share common boundary. There exists an isotopy of $\mathcal{F}_{\epsilon,r}$ to $F_{\epsilon,r}$ within $B_\epsilon(0)$ which is the identity on the boundary of each surface.

3 Multicurves

Definition 1 *Let $f : X \rightarrow S$ be a nonconstant proper holomorphic map from a connected complex surface X to a Riemann surface (i.e. complex curve) S . We will say that f is a **nonsingular holomorphic fibration** if f is a submersion.*

Clearly the mapping f as above is a real-analytic fibration, however in most cases it does not determine a locally trivial holomorphic bundle. If f is not a submersion we will still think of it as a **singular** fibration, we shall use the notation \mathcal{F}_t to denote the fiber $f^{-1}(t)$ of f over $t \in S$.

Definition 2 Let $f : X \rightarrow D^2$ be a nonconstant proper holomorphic map with connected fibers where X is a 2-dimensional complex surface and D^2 is the unit disk in \mathbb{C} . We assume that the origin is the only critical value of f . The singular fiber $C = f^{-1}(0)$ is called a **multicurve** if it is a smooth curve of the multiplicity > 1 . In other words, the germ of f at each point $c \in C$ is equivalent to the map $(z, w) \mapsto z^n, n > 0, z, w \in \mathbb{C}$. The number n is the multiplicity of C .

Let $t \in D^2 - 0$. Define the maps

$$\iota_* : H_2(f^{-1}(t)) \rightarrow H_2(X) \cong H_2(C)$$

$$\iota_{\#} : \pi_1(f^{-1}(t)) \rightarrow \pi_1(X) \cong \pi_1(C)$$

induced by the inclusion $\iota : f^{-1}(t) \hookrightarrow X$.

Lemma 4 If C is a multicurve then the map ι_* is not surjective. Assume that C is a non-simply-connected multicurve. Then the map $\iota_{\#}$ is not onto.

Proof. Consider $Y = f^{-1}(D) \subset X$ where $D = \{z \in \mathbb{C} : |z| \leq |t|\}$ is the closed disk in D^2 containing t . The inclusion $Y \hookrightarrow X$ is a homotopy-equivalence so we restrict our discussion to Y . The map $C \hookrightarrow Y$ is a homotopy-equivalence, thus the fundamental class of C generates $H_2(Y)$. The dual generator of $H_2(Y, \partial Y)$ is represented by 2-disk $\Delta \subset Y$ which is transversal to the fibers of f and $\partial\Delta \subset \partial Y$. Since C is a multicurve, the algebraic intersection number $[f^{-1}(t)] \cdot [\Delta] = n > 1$, where n is the multiplicity of C . Thus $[f^{-1}(t)] = n[C]$ which proves the first assertion.

The map $\iota_{\#}$ is injective (since ι is homotopic to a covering $f^{-1}(t) \rightarrow C$). Thus $n = |\pi_1(C) : \iota_{\#}(\pi_1(f^{-1}(t)))|$, this proves the second assertion. \square

4 Proof of the main theorem

If $\pi_1(X)$ fits into short exact sequence

$$1 \rightarrow K \rightarrow \pi \rightarrow Q = \pi_1(S) \rightarrow 1$$

where S is a hyperbolic Riemann surface then it follows from Kodaira's classification theorem that X is a complex-algebraic surface. If X is assumed to be Kähler, Q torsion-free and $\beta_1^{(2)}(Q) \neq 0$, then Q is the fundamental group of a hyperbolic Riemann surface, moreover if \tilde{X} is the covering of X corresponding to K then there is a discrete faithful conformal action of Q on \mathbb{H}^2 and a Q -equivariant proper holomorphic map

$$\tilde{f} : \tilde{X} \rightarrow \mathbb{H}^2$$

with connected fibers (see [1]). In particular, the projection $\pi_1(X) \rightarrow Q$ is induced by a holomorphic map $f : X \rightarrow S$, for the complex structure on S given by \mathbb{H}^2/Q .

