Universality theorems for configuration spaces of planar linkages

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Abstract

We prove realizability theorems for vector-valued polynomial mappings, real-algebraic sets and compact smooth manifolds by moduli spaces of planar linkages. We also establish a relation between universality theorems for moduli spaces of mechanical linkages and projective arrangements.

1. Introduction

This paper deals with moduli spaces of planar linkages. An *abstract linkage* (L, ℓ) is a graph L with a positive real number $\ell(e)$ assigned to each edge e. We assume that we have chosen a distinguished oriented edge $e^* = [v_1 v_2]$ in L. The *moduli space* $\mathcal{M}(\mathcal{L})$ of planar realizations of $\mathcal{L} := (L, \ell, e^*)$ is the set[1] of maps ϕ from the vertex set of L into the Euclidean plane $\mathbb{R}^2$ (which will be identified with the complex plane $\mathbb{C}$) such that

- $|\phi(v) - \phi(w)|^2 = (\ell[vw])^2$ for each edge $[vw]$ of L.
- $\phi(v_1) = (0, 0)$.
- $\phi(v_2) = (\ell(e^*), 0)$.

Clearly these conditions give $\mathcal{M}(\mathcal{L})$ a natural structure of a real-algebraic set in $\mathbb{R}^{2r}$ where r is the number of vertices in L.

It is important to note that the ideal I in the polynomial ring $\mathbb{R}[X_1, Y_1, ..., X_r, Y_r]$ (here X_i, Y_i are the coordinates of $\phi(v_i)$) generated by the above equations can be strictly contained in the ideal of *all* polynomials vanishing on the set $\mathcal{M}(\mathcal{L}) \subset \mathbb{R}^{2r}$. We let $\mathfrak{M}(\mathcal{L})$ denote the affine subscheme of $\mathbb{R}^{2r}$ corresponding to the ideal I. We will in fact add more functions to the ideal I corresponding to certain degenerate triangles – see Convention 3.8.

In Definition 3.15 we define *functional linkages*. The abstract linkage corresponding to a functional linkage comes with two sets of vertices: the *inputs* $(P_1, ..., P_m)$ and the *outputs* $(Q_1, ..., Q_n)$. We let $p : \mathcal{M}(\mathcal{L}) \to \mathbb{A}^m$ (the *input map*) and $q : \mathcal{M}(\mathcal{L}) \to \mathbb{A}^n$ (the *output map*) be the *forgetful maps* that record only the positions of the images of P_i's and Q_j's under realizations ϕ. Here the affine line $\mathbb{A}$ is either $\mathbb{C} \cong \mathbb{R}^2$ (in which case we refer to $\mathcal{L}$ as a *complex functional linkage*) or $\mathbb{R} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ (in which case we refer to $\mathcal{L}$ as *real functional linkage*).

We say that $\mathcal{L}$ as above is a *functional linkage* [2] for a mapping $f : \mathbb{A}^m \to \mathbb{A}^n$ if there is a commutative diagram

[1] A priori this set could be empty.
[2] See Definition 3.15 for more precise definition.

$$\begin{array}{ccc} & \mathcal{M}(\mathcal{L}) & \\ \swarrow p & & q \searrow \\ \mathbb{A}^m & \xrightarrow{f} & \mathbb{A}^n \end{array}$$

and p is a *regular topological branched cover* of a bounded domain in $\mathbb{A}^m$. We prove

Theorem A. *Let $f : \mathbb{A}^m \to \mathbb{A}^n$ be a polynomial map with real coefficients (where $\mathbb{A}$ is either $\mathbb{C}$ or $\mathbb{R}$). Let $\mathcal{O}$ be a point in $\mathbb{A}^m$ and $r > 0$. Then there is a functional linkage $\mathcal{L}$ for f such that the r-ball $B_r(\mathcal{O})$ is in the interior of the image of p and p is an analytically trivial covering over $B_r(\mathcal{O})$. The same conclusion holds for polynomial maps whose coefficients are not required to be real if we use more general definition of functional linkage: instead of the moduli space $\mathcal{M}(\mathcal{L})$ we consider the space of* **relative** *realizations $C(\mathcal{L}, Z)$, see Definition 3.15 for details.*

