Krull Dimensions of Rings of Holomorphic Functions

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Abstract. We prove that the Krull dimension of the ring of holomorphic functions of a connected complex manifold is at least the cardinality of continuum if and only if it is > 0.

Let R be a commutative ring. Recall that the *Krull dimension* $\dim(R)$ of R is the supremum of cardinalities lengths of chains of distinct proper prime ideals in R. Our main result is:

Theorem 1. *Let M be a connected complex manifold and $H(M)$ be the ring of holomorphic functions on M. Then the Krull dimension of $H(M)$ either equals* 0 *(if and only if $H(M) = \mathbb{C}$) or is infinite, if and only if M admits a nonconstant holomorphic function $M \to \mathbb{C}$. More precisely, unless $H(M) = \mathbb{C}$, $\dim H(M) \geq \mathfrak{c}$, i.e., the ring $H(M)$ contains a chain of distinct prime ideals whose length has cardinality of continuum.*

Our proof of this theorem mostly follows the lines of the proof by Sasane [**S**], who proved that for each nonempty domain $M \subset \mathbb{C}$ the Krull dimension of $H(M)$ is infinite (he did not prove that $\dim H(M) \geq \mathfrak{c}$).

Remark 2. We note that Henricksen [**H**] was the first to prove that the Krull dimension of the ring of entire functions on $\mathbb{C}$ has cardinality at least continuum.

In our proof we will use the Axiom of Choice in two ways: (a) to establish existence of certain maximal ideals and (b) to get existence of a nonprincipal ultrafilter ω on $\mathbb{N}$ and, hence of the ordered field ${}^*\mathbb{R}$ of *nonstandard real* (or, *hyperreal*) numbers. The field ${}^*\mathbb{R}$ contains ${}^*\mathbb{N}$, the *nonstandard natural* (or *hypernatural*) numbers.

The field ${}^*\mathbb{R}$ is a certain quotient of the countable direct product $\prod_{k\in\mathbb{N}} \mathbb{R}$; we will denote the equivalence class (in ${}^*\mathbb{R}$) of a sequence (x_k) in $\mathbb{R}$ by $[x_k]$. Accordingly, ${}^*\mathbb{N}$ consists of equivalence classes $[n_k]$ of sequences of natural numbers. Roughly speaking, we will use ${}^*\mathbb{N}$ and a certain order relation on it to compare rates of growth of sequences of natural numbers.

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Definition 3. A commutative unital ring R is *ample* if there exists a sequence of valuations ν_k on R such that for each $\beta \in {}^*\mathbb{N}$, there exists an $a = a_\beta \in R$ with the property

$$[\nu_k(a)] = \beta. \tag{1}$$

The main technical result of this paper is:

Theorem 4. *For each ample ring R,* $\dim(R) \geq \mathfrak{c}$. *In particular, R has infinite Krull dimension.*

This theorem and its proof are inspired by Theorem 2.2 of [**S**], although some parts of the proof resemble the ones of [**H**].

We will verify, furthermore, that whenever M is a connected complex manifold which has a nonconstant holomorphic function, the ring $H(M)$ is ample. This, combined with Theorem 4, will immediately imply Theorem 1.

Remark 5. 1. We refer the reader to Section 5.3 of [**Cla**] for further discussion of algebraic properties of rings of holomorphic functions and for a shorter proof of Theorem 4, which uses ultralimits and but not the hyperreal numbers.

2. Theorem 1 shows that for every Stein manifold M (of positive dimension), the ring $H(M)$ has infinite Krull dimension. In particular, this applies to any noncompact connected Riemann surfaces (since every such surface is Stein, [**BS**]).

3. Noncompact connected complex manifolds M of dimension > 1 can have $H(M) = \mathbb{C}$; for instance, take M to be the complement to a finite subset in a compact connected complex manifold (of dimension > 1).

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1. Hyperreal numbers

We refer the reader to [**Go**] for a detailed treatment of hyperreal numbers, but we include a brief introduction below. A nonprincipal ultrafilter on $\mathbb{N}$ can be regarded as a finitely-additive probability measure on $\mathbb{N}$ which vanishes on each finite subset and takes the value 0 or 1 on each subset of $\mathbb{N}$. The existence of nonprincipal ultrafilters (the *ultrafilter lemma*) follows from the Axiom of Choice. Subsets of full measure are called *ω-large.* Using ω, one defines the following equivalence relation on the product

$$\prod_{k\in\mathbb{R}} \mathbb{R}.$$

Two sequences (x_k) and (y_k) are equivalent if $x_k = y_k$ for an ω-all k, i.e., the set

$$\{k : x_k = y_k\}$$

is ω-large. The quotient by this equivalence relation, denoted

$${}^*\mathbb{R} = \prod_{k\in\mathbb{N}} \mathbb{R}/\omega,$$

is the set of hyperreal numbers. Let $[x_k]$ be the equivalence class of the sequence (x_k).