The i -th L_2 -Betti number $\beta_i^{(2)}(G)$ of a finitely presentable group G is the dimension of the i -th reduced L_2 -cohomology group $\bar{\ell}_2 H^i(G)$, we refer the reader to [9, Chapter 8] and [1] for the precise definitions. For our purposes it is enough to know that $\beta_i^{(2)}(Q) > 0$ for each 2-dimensional finitely presentable group Q provided that $\chi(Q) < 0$ (see [9, Chapter 8]). In particular, if Q is the fundamental group of a hyperbolic Riemann surface of finite type then $\beta_1^{(2)}(Q) > 0$. Thus, in any case we have a holomorphic map $f : X \rightarrow S$.

We start the proof with the simple case when f is a *holomorphic Morse function*, i.e. the germ of f at each critical point is equivalent to $(z, w) \mapsto zw$. The proof in this case is easier and it illustrates the idea of the proof in the general case.

Let d denote the hyperbolic metric on the unit disk in \mathbb{C} . We will suppose that the origin 0 is a regular value of \tilde{f} . Direct computations show that the function

$$\gamma := \tilde{f} \circ d(0, \tilde{f}(x))$$

is a real Morse function on \tilde{X} away from $\tilde{f}^{-1}(0)$ and the Morse index of γ at each critical point in $\tilde{X} - \tilde{f}^{-1}(0)$ is two. It is clear that $r \in \mathbb{R}_+$ is a critical value of γ if and only if there is a critical value $z \in \mathbb{H}^2$ of \tilde{f} within the distance r from the origin. Let \mathcal{F} denote the generic fiber of \tilde{f} . Thus the space \tilde{X} is obtained by attaching 2-handles to $\mathcal{F} \times D^2$. Each singular fiber of \tilde{f} is obtained from \mathcal{F} by ‘‘pinching’’ a certain collection of disjoint simple loops. Since \tilde{X} is aspherical, each of these loops is homotopically nontrivial and no two such loops are homotopic to each other. (Otherwise \tilde{X} contains a rational curve which then lifts to a homologically nontrivial 2-cycle in the universal cover of X .)

We now claim that the group $\pi_1(\tilde{X})$ is finitely generated but not finitely presentable. Our proof follows an argument of Bestvina and Brady [3]. Since \tilde{X} is obtained from $\mathcal{F} \times D^2$ by attaching only 2-handles, the fundamental group of \tilde{X} is the quotient of $\pi_1(\mathcal{F})$. Recall that $\pi_1(\tilde{X})$ is finitely presentable, the epimorphism

$$\pi_1(\mathcal{F}) \rightarrow \pi_1(\tilde{X})$$

determines a finite generating set for $\pi_1(\tilde{X})$ (i.e. the generators of $\pi_1(\mathcal{F})$).

Lemma 5 *Let G be a finitely presentable group and $\{y_1, \dots, y_m\}$ be a finite generating set for G . Then there is a finite number of relators R_1, \dots, R_k such that $\langle y_1, \dots, y_m | R_1, \dots, R_k \rangle$ is a presentation of G .*

Proof. Let $\langle x_1, \dots, x_s | Q_1, \dots, Q_n \rangle$ be a finite presentation of G . There is a finite sequence of *Tietze transformations* (see for instance [14, §1.5]) which transform the generating set $X = \{x_1, \dots, x_s\}$ to $Y = \{y_1, \dots, y_m\}$, simultaneously they transform system of relators Q_1, \dots, Q_n for X to a system of relators for Y . On each step a finite presentation is transformed to a finite presentation. Hence, in the end we get a finite system of relators R_1, \dots, R_k for the generating set X . \square