Let $S \subset \mathbb{R}^m$ be a compact real-algebraic set, i.e. it is the zero set of a polynomial function $f : \mathbb{R}^m \to \mathbb{R}^n$. The set S is contained in an r-ball $B_r(\mathcal{O})$ centered at $\mathcal{O}$. We then apply Theorem A (the real case) and construct a functional linkage $\mathcal{L}$ for the pair $(f, B_r(\mathcal{O}))$. We let $\mathcal{L}_0$ be the abstract linkage obtained from $\mathcal{L}$ by gluing the output vertices to the base-vertex v_1. Let p_0 denote the restriction of the input mapping p of $\mathcal{L}$ to $\mathcal{M}(\mathcal{L}_0)$, this map is the "input map" of $\mathcal{L}_0$. We show in §10 that p_0 is an analytically trivial polynomial covering over S. We obtain

Theorem B. *Let X be any compact real-algebraic subset of $\mathbb{R}^m$. Then there is a linkage $\mathcal{L}_0$ so that $\mathcal{M}(\mathcal{L}_0)$ is Nash isomorphic to a disjoint union of a finite number of copies of X.*

Remark 1.1. *Nash isomorphism is defined in §2. Nash isomorphism implies real analytic isomorphism.*

Similarly we have

Theorem B'. *Let X be any complex-algebraic subset of $\mathbb{C}^m$ and U be an open (in the classical topology) bounded subset of X. Then there is a linkage $\mathcal{L}_0$ so that the input map p_0 is an analytically trivial polynomial covering over U.*

Remark 1.2. *Here we treat X as a* **real** *algebraic set.*

Now let M be a compact smooth manifold. By work of Seifert, Nash, Palais and Tognoli (see [AK] and [T]), M is diffeomorphic to a real algebraic set S, hence as a corollary of Theorem B we get

Corollary C. *Let M be a smooth compact manifold. Then there is a linkage $\mathcal{L}_0$ whose moduli space is diffeomorphic to a disjoint union of a number of copies of M.*

Remark 1.3. *It is not true that for any compact smooth manifold M there exists $\mathcal{L}_0$ such that $\mathcal{M}(\mathcal{L}_0)$ is diffeomorphic to M. This is because $\mathcal{M}(\mathcal{L}_0)$ admits a $\mathbb{Z}/2$-action (coming from $O(2)/SO(2)$) which is nontrivial provided that $\mathcal{L}_0$ is connected and $\mathcal{M}(\mathcal{L}_0)$ is not a point. Hence, if M is compact, distinct from a point and does not admit a nontrivial $\mathbb{Z}/2$-action then M cannot be diffeomorphic to the moduli space of a planar linkage. Examples of 3-manifolds M such that $Diff(M)$ contains no nontrivial finite subgroups see in the papers of S. Kojima [Ko1], [Ko2].*

Question 1.4. *Given a compact real-algebraic set S is it possible to construct a marked abstract linkage $\mathcal{L}$ so that the output map $C(\mathcal{L}, Z) \to M$ is a 2-fold analytically trivial covering? H. King [K], generalizing the methods of this paper, proved that the answer is* *positive if one generalizes the notion of abstract linkage to* **cabled linkage**, *i.e. certain points are connected by* **cables**: *they are not allowed to move more than certain distance apart. Moreover, King's theorem applies to real* **quasi-algebraic sets**, *i.e. sets determined by algebraic equations and non-strict algebraic inequalities.*

Universality theorems similar to Theorems A, B and Corollary C hold for moduli spaces of realizations of abstract arrangements in $\mathbb{P}^2$, see [Mn] [3], [KM2] and §12 of this paper for more details (the main framework of the present paper is analogous to [KM2]). However stronger realizability theorems hold in this case. In particular we get functional arrangements such that the input map is *injective* (see [KM2]). Let $\mathcal{M}(\mathcal{L}, \mathbb{RP}^2)$, $\mathcal{M}(\mathcal{L}, \mathbb{R}^3)$ denote the moduli spaces of realizations of a linkage $\mathcal{L}$ in $\mathbb{RP}^2$ and $\mathbb{R}^3$.