The binary operations on sequences of real numbers project to binary operations on ${}^*\mathbb{R}$ making ${}^*\mathbb{R}$ a field. The total order $\leq$ on ${}^*\mathbb{R}$ is defined by $[x_k] \leq [y_k]$ if and only if $x_k \leq y_k$ for an ω-all $k \in \mathbb{N}$. With this order, ${}^*\mathbb{R}$ becomes an ordered field.

The set of real numbers embeds into ${}^*\mathbb{R}$ as the set of equivalence classes of constant sequences; the image of a real number x under this embedding is still denoted by x. We set ${}^*\mathbb{R}_+ := \{\alpha \in {}^*\mathbb{R} : \alpha > 0\}$.

The projection of

$$\prod_{k\in\mathbb{N}} \mathbb{N} \subset \prod_{k\in\mathbb{N}} \mathbb{R}$$

to ${}^*\mathbb{R}$ is denoted ${}^*\mathbb{N}$, this is the set of *hypernatural numbers*. We define a further equivalence relation $\sim_u$ on ${}^*\mathbb{R}$ by:

$$\alpha \sim_u \beta$$

if there exist positive real numbers a, b such that

$$a\alpha \leq \beta \leq b\alpha.$$

The equivalence class (α) of $\alpha \in {}^*\mathbb{R}$ (for this equivalence relation) is a multiplicative analogue of the *galaxy* $gal(\alpha)$ of α, see [**Go**]:

Definition 6. The *galaxy* $gal(\alpha)$ of a hyperreal number $\alpha \in {}^*\mathbb{R}$ is the union

$$\bigcup_{n\in\mathbb{N}} [\alpha - n, \alpha + n] \subset {}^*\mathbb{R}.$$

In other words, $\beta \in gal(\alpha)$ if and only if there exists a real number a such that $\alpha - a \leq \beta \leq \alpha + a$.

The next lemma is immediate:

Lemma 7. *For* $\alpha \in {}^*\mathbb{R}_+$*, the equivalence class* (α) *of* α *equals* $\exp(gal(\log(\alpha)))$.

We let ${}^u\mathbb{R}$ denote the quotient ${}^*\mathbb{R}/\sim_u$ and ${}^u\mathbb{N}$ the projection of ${}^*\mathbb{N}$ to ${}^u\mathbb{R}$. Define the total order $\gg$ on ${}^u\mathbb{R}$ by

$$(\beta) \gg (\alpha)$$

if for every real number c, $c\alpha < \beta$. By abusing the notation, we will simply say that $\beta \gg \alpha$, with $\alpha, \beta \in {}^*\mathbb{R}$.

For the reader who prefers to think in terms of sequences of (positive) real numbers, the relation $(\beta) \gg (\alpha)$ is an analogue of the relation

$$(a_n) = o((b_n)), \quad n \to \infty.$$

Remark 8. The equivalence relation $\sim_u$ and the order $\gg$ are similar to the ones used by Henricksen in [**H**].

Proposition 9. *The set* ${}^u\mathbb{N}$ *has the cardinality of continuum.*

Proof. Note first that ${}^*\mathbb{R}$ has cardinality of continuum, hence, the cardinality of ${}^u\mathbb{N}$ is at most $\mathfrak{c}$. The proof of the proposition then reduces to two lemmata.

Lemma 10. *The set* $gal({}^*\mathbb{R}_+)$ *of galaxies* $\{gal(\alpha) : \alpha \in {}^*\mathbb{R}_+\}$ *has the cardinality of continuum.*

Proof. For each $\alpha = [a_k] \in {}^*\mathbb{R}_+$, the galaxy $gal(\alpha)$ contains the hypernatural number $\lceil\alpha\rceil = [b_k]$, where $b_k = \lceil a_k \rceil$. For each hypernatural number $\beta \in {}^*\mathbb{N}$, and natural number $n \in \mathbb{N}$, the intersection

$$[\beta - n, \beta + n] \cap {}^*\mathbb{N}$$

is finite, equal $\{\beta - n, ..., \beta + n\}$. Therefore, $gal(\beta) \cap {}^*\mathbb{N} = \{\beta\} + \mathbb{Z}$. It follows that the map

$${}^*\mathbb{N} \to gal({}^*\mathbb{R}_+), \quad \beta \mapsto gal(\beta)$$

is a bijection modulo $\mathbb{Z}$. Lastly, the set of hypernatural numbers ${}^*\mathbb{N}$ has the cardinality of continuum. □