Therefore there are finitely many elements $\alpha_1, \dots, \alpha_n$ of $\pi_1(\mathcal{F})$ which normally generate the kernel $\text{Ker}(\phi)$ of

$$\phi : \pi_1(\mathcal{F}) \rightarrow \pi_1(\tilde{X})$$

We shall identify α_j and the corresponding loops on \mathcal{F} . Thus there is a closed metric disk D centered at the origin in $\mathbb{H}^2 = \tilde{S}$ such that each $\alpha_j, j = 1, \dots, n$, is contractible in $U = \tilde{f}^{-1}(D)$. This implies that each $\alpha \in \text{Ker}(\phi)$ is contractible in $\tilde{f}^{-1}(D)$. We will assume that the boundary of D contains no critical values of γ . However we have infinitely many critical values of \tilde{f} outside of the disk D . Let z be one of them and D' be a closed topological disk in \mathbb{H}^2 which contains both D and z and does not contain any critical values of \tilde{f} which are not in $\{z\} \cup D$. Homotopically the Morse surgery corresponding to z amounts to attaching 2-cells along certain loops $\alpha \subset \mathcal{F}$. Thus $\alpha \in \text{Ker}(\phi)$, which implies that α is contractible in U . It follows that we get an immersed homotopically nontrivial 2-sphere $\zeta \subset \tilde{f}^{-1}(D')$. The space \tilde{X} is obtained from $\tilde{f}^{-1}(D')$ by attaching only 2-handles, thus the homotopy class $[\zeta]$ is nontrivial in $\pi_2(\tilde{X})$ which contradicts asphericity of \tilde{X} . This concludes the proof in the case when \tilde{f} is a complex Morse function.

Remark 6 *J. Kollár had suggested an argument which reduces the general case to the case of holomorphic Morse function provided that no irreducible component of each singular fiber of \tilde{f} has multiplicity > 1 . Namely, perturb $\tilde{f} : \tilde{X} \rightarrow \mathbb{H}^2$ in a Q -equivariant manner to a smooth map $g : \tilde{X} \rightarrow \mathbb{H}^2$ with connected fibers so that:*

(a) *The sets of critical values of g and \tilde{f} are equal.*

(b) *If s is a critical value of g (and \tilde{f}) and $C_s \subset \mathbb{H}^2$ is a small circle around s then the 3-manifolds $g^{-1}(C_s), \tilde{f}^{-1}(C_s)$ are homeomorphic.*

(c) *The mapping g is a holomorphic Morse function near each singular fiber.*

Then apply the same arguments as before to the function g to conclude that neither \tilde{f} nor g has critical points.

However, technically it seems (at least to me) easier to apply the direct topological arguments below than to analyze the special case when a singular fiber of \tilde{f} has an irreducible component of multiplicity > 1 .

We now consider the general case. We will run essentially the same arguments as in the case of holomorphic Morse function. Let $\Sigma = \Sigma(\tilde{f})$ denote the set of critical values of the holomorphic function \tilde{f} , $\tilde{S}' := \tilde{S} - \Sigma$ and $\tilde{X}' := \tilde{f}^{-1}(S')$.

Lemma 7 (1) *The fundamental group of a generic fiber \mathcal{F} of \tilde{f} maps onto $K = \pi_1(\tilde{X})$. (2) *No singular fiber of \tilde{f} is a multicurve, i.e. a singular fiber of \tilde{f} cannot be a smooth complex curve.**

Proof. The restriction \tilde{f}' of \tilde{f} to \tilde{X}' is a (nonsingular) fibration with connected fibers, thus $\pi_1(\mathcal{F})$ is the kernel of the homomorphism

$$\pi_1(\tilde{f}') : \pi_1(\tilde{X}') \rightarrow \pi_1(\tilde{S}')$$

In particular, the subgroup $\pi_1(\mathcal{F})$ is normal in $\pi_1(\tilde{X}')$. For each puncture $s_i \in \Sigma$ choose a small loop on \tilde{S}' going once around s_i and choose a homeomorphic lift γ_i of this loop to \tilde{X}' . Then the group $\pi_1(\tilde{X}')$ is generated by $\pi_1(\mathcal{F})$ and by the loops $\gamma_i, s_i \in \Sigma$. Let D_{s_i} denote a small metric disk on \mathbb{H}^2 centered at $s_i \in \Sigma$ (so that $D_{s_i} \cap \Sigma = \{s_i\}$). If for some s_i the fundamental group of $\pi_1(\partial \tilde{f}^{-1}(D_{s_i}))$ does not map