In §13 we establish a relation between the two kinds of universality theorems using moduli spaces of spatial Euclidean linkages, namely we show that the universality theorem for arrangements (Theorem 12.7) implies the following

Theorem D. *Let S be a compact real algebraic set defined over $\mathbb{Z}$. Then there exist abstract linkages $\mathcal{L}, \mathcal{L}'$ and Zariski open and closed subsets $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2) \subset \mathcal{M}(\mathcal{L}, \mathbb{RP}^2)$, $\mathcal{M}_0(\mathcal{L}', \mathbb{R}^3) \subset \mathcal{M}(\mathcal{L}', \mathbb{R}^3)$, so that:*

(1) $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$ is entire birationally isomorphic to S.

(2) $\mathcal{M}_0(\mathcal{L}', \mathbb{R}^3)$ is an analytically trivial entire rational covering of S.

Remark 1.5. *It seems surprising that one can prove a somewhat stronger realization theorem for arrangements than for linkages. One explanation for this is that the image of the input map of any connected functional linkage is bounded. By a theorem of Sullivan [Sul] a manifold with nonempty boundary cannot be an algebraic set* [4]*. Thus (unlike the case of functional arrangements)* **there are no functional linkages if we require the input map to be injective**.

There is a long history of previous work on mechanical linkages (see §14 for a more detailed discussion). In particular, versions of Theorem A (for polynomial functions $\mathbb{R} \to \mathbb{R}^2$) and of Theorem E below were first formulated by A. B. Kempe in 1875 [Ke1], however, his statement of the theorem was vague and as far as we can tell, his proof requires corrections (due to possible degenerate configurations).

Theorem E. (See §11.) *Let $f = f(z, \bar{z}), f : \mathbb{C} \to \mathbb{R}$ be a polynomial function of the variables $z, \bar{z}$ and $\Gamma := f^{-1}(0) \subset \mathbb{C}$ be a real-algebraic curve. Pick an open (in the classical topology) bounded subset $U \subset \Gamma$. Then there is a closed $\mathbb{C}$-functional linkage $\mathcal{L}_0$ so that the input map $p_0 : C(\mathcal{L}_0, Z) \to \mathbb{C}$ is an analytically trivial polynomial covering over U. Thus we can "draw" arbitrary algebraic curves in $\mathbb{R}^2$ using planar linkages.*

The main problem with Kempe's proof is that it works well only for a certain subset of the moduli space, however near certain "degenerate" configurations the moduli space splits into several components and the linkage fails to describe the desired polynomial function (see for instance §3.1). We use the "rigidified parallelograms" to resolve this problem and get rid of the undesirable components.

Kempe's methods were also insufficient to prove Theorem B and Corollary C even if the problem of "degenerate configurations" is somehow resolved. The second obstacle in proving

[3] Note that N. Mnev in [Mn] proves other interesting universality theorems for configuration spaces of points in $\mathbb{P}^n$ and convex polytopes in $\mathbb{R}^k$ of fixed combinatorial type. His results were generalized later by J. Richter-Gebert in [R], who proved a universality theorem for 4-dimensional convex polytopes.

[4] The Euler characteristic of the link of a boundary point is 1. But, by [Sul], if X is a real algebraic set then the Euler characteristic of the link of any $x \in X$ is even.

Theorem B is that the restriction p_0 of the regular ramified covering $p : \mathcal{M}(\mathcal{L}) \to Dom(\mathcal{L})$ to $\mathcal{M}(\mathcal{L}_0)$ a priori does not have to be an analytically trivial covering:

(a) It is possible that $\mathcal{M}(\mathcal{L}_0)$ intersects the ramification locus of p,

(b) if p_0 is a covering it might be nontrivial,

(c) even if p_0 is a *topologically* trivial covering it can fail to be *analytically trivial* covering because of "quasiwalls" (see §6 for various examples): think of the function $x^3 : \mathbb{R} \to \mathbb{R}$.