Lemma 11. *The map $\lambda : {}^*\mathbb{N} \to gal({}^*\mathbb{R}_+)$, $\lambda : \beta \mapsto gal(\log(n))$, is surjective.*

Proof. For each $\alpha \in {}^*\mathbb{R}_+$ let $\beta = \lceil \exp(\alpha) \rceil \in {}^*\mathbb{N}$. Since $\log(x+1) - \log(x) \leq 1$ for $x \geq 1$, we have that

$$\log(\beta) \in gal(\alpha).$$

□

Now, we can finish the proof of the proposition. The map $\lambda : {}^*\mathbb{N} \to gal({}^*\mathbb{R}_+)$ descends to a map $\mu : {}^u\mathbb{N} \to gal({}^*\mathbb{R}_+)$. According to Lemma 11, the map μ is surjective. By Lemma 10 the set $gal({}^*\mathbb{R}_+)$ has the cardinality of continuum.

We will prove Theorem 4 in the next section by showing that for each ample ring R, the ordered set $({}^u\mathbb{N}, \gg)$ embeds into the poset of prime ideals in R reversing the order:

$$(\beta) \gg (\alpha) \Rightarrow P_\beta \subsetneq P_\alpha$$

for certain prime ideals $P_\gamma \subset R$ determines by $(\gamma) \in {}^u\mathbb{N}$. Proposition 9 will then imply that the Krull dimension of R is at least $\mathfrak{c}$.

2. Krull dimension of ample rings

Recall that a *valuation* on a unital ring R is a map $\nu : R \to \mathbb{R}_+ \cup \{\infty\}$ such that:
1. $\nu(a + b) \geq \min(a, b)$,
2. $\nu(ab) = \nu(a) + \nu(b)$.
3. $\nu(a) = \infty \iff a = 0$.
4. $\nu(1) = 0$.

For the following lemma, see Theorem 10.2.6 in [**Coh**] (see also Proposition 4.8 of [**Cla**] or Theorem 1 in [**K**]).

Lemma 12. *Let I be an ideal in a commutative ring A and $M \subset A \backslash I$ be a subset closed under multiplication. Then there exists an ideal $J \subset A$ containing I and disjoint from M, so that J is maximal with respect to this property. Furthermore, J is a prime ideal in A.*

Let R be an ample ring and ν_k the corresponding sequence of valuations on R. For each $\beta \in {}^*\mathbb{N}$ we define

$$I_\beta := \{a \in R | \ [\nu_k(a)] \gg [\beta]\} \subset R.$$

Lemma 13. *Each I_α is an ideal in R.*

PROOF. We will check that I_α is additive since it is clearly closed under multiplication by elements of R. Take $p', p'' \in I_\alpha$,

$$[\nu_k(p')] \gg \alpha, [\nu_k(p'')] \gg \alpha.$$

By the definition of a valuation,

$$n_k := \nu_k(p' + p'') \geq \min(\nu_k(p'), \nu_k(p'')),$$

for each $k \in \mathbb{N}$. For $m \in \mathbb{N}$, define the ω-large sets

$$A' = \{k : \nu_k(p') \geq m\alpha\}, \quad A'' = \{k : \nu_k(p'') \geq m\alpha\}.$$

Therefore, their intersection $A = A' \cap A''$ is ω-large as well, which implies that

$$\forall m \in \mathbb{N}, [n_k] \geq m\alpha \Rightarrow [n_k] \gg \alpha.$$

□

Then for each $\gamma \gg \beta$, the element a_γ as in Definition 3, belongs to I_β. It follows that $I_\beta \neq 0$ for every β. Define the subsets

$$M_\beta := \{a \in R | \exists n \in \mathbb{N}, [\nu_k(a)] \leq n\beta\} \subset R;$$

each M_β is closed under the multiplication. It is immediate that whenever $\alpha \leq \beta$, we have the inclusions

$$I_\beta \subset I_\alpha, \quad M_\alpha \subset M_\beta.$$

It is also clear that $I_\beta \cap M_\beta = \emptyset$. At the same time, for each $\beta \gg \alpha$,

$$a_\beta \in I_\alpha \cap M_\beta.$$

For each α we let $\mathcal{J}_\alpha$ denote the set of ideals $P \subset R$ such that

$$I_\alpha \subset P, P \cap M_\alpha = \emptyset.$$

By Lemma 12, every maximal element $P \in \mathcal{J}_\alpha$ is a prime ideal.