onto $\pi_1(\tilde{f}^{-1}(D_{s_i}))$ then it is true for infinitely many points $s \in \Sigma$ (all the points in the Q -orbit of s_i), thus the group K cannot be finitely generated. Thus the map

$$\pi_1(\tilde{X}') \rightarrow \pi_1(\tilde{X})$$

is onto. Since γ_i -s belong to the kernel of this map we conclude that the group $\pi_1(\mathcal{F})$ maps onto $\pi_1(\tilde{X})$.

If $\mathcal{F}_{s_i} = \tilde{f}^{-1}(s_i)$, $s_i \in \Sigma$, is a multicurve then

$$\pi_1(\partial\tilde{f}^{-1}(D_{s_i})) \rightarrow \pi_1(\tilde{f}^{-1}(D_{s_i})) = \pi_1(\mathcal{F}_{s_i})$$

is not onto (Lemma 4), which contradicts our assumptions. This proves the second assertion of Lemma. \square

Now suppose that $f : X \rightarrow S$ is not a nonsingular holomorphic fibration. Thus the map \tilde{f} has at least one fiber which is not a smooth complex curve. (By Lemma 7 each singular fiber has to be of this type.) Our goal is to show that this assumption leads to a contradiction. Let $T \subset \mathbb{H}^2$ be a locally finite embedded tree whose vertex set is Σ (this tree of course is not $\pi_1(S)$ -invariant). We can assume that edges of T are geodesics in \mathbb{H}^2 . For each vertex $s \in \Sigma$ of T we choose a small closed metric disk D_s centered at s such that $D_s \cap T$ is equal to the intersection of D_s and open edges of T emanating from s . If $T' \subset T$ is a subtree then $N(T')$ will denote the union of T' and disks D_s for those vertices s of T which belong to T' . Let $Y(T') := \tilde{f}^{-1}(N(T'))$.

Since \tilde{f} is a smooth fibration away from singular fibers it follows that the inclusion

$$Y(T) \hookrightarrow \tilde{X}$$

is a homotopy-equivalence. Therefore we restrict our attention to the topology of $Y(T)$.

Let T' be a finite subtree of T which is the convex hull of its vertices.

Lemma 8 *The homomorphism*

$$\pi_2(Y(T')) \rightarrow \pi_2(Y(T))$$

is injective.

Proof. It is enough to prove this assertion for the lifts of $Y(T')$, $Y(T)$ to the universal cover of X . Since X is aspherical, its universal cover \tilde{X} cannot contain compact complex curves, hence the lift of $\tilde{f}^{-1}(t)$, $t \in T - \Sigma$ to \tilde{X} is a noncompact surface. Therefore this lift has trivial H_2 and the assertion follows from the Meyer-Vietors sequence. \square

Let $s \in \Sigma - T'$ be a vertex of T which is connected to T' by an edge $[ss']$, $s' \in \Sigma \cap T'$. Note that the inclusions

$$Y(T') \hookrightarrow Y(T' \cup [s's]), \quad Y(s) \hookrightarrow Y([s's])$$

are homotopy-equivalences. Here and in what follows $[s's')$ denotes the half-open edge connecting s to s' : $s \in [s's')$, $s' \notin [s's')$.

Lemma 9 *Suppose that $\pi_1(Y(T')) \rightarrow \pi_1(Y(T))$ is a monomorphism. Then $\pi_2(Y(T' \cup [ss'])) \neq 0$.*

Proof. Let $t \in [ss']$ be the midpoint. Then there is a subsurface $\mathcal{F}' \subset \mathcal{F}_t$ such that:

- (a) No boundary loop of \mathcal{F}' is nil-homotopic in \mathcal{F}_t .
- (b) The image of $\pi_1(\mathcal{F}')$ in $\pi_1(Y([ss']))$ is trivial.