Both problems of degenerate configurations and reflection symmetries of linkages were neglected (or incorrectly resolved) in the previous work we have seen (e.g. [B], [HJW] and [JS]). The first precise formulation of a theorem of the above type was given by W. Thurston– who stated a version of Corollary C about 20 years ago and has given lectures on it since. He realized that such a theorem would follow by combining the 19-th century work on linkages (i.e. Kempe's theorem) with the work of Seifert, Nash, Palais and Tognoli. However, Thurston did not write up a proof so we have no way of knowing whether he overcame the problems discussed above in the 19-th century work on linkages. There is also ambiguity concerning which theorem Thurston formulated in his lectures, we heard three different versions from three sources. According to the most recent (April 1997) oral communication from Thurston, he can also prove Corollary C.

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Contents

2. Some real algebraic geometry

In this section we review those notions from the theory of real affine schemes and real analytic spaces that we will need in the rest of the paper. The discussion is complicated because (as noted in the introduction) the natural system of equations defining $\mathcal{M}(\mathcal{L})$ defines an ideal in $\mathbb{R}[X_1, Y_1, \ldots, X_r, Y_r]$ that is not reduced. Also it is remarkable that the abstract notion of fiber product of schemes corresponds (under the realization functor) to the operation of gluing of linkages, see Theorem 4.1. The reader who interested in proving only the *topological* version of Corollary C can skip this section.

We begin by defining the categories of affine schemes and algebraic sets and the forgetful functor Φ from affine schemes to affine algebraic sets.

An *affine subscheme* $\mathfrak{X}$ of $\mathbb{R}^n$ is a locally-ringed space (i.e. a topological space equipped with a sheaf of local rings) of the form $Spec\mathbb{R}[X_1, ..., X_n]/I$ where I is an ideal (see [H, page 72] for the definition of *Spec*). An *affine scheme* defined over $\mathbb{R}$ is a locally-ringed space of the above form for some n. If $\mathfrak{X}$ is as above we shall use the notation $\mathbb{R}[\mathfrak{X}]$ to denote the ring $\mathbb{R}[X_1, ..., X_n]/I$ which we will call the *coordinate ring* of $\mathfrak{X}$.

If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ we will abuse notation and use $\mathbf{k}^n$ to denote the corresponding scheme over $\mathbb{R}$. (In particular $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$.)

If $\mathfrak{X}$ is an affine scheme and $x \in \mathfrak{X}$ is a point then $T_x\mathfrak{X}$ will denote the Zariski tangent space of $\mathfrak{X}$ at x.

Let $\mathfrak{X} = Spec\mathbb{R}[X_1, ..., X_n]/I$ and $\mathfrak{Y} = Spec\mathbb{R}[Y_1, ..., Y_m]/J$. Then according to [H] a morphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ consists of a pair $(f, \tilde{f})$ where $f : \mathfrak{X} \to \mathfrak{Y}$ is a map of sets and $\tilde{f} : \mathcal{O}_{\mathfrak{Y}} \to f_*\mathcal{O}_{\mathfrak{X}}$ is a map of sheaves. However for the case in hand, $\tilde{f}$ is determined by the associated map of global sections

$$\tilde{f} : \mathbb{R}[Y_1, ..., Y_m]/J \to \mathbb{R}[X_1, ..., X_n]/I.$$

Thus we may identify $\tilde{f}$ with a morphism $\tilde{f} : \mathbb{R}[Y_1, ..., Y_m] \to \mathbb{R}[X_1, ..., X_n]$ with $\tilde{f}(J) \subset I$. If x is a point in $\mathfrak{X}$ then we will use the notation $D_x f$ to denote the map of Zariski tangent spaces $D_x f : T_x\mathfrak{X} \to T_{f(x)}\mathfrak{Y}$ induced by f.