LEMMA 14. *Every $\mathcal{J}_\alpha$ contains unique maximal element, which we will denote P_α in what follows.*

PROOF. Suppose that P', P'' are two maximal elements of $\mathcal{J}_\alpha$. We define the ideal $P = P' + P''$. Clearly, P contains I_α. To prove that P is disjoint from M_α, take $p' \in P', p'' \in P''$, since $p' \notin M_\alpha, p'' \notin M_\alpha$. Then the same proof as in Lemma 13 shows that $[\nu_k(p' + p'')] \gg \alpha$ which means that $p' + p'' \notin M_\alpha$. Thus, $P \in \mathcal{J}_\alpha$ and, in view of maximality of P', P'', we obtain

$$P' = P = P''.$$

□

For each $\beta \gg \alpha$ we define the ideal $Q_{\alpha\beta} := I_\alpha + P_\beta$.

LEMMA 15. $Q_{\alpha\beta} \cap M_\alpha = \emptyset$.

PROOF. The proof is similar to the one of the previous lemma. Let $q = c + p$, $c \in I_\alpha, p \in P_\beta$. Since $p \notin M_\beta$, $p \notin M_\alpha$ as well. Therefore,

$$[\nu_k(p)] \gg \alpha.$$

Since $c \in I_\alpha$,

$$[\nu_k(c)] \gg \alpha.$$

Hence,

$$[\nu_k(c + p)] \gg \alpha$$

as well. Thus, $q \notin M_\alpha$. □

COROLLARY 16. *$Q_{\alpha\beta} \in \mathcal{J}_\alpha$. In particular, $Q_\alpha \subset P_\alpha$.*

PROOF. It suffices to note that $I_\alpha \subset Q_{\alpha\beta}$ according to the definition of $Q_{\alpha\beta}$. □

LEMMA 17. *The inequality $\beta \gg \alpha$ implies $P_\beta \subset P_\alpha$ and this inclusion is proper.*

PROOF. By the definition of $Q_{\alpha\beta}$ and Corollary 16, we have the inclusions

$$P_\beta \subset Q_\alpha \subset P_\alpha.$$

We now claim that $P_\beta \neq Q_{\alpha\beta} = I_\alpha + P_\beta$. Recall that $a_\alpha \in I_\alpha \subset Q_{\alpha\beta}$ and $a_\alpha \in M_\beta$, while $M_\beta \cap P_\beta = \emptyset$. Thus, $a_\alpha \in Q_{\alpha\beta} \setminus P_\beta$. □

According to Proposition 9, the set ${}^*\mathbb{N}$ of hypernatural numbers contains a subset S of cardinality continuum such that for all $\alpha < \beta$ in S, we have $\beta \gg \alpha$. The map

$$\alpha \mapsto P_\alpha$$

sends each $\alpha \in S$ to a prime ideal in R; $\alpha < \beta$ implies that $P_\beta \subsetneq P_\alpha$.

We conclude that the ring R contains the (descending) chain of distinct prime ideals $P_\alpha, \alpha \in S$; the length of this chain has the cardinality of continuum. In particular, $\dim(R) \geq \mathfrak{c}$. Theorem 4 follows. □

3. Ampleness of rings of holomorphic functions

We will need the following classical result, see e.g. [**Con**, Ch. VII, Theorem 5.15]:

THEOREM 18. *Let $D \subset \mathbb{C}$ be a domain, and let $c_k \in D$ be a sequence which does not accumulate anywhere in D and let m_k be a sequence of natural numbers. Then there exists a holomorphic function g in D which has zeroes only at the points c_k and such that m_k is the order of zero of g at c_k, $k \in \mathbb{N}$.*

COROLLARY 19. *If M is a connected complex manifold which admits a non-constant holomorphic function $h : M \to \mathbb{C}$, then the ring $H(M)$ is ample.*

PROOF. We let D denote the image of h. Pick a sequence $c_k \in D$ which converges to a point in $\hat{\mathbb{C}} \setminus D$ and which consists of regular values of h. (Here $\hat{\mathbb{C}}$ is the Riemann sphere.) For each c_k the preimage $C_k := h^{-1}(c_k)$ is a complex submanifold in M; in each C_k pick a point b_k. Define valuations

$$\nu_k : H(M) \to \mathbb{Z}_+ \cup \{\infty\}$$

by $\nu_k(f) := ord_{b_k}(f)$, the total order of f at b_k, cf. [**Gu**, Chapter C, Definition 1].

Now, given $\beta \in {}^*\mathbb{N}$, $\beta = [m_k]$, we let $g = g_\beta$ denote a holomorphic function on D as in Theorem 18. Define $a = a_\beta := g \circ h \in H(M)$. Then $\nu_k(a) = m_k$, which implies that the ring $H(M)$ is ample. □

Ampleness of $H(M)$ together with Theorem 4 imply Theorem 1.

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