The subsurface \mathcal{F}' appears as follows: let $p \in \mathcal{F}_s$ be a singular point, then \mathcal{F}' is a part of \mathcal{F}_t corresponding to the Milnor fiber in $S_\epsilon(p)$, see section 2. If a boundary loop of \mathcal{F}' is nil-homotopic in \mathcal{F}_t then \tilde{X} contains a rational complex curve which contradicts the assumption that $\pi_2(X) = 0$.

Therefore, the assumption of Lemma implies that the image of $\pi_1(\mathcal{F}')$ in $\pi_1(Y(T' \cap [ss']))$ is trivial. Consider the total lift $\hat{\mathcal{F}}'$ of \mathcal{F}' to the universal cover \hat{X} of X , then $\hat{\mathcal{F}}'$ is contained in the lift $\hat{\mathcal{F}}_t$ of \mathcal{F}_t to \hat{X} . Note that no component of $\hat{\mathcal{F}}_t - \hat{\mathcal{F}}'$ is bounded (otherwise after degeneration of $\hat{\mathcal{F}}_s$ to a singular fiber we will get a compact complex curve in \hat{X} which is impossible). If \mathcal{F}' is not a planar surface then $\hat{\mathcal{F}}'$ contains a non-separating loop, otherwise a component of $\partial\hat{\mathcal{F}}'$ is not nil-homologous in $\hat{\mathcal{F}}_t$. In the both cases we apply Meyer-Vietors arguments to get a homologically nontrivial spherical cycle in $\hat{Y}(T' \cup [ss'])$, thus $\pi_2(Y(T' \cup [ss'])) \neq 0$. \square

Since K is assumed to be finitely-presentable, there are finitely many elements $\alpha_i \in \pi_1(\mathcal{F})$ which normally generate the kernel of $\pi_1(\mathcal{F}) \rightarrow \pi_1(Y)$. Thus there is a finite subtree $T' \subset T$ such that all the loops α_i are nil-homotopic in $Y(T')$. Since $\pi_1(Y(T'))$ maps onto $\pi_1(Y(T))$ (Lemma 7) it follows that $\pi_1(Y(T')) \rightarrow \pi_1(Y(T))$ is an isomorphism. Hence for an edge $[ss']$ of T which has one vertex in T' and the other vertex in $T - T'$ we have:

$$\pi_2(Y(T' \cup [ss'])) \neq 0$$

(according to Lemma 9). Now we apply Lemma 8 to conclude that $\pi_2(Y(T)) \neq 0$. However $\pi_2(Y(T)) \cong \pi_2(\tilde{X}) = 0$ since X is aspherical. This contradiction proves Theorem 2. \square

5 Complex-hyperbolic surfaces

Let $B \subset \mathbb{C}^2$ be the unit ball. We will give B the *Kobayashi metric*, this metric can be described as follows. Let $p, q \in B$ be distinct points, there is a unique complex line $L \subset \mathbb{C}^2$ so that $p, q \in B \cap L$. Now identify $B \cap L$ with the hyperbolic plane \mathbb{H}^2 where the curvature is normalized to be -1 . Finally let $d(p, q) := d_{\mathbb{H}^2}(p, q)$. Then the *complex-hyperbolic plane* $\mathbb{H}_{\mathbb{C}}^2$ is the unit ball B with the Kobayashi distance d . It turns out that the Kobayashi distance d is induced by a Riemannian metric ρ on B . Below we list some properties of the complex-hyperbolic plane $\mathbb{H}_{\mathbb{C}}^2$, we refer to [8], [2], [5], [20] for detailed discussion.

- (a) ρ is Kähler.
- (b) The sectional curvature of ρ is pinched between the constants -1 and $-1/4$.