An affine scheme as above is said to be *reduced* if I is equal to its radical $\sqrt{I}$. Recall that

$$\sqrt{I} = \{g \in \mathbb{R}[X_1, ..., X_n] : g^k \in I \ \text{ for some } \ k\}.$$

An affine scheme $\mathfrak{X}$ over $\mathbb{R}$ is said to be *real reduced* if it is reduced and moreover I as above has the property:

Suppose $g_1, ..., g_\ell \in \mathbb{R}[X_1, ..., X_n]$ satisfy $g_1^2 + ... + g_\ell^2 \in I$. Then $g_1, ..., g_\ell \in I$.

Let $I \subset \mathbb{R}[X_1, ..., X_n]$ be an ideal. We define the *real radical* $\sqrt[r]{I}$ by

$$g, g_1, ..., g_\ell \in \sqrt[r]{I} \iff g^{2k} + g_1^2 + ... + g_\ell^2 \in I \text{ for some } k.$$

Note that $I \subset \sqrt{I} \subset \sqrt[r]{I}$. Thus $\mathfrak{X} = Spec\mathbb{R}[X_1, ..., X_n]/I$ is real reduced if and only if $I = \sqrt[r]{I}$. An ideal I is *real radically closed* if $I = \sqrt[r]{I}$.

We now define the category of real algebraic sets. Let I be an ideal in $\mathbb{R}[X_1, ..., X_n]$. We define a subset $Z(I) \subset \mathbb{R}^n$ by

$$Z(I) = \{x \in \mathbb{R}^n : g(x) = 0, g \in I\}.$$

We note that $Z(I) = Z(\sqrt[r]{I})$. A subset $X \subset \mathbb{R}^n$ is said to be an *algebraic subset* if there exists $I \subset \mathbb{R}[X_1, ..., X_n]$ with $X = Z(I)$. A set X is said to be a real algebraic set if it is an algebraic subset of $\mathbb{R}^n$ for some n. Let X and Y be algebraic sets. Then a map $f : X \to Y$ is a *morphism* if there exist embeddings as above $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ such that f is the restriction of a polynomial mapping $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$.

Similarly we define *semi-algebraic subsets* of $\mathbb{R}^n$. The collection of semi-algebraic subsets in $\mathbb{R}^n$ is the boolean algebra containing all sets of the form $\{x \in \mathbb{R}^n | f(x) > 0\}$ for arbitrary polynomial functions f. For instance, any algebraic set $\{x | f(x) = 0\}$ (where f is a polynomial) is semi-algebraic since it is the complement of $\{x | f(x) > 0\} \cup \{x | f(x) < 0\}$. A morphism between the semi-algebraic subsets $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ is a map $f : X \to Y$ which is the restriction of a polynomial mapping $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$.

Let $X \subset \mathbb{R}^n$ be any subset. We define an ideal $\mathcal{I} = \mathcal{I}(X) \subset \mathbb{R}[X_1, ..., X_n]$ by

$$\mathcal{I}(X) := \{g \in \mathbb{R}[X_1, ..., X_n] : g(x) = 0 \text{ for all } x \in X\}.$$

If $X = Z(J)$ for some ideal $J \subset \mathbb{R}[X_1, ..., X_n]$, then $\mathcal{I} = \sqrt[r]{J}$. This is the real Nullstellensatz for polynomials [BE]. Equivalently

Theorem 2.1. $\mathcal{I}(Z(I)) = \sqrt[r]{I}$.

Corollary 2.2. *Z and $\mathcal{I}$ give an order reversing bijection between real radically closed ideals in $\mathbb{R}[X_1, ..., X_n]$ and algebraic subsets in $\mathbb{R}^n$.*

We define a functor Φ from the category of real affine subschemes of $\mathbb{R}^n$ to algebraic subsets of $\mathbb{R}^n$ by $\Phi(\mathfrak{X}) = X$ with $X = Z(I)$, where $\mathfrak{X} = Spec\mathbb{R}[X_1, ..., X_n]/I$. If $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a morphism then $f = \Phi(\mathfrak{f}) : X \to Y$ is the associated map of sets. We have a right inverse Ψ to Φ. Let X be an algebraic subset of $\mathbb{R}^n$. Define a real reduced scheme $\mathfrak{X}_{can} = \Psi(X)$ by

$$\mathfrak{X}_{can} = Spec\mathbb{R}[X_1, ..., X_n]/\mathcal{I}(X).$$

If $f : X \to Y$ is a morphism we define $\mathfrak{f} : \mathfrak{X}_{can} \to \mathfrak{Y}_{can}$ by $\mathfrak{f} = (f, f^*)$ where f^* is the pullback map on functions.