(c) The group of biholomorphic automorphisms of B equals the identity component in the isometry group of $\mathbb{H}_{\mathbb{C}}^2$ which is isomorphic to the $PU(2, 1)$ so that B is the symmetric space for the group $PU(2, 1)$: $B = PU(2, 1)/K$ where $K \cong U(2)$ is a maximal compact subgroup in $PU(2, 1)$.

(d) Let Γ be a torsion-free uniform lattice in $PU(2, 1)$. The quotient B/Γ is a compact Kähler surface which is actually a smooth complex algebraic surface. The quotient B/Γ is called a *complex-hyperbolic surface*.

(e) For each compact complex-hyperbolic surface we have the following identity between the Chern classes: $c_1^2 = 3c_2$, i.e. $\chi = 3\tau$ where χ is the Euler characteristic and τ is the signature.

(f) If X is a smooth compact complex algebraic surface for which the equality $c_1^2 = 3c_2$ holds, then the universal cover of X is biholomorphic to either $\mathbb{H}_{\mathbb{C}}^2$, or \mathbb{C}^2 , or the complex-projective plane $\mathbb{P}_{\mathbb{C}}^2$.

The key fact about complex-hyperbolic surfaces which will be used in this paper is the following recent theorem of K. Liu [12]:

Theorem 10 *Let X be a compact complex-hyperbolic surface. Then X does not admit nonsingular holomorphic fibrations over complex curves.*

6 Incoherent example

Recall that a group Γ is called *coherent* if every finitely-generated subgroup $\Gamma' \subset \Gamma$ is also finitely presentable. Examples of coherent groups include free groups, surface groups, 3-manifold groups (see [19]) and certain groups of cohomological dimension 2 (see [7], [15]). The simplest example of noncoherent group is $\mathbb{F}_2 \times \mathbb{F}_2$, where \mathbb{F}_2 is the free group on two generators. (The finitely generated infinitely presentable subgroup in $\mathbb{F}_2 \times \mathbb{F}_2$ is the kernel of the homomorphism $\phi : \mathbb{F}_2 \times \mathbb{F}_2 \rightarrow \mathbb{Z}$ where ϕ maps each free generator of each \mathbb{F}_2 to the generator of \mathbb{Z} .) Thus there is a uniform lattice in the Lie group $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ which is not coherent. The first example of noncoherent discrete geometrically finite subgroup of $Isom(\mathbb{H}^4)$ was constructed in [11], [18]. Later on this example was generalized in [4] to a uniform lattice in $Isom(\mathbb{H}^4)$.

As an application of the main result of this paper we show that certain uniform lattices in $PU(2, 1) = Isom(\mathbb{H}_{\mathbb{C}}^2)$ are not coherent (these are the first known examples of incoherent discrete subgroups of $PU(2, 1)$). The groups which we consider were known before (see [13], [2], [5]) however their incoherence was unknown.

Lemma 11 *Suppose that X is a compact complex-hyperbolic surface whose fundamental group π fits into a short exact sequence*

$$1 \rightarrow K \rightarrow \pi \rightarrow Q = \pi_1(S) \rightarrow 1$$

where S is a compact hyperbolic Riemann surface and the group K is finitely generated. Then K is not finitely presentable.

Proof. Suppose that K is finitely presentable. The surface X is aspherical since its universal cover is the complex ball. Then by Theorem 2 the projection $\pi \rightarrow Q$ is induced by a nonsingular holomorphic fibration of the surface X . On the other hand, complex-hyperbolic surfaces do not admit such fibrations by [12]. \square

Now we describe an example the fundamental group of a complex-hyperbolic surface satisfying the conditions of Lemma 11 following [13]. Define automorphisms ϕ, ψ of the free group on three generators A_1, A_2, A_3 by

$$\phi(A_1) = A_1 A_2 A_1^{-1}, \phi(A_2) = A_1 A_3 A_1^{-1}, \phi(A_3) = A_1$$

$$\psi(A_1) = (A_1 A_2) A_3 (A_1 A_2)^{-1}, \psi(A_2) = A_1 A_2 A_1^{-1}, \psi(A_3) = A_1$$

Ron Livne [13] constructed a uniform lattice $\Gamma_{d,N}$ in $PU(2,1)$ with the presentation

$$\langle x, y, A_1, A_2, A_3 \mid x A_i x^{-1} = \phi(A_i), y A_i y^{-1} = \psi(A_i) \quad (1 \leq i \leq 3),$$

$$x^3 = y^2 = A_1 A_2 A_3, (A_1 A_2 A_3)^{2d} = A_1^2 = A_2^2 = A_3^2 = (y x^{-1})^N = 1 \rangle$$

where $(N, d) \in \{(7, 7), (8, 4), (9, 3), (12, 2)\}$. Note that the subgroup K_d generated by A_1, A_2, A_3 in $\Gamma_{d,N}$ is normal and finitely generated, the quotient $\Gamma_{d,N}/K_d$ is the hyperbolic triangle group

$$\Delta_N := \langle x, y \mid x^3 = y^2 = (y x^{-1})^N = 1 \rangle$$

since $N \geq 7$. Now fix a pair (N, d) from the above list and let $\Gamma := \Gamma_{d,N}$, $\Delta := \Delta_N$, $K := K_d$. Let $\Delta' < \Delta$ be a torsion-free subgroup of finite index and $\Gamma' < \Gamma$ be the pull-back of Δ' to Γ . Then Γ' fits into short exact sequence

$$1 \rightarrow K \rightarrow \Gamma' \rightarrow \Delta' \rightarrow 1$$

The group Γ' still has torsion, so let π be a torsion-free subgroup of Γ' , $K' := \pi \cap K$, $Q := \pi/K'$. Clearly K' is finitely generated and Q is the fundamental group of a compact hyperbolic Riemann surface. The group π acts freely discretely cocompactly on $\mathbb{H}_{\mathbb{C}}^2$ and hence is the fundamental group of the compact complex-hyperbolic surface $X = \mathbb{H}_{\mathbb{C}}^2/\pi$. By Lemma 11 the group K' is not finitely presentable.

Remark 12 *Bill Goldman had told me long ago about Livne's example as a candidate for non-coherence, however until recently I did not know how to prove that the group K is not finitely presentable.*

Note that the group K is not geometrically finite and its limit set is the whole sphere at infinity of $\mathbb{H}_{\mathbb{C}}^2$ (since K is normal in Γ).

Question 13 *Let $\Gamma \subset PU(2,1)$ be a finitely generated discrete subgroup whose limit set is not the whole sphere at infinity of $\mathbb{H}_{\mathbb{C}}^2$. Is Γ finitely-presentable? Is Γ geometrically finite?*

Remark 14 *There are several reasons why it is difficult to construct finitely generated geometrically infinite subgroups of $PU(2, 1)$. One of them is the following result due to M. Ramachandran:*

Let Γ be a discrete subgroup of $PU(2, 1)$ which does not contain parabolic elements and which acts cocompactly on a component Ω_0 of the domain of discontinuity $\Omega(\Gamma) \subset \partial_\infty \mathbb{H}_\mathbb{C}^2$. Then Γ is geometrically finite and $\Omega_0 = \Omega(\Gamma)$.

(Instead of assuming that Γ contains no parabolic elements it is enough to assume that each maximal parabolic subgroup of Γ is isomorphic to a lattice in the 3-dimensional Heisenberg group.)

Question 15 *Is there a compact **real-hyperbolic** 4-manifold X whose fundamental group fits into a short exact sequence:*

$$1 \rightarrow K \rightarrow \pi_1(X) \rightarrow Q \rightarrow 1$$

where K is finitely presentable or even a surface group and Q is a hyperbolic surface group ?

More generally:

Question 16 *Is there a Gromov-hyperbolic group π which fits into a short exact sequence:*

$$1 \rightarrow K \rightarrow \pi \rightarrow Q \rightarrow 1$$

where K and Q are closed hyperbolic surface groups?