Remark 2.3. *As noted in the introduction if $\mathcal{L}$ is an abstract linkage then there is a canonical affine subscheme $\mathfrak{M}(\mathcal{L})$ of $\mathbb{R}^{2r-4}$ such that*

$$\Phi(\mathfrak{M}(\mathcal{L})) = \mathcal{M}(\mathcal{L}).$$

However $\mathfrak{M}(\mathcal{L})$ is not necessarily reduced or real reduced. So a general polynomial map $f : \mathcal{M}(\mathcal{L}_1) \to \mathcal{M}(\mathcal{L}_2)$ will not necessarily be of the form $\Phi(\mathfrak{f})$ for a morphism $\mathfrak{f} : \mathfrak{M}(\mathcal{L}_1) \to \mathfrak{M}(\mathcal{L}_2)$.

Suppose that X and Y are algebraic sets and that $f : X \to Y$ is a morphism. Suppose further that we have chosen affine schemes $\mathfrak{X}, \mathfrak{Y}$ with $\Phi(\mathfrak{X}) = X$ and $\Phi(\mathfrak{Y}) = Y$. We will say f is a *scheme-theoretic* morphism if there is a morphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ such that $\Phi(\mathfrak{f}) = f$.

The next definitions follow [AK]. Let $X \subset \mathbb{R}^n$ be an algebraic set. We define an *entire rational* function $f : X \to \mathbb{R}$ to be a function which is locally (in a Zariski open neighborhood of each point of X) the quotient of polynomials. Now let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be algebraic subsets. An *entire rational map* $X \to Y$ is a mapping of sets where components are entire rational functions. A *entire birational isomorphism* $f : X \to Y$ is an entire rational map which has entire rational inverse (in particular f is a homeomorphism). Note that the notion of *entire birational isomorphism* is more restrictive than birational isomorphism (a birational map does not have to be defined everywhere).

In what follows we will need the notion of the fiber product of affine schemes defined over $\mathbb{R}$. Suppose that we have a diagram of real affine schemes

$$\begin{array}{ccc} & & \mathfrak{Y} \\ & & \mathfrak{g} \downarrow \\ \mathfrak{X} & \xrightarrow{\mathfrak{f}} & \mathfrak{Z} \end{array}$$

The categorical fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is an affine scheme $\mathfrak{W}$ defined over $\mathbb{R}$ with morphisms $\pi_1 : \mathfrak{W} \to \mathfrak{X}$ and $\pi_2 : \mathfrak{W} \to \mathfrak{Y}$ such that we have a diagram (called a *fiber square*)

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{\pi_2} & \mathfrak{Y} \\ \pi_1 \downarrow & & \mathfrak{g} \downarrow \\ \mathfrak{X} & \xrightarrow{\mathfrak{f}} & \mathfrak{Z} \end{array}$$

satisfying the universal property that:

For any affine scheme $\mathfrak{V}$ defined over $\mathbb{R}$ the natural map of sets

$$Mor(\mathfrak{V}, \mathfrak{W}) \to Mor(\mathfrak{V}, \mathfrak{X}) \times Mor(\mathfrak{V}, \mathfrak{Y})$$

is an injection with image the subset of pairs (α, β) satisfying $\mathfrak{f} \circ \alpha = \mathfrak{g} \circ \beta$.