Note that Lee Mosher [17] constructed similar example when K is a closed hyperbolic surface group and Q is a free nonabelian group.

Question 17 *Let Γ_g be the mapping class group of a compact surface of genus g . Is there g and a finitely generated non-free subgroup Q of Γ_g which consists only of the identity and pseudo-Anosov elements?*

Mosher's example comes from a "Schottky-type" subgroup Q in Γ_g where K is the fundamental group of a genus g surface.

References

- [1] D. ARAPURA, P. BRESSLER, AND M. RAMACHANDRAN, *On the fundamental group of a compact Kähler manifold*, Duke Math. J., 68 (1993), pp. 477–488.
- [2] G. BARTHEL, F. HIRZEBRUCH, AND T. HÖFER, *Geradenkonfigurationen und Algebraische Flächen*, Aspects of Mathematics, D4, Friedr. Vieweg & Sohn, Braunschweig, 1987.
- [3] M. BESTVINA AND N. BRADY, *Morse theory and finiteness properties of groups*, Inventiones Math., 129 (1997), pp. 445–470.

- [4] B. BOWDITCH AND G. MESS, *A 4-dimensional Kleinian group*, Transactions of AMS, 14 (1994), pp. 391–405.
- [5] P. DELIGNE AND G. D. MOSTOW, *Commensurabilities among lattices in $PU(1, n)$* , vol. 132 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1993.
- [6] D. EISENBUD AND W. NEUMANN, “*Three-dimensional link theory and invariants of plane curve singularities*”, vol. 110 of Ann. of Math. Stud., Princeton Univ. Press, 1985.
- [7] M. FEIGN AND M. HANDEL, *Mapping tori of free groups are coherent*. Preprint, 1997.
- [8] W. GOLDMAN, *Complex hyperbolic geometry*, Oxford Mathematical Monographs, To appear.
- [9] M. GROMOV, *Asymptotic invariants of infinite groups*, in “Geometric groups theory”, volume 2, Proc. of the Symp. in Sussex 1991, G. Niblo and M. Roller, eds., vol. 182 of Lecture Notes series, Cambridge University Press, 1993.
- [10] J. HILLMAN, *Complex surfaces which are fibre bundles*, Topology and its Applications, (to appear).
- [11] M. KAPOVICH AND L. POTYAGAILO, *On absence of Ahlfors’ finiteness theorem for Kleinian groups in dimension 3*, Topology and its Applications, 40 (1991), pp. 83–91.
- [12] K. LIU, *Geometric height inequalities*, Math. Res. Lett., 3 (1996), pp. 693–702.
- [13] R. LIVNE, *On certain covers of the universal elliptic curve*. Ph. D. Thesis, Harvard University, 1981.
- [14] W. MAGNUS, A. KARRASS, AND D. SOLITAR, *Combinatorial group theory: presentations in terms of generators and relations*, vol. 13 of Pure and Applied Mathematics, Interscience Publ., New York, 1966.
- [15] J. MCCAMMOND AND D. WISE, *Coherent groups and perimeter of 2-complexes*. Preprint, 1997.
- [16] J. MILNOR, *Singular points of complex hypersurfaces*, vol. 61 of Annals of Mathematics Studies, Princeton University Press, Princeton, 1968.
- [17] L. MOSHER, *Hyperbolic-by-hyperbolic hyperbolic groups*, Proceedings of AMS, (to appear).
- [18] L. POTYAGAILO, *The problem of finiteness for Kleinian groups in 3-space*, in Knots 90 (Osaka, 1990), de Gruyter, Berlin, 1992, pp. 619–623.
- [19] P. SCOTT, *Compact submanifolds of 3-manifolds*, Journ. of the LMS, 6 (1973), pp. 437–448.

- [20] YAU, S.-T., *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. USA, 74 (1977), pp. 1789–1799.

Michael Kapovich: Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA ; kapovich@math.utah.edu