Remark 2.4. *We will also use the notation $\mathfrak{X} \times_{\mathfrak{f}=\mathfrak{g}} \mathfrak{Y}$ for the fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$.*

We now have

Lemma 2.5. *The categorical fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ exists (as an affine scheme defined over $\mathbb{R}$) and is unique up to canonical isomorphism.*

Proof: Choose representations

$$\mathfrak{X} = Spec\mathbb{R}[X_1, ..., X_n]/I, \mathfrak{Y} = Spec\mathbb{R}[Y_1, ..., Y_m]/J, \mathfrak{Z} = Spec\mathbb{R}[Z_1, ..., Z_\ell]/K.$$

Then define

$$\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} := Spec(\mathbb{R}[X_1, ..., X_n]/I \otimes_{\mathbb{R}[Z_1, ..., Z_\ell]/K} \mathbb{R}[Y_1, ..., Y_m]/J).$$

This proves existence. Uniqueness is obvious. □

Remark 2.6. *In down-to-earth terms $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is represented by the subscheme of $\mathbb{R}^{n+m}$ (with coordinates $(X_1, ..., X_n, Y_1, ..., Y_m)$) defined by the union of equations defining $\mathfrak{X}$ and $\mathfrak{Y}$ together with the equations*

$$f^* Z_i = g^* Z_i, i = 1, ..., \ell.$$

The fiber product operation is a functor on the category of "affine schemes over $\mathfrak{Z}$" in the following sense:

Given two commutative diagrams

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "\mathfrak{f}"] \arrow[dr] & & \mathfrak{X}' \arrow[dl] \\ & \mathfrak{Z} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{Y} \arrow[rr, "\mathfrak{g}"] \arrow[dr] & & \mathfrak{Y}' \arrow[dl] \\ & \mathfrak{Z} & \end{tikzcd}$$

we obtain a morphism

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \arrow[rr, "\mathfrak{f} \times_{\mathfrak{Z}} \mathfrak{g}"] \arrow[dr] & & \mathfrak{X}' \times_{\mathfrak{Z}} \mathfrak{Y}' \arrow[dl] \\ & \mathfrak{Z} & \end{tikzcd}$$

This is easily seen when reformulated in terms of tensor products of $\mathbb{R}[\mathfrak{Z}]$-algebras. Then the fiber product $\mathfrak{f} \times_{\mathfrak{Z}} \mathfrak{g}$ is just the tensor product of $\mathbb{R}[\mathfrak{Z}]$-algebra homomorphisms.

We next define a functor Φ from the category of affine schemes over $\mathbb{R}$ to real analytic spaces. The definition is complicated because we do not want to assume that our real analytic spaces are reduced- we do not want to loose track of nilpotents.

Let $\mathfrak{X} = Spec(\mathbb{R}[X_1, ..., X_n]/I)$ be a subscheme of $\mathbb{R}^n$. Our goal is to define the sheaf $\mathcal{O}^{an}$ (in the classical topology) of real analytic "functions" (they may be nilpotent) over the algebraic set X that underlies $\mathfrak{X}$. We first define $\mathcal{O}^{an}_{\mathbb{R}^n}$ for the affine space $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be an open subset. Then $\mathcal{O}^{an}_{\mathbb{R}^n}(U)$ is defined to be the ring of all analytic functions from U to $\mathbb{R}$. Let $\iota : X \to \mathbb{R}^n$ be the inclusion. We define a sheaf $\iota^{-1}\mathcal{O}^{an}_{\mathbb{R}^n}$ over X as follows. Let $U \subset X$ be open (in the classical topology). Then

$$\iota^{-1}\mathcal{O}^{an}_{\mathbb{R}^n}(U) = \varinjlim_{V \supset U} \mathcal{O}^{an}_{\mathbb{R}^n}(V)$$

where the direct limit is taken over all open subsets $V \subset \mathbb{R}^n$ that contain U with $V_1 < V_2$ if $V_1 \supset V_2$ (see [H, page 65]). Thus an element of $\mathcal{O}^{an}_{\mathbb{R}^n}(U)$ is a germ along U of a real-analytic function defined on a neighborhood of U in $\mathbb{R}^n$. Note that if $I = (f_1, ..., f_N)$ then the restrictions of $f_1, ..., f_N$ give rise to elements (again denoted $f_1, ..., f_N$) of $\iota^{-1}\mathcal{O}^{an}_{\mathbb{R}^n}(U)$. We define $\mathcal{J}^{an}(U) \subset \iota^{-1}\mathcal{O}^{an}_{\mathbb{R}^n}(U)$ to be the ideal generated by $f_1, ..., f_N$. We then define a presheaf $\tilde{\mathcal{O}}^{an}$ on X by

$$\tilde{\mathcal{O}}^{an}(U) = \iota^{-1}\mathcal{O}^{an}_{\mathbb{R}^n}(U)/\mathcal{J}^{an}(U).$$

We let $\mathcal{O}^{an}$ be the sheaf on X associated to the presheaf $\tilde{\mathcal{O}}^{an}$. The set X equipped with the sheaf of local rings $\mathcal{O}^{an}$ will be called the *analytic space associated to the affine scheme* $\mathfrak{X}$ and denoted $\mathfrak{X}^{an}$.

Remark 2.7. *This definition states that an analytic function $f \in \mathcal{O}^{an}_{\mathbb{R}^n}$ "vanishes" on $X \cap U$ if it may be written as $f = \sum_{i=1}^N g_i f_i$ where $I = (f_1, ..., f_N)$ and $g_1, ..., g_N$ are analytic functions on U. This is a stronger requirement than requiring that the induced* **function** *f on $X \cap U$ is identically zero. An element of $\mathcal{O}^{an}(U)$ corresponds to a cover $U = \cup_{i \in I} U_i$ and a collection of elements $g_i \in \iota^{-1}\mathcal{O}^{an}_{\mathbb{R}^n}(U_i)$, $i \in I$, such that the restrictions of g_i and g_j in $\iota^{-1}\mathcal{O}^{an}_{\mathbb{R}^n}(U_i \cap U_j)$ agree in the sense that their difference lies in $\mathcal{J}^{an}(U_i \cap U_j)$. The point is that an analytic function on U might not be the restriction of an analytic function defined in a neighborhood of U in $\mathbb{R}^n$ (see Lemma 2.8 below).*

We amplify the previous remark by a sobering example. Let X be the "Whitney's umbrella"

$$X = \{(x, y, z) \in \mathbb{R}^3 : x^2 - zy^2 = 0\}.$$

We give X the affine scheme structure $\mathfrak{X}_{can}$ and define I, $\mathcal{O}^{an}, \mathcal{J}^{an}$ as above. Consider a point $p = (0, 0, z) \in X$ with $z < 0$. If U is a sufficiently small neighborhood of this point then $X \cap U$ is contained in the z-axis and the coordinate functions x and y vanish on $U \cap X$. Nevertheless, *they are not in* $\mathcal{J}_p^{an}$ *as defined above.* Hence I is real-reduced but $\mathcal{J}^{an}$ is not real-reduced (i.e. $\mathcal{J}^{an} \neq \mathcal{J}_{can}$, see the next paragraph). However $x, y \in \sqrt[r]{\mathcal{J}_p^{an}}$.

If X is an algebraic set we define the sheaf of analytic functions $\mathcal{O}_{can}$ on X to be the sheaf associated to the presheaf $\tilde{\mathcal{O}}_{can}$ defined as follows. Let $U \subset X$ be open. Then $f \in \tilde{\mathcal{O}}_{can}(U)$ if there is a neighborhood V of U in $\mathbb{R}^n$ and an analytic function $\tilde{f} \in \mathcal{O}_{\mathbb{R}^n}^{an}(V)$ that restricts to f. We define $\mathcal{J}_{can} \subset \iota^{-1}(\mathcal{O}_{\mathbb{R}^n}^{an})$ to be the sheaf of analytic functions on $\mathbb{R}^n$ that vanish on X in the usual sense. Unless otherwise indicated we will assume that an algebraic set is given the above analytic structure.

We can also use Whitney's umbrella X to give an example of a real analytic function on an algebraic set $Y \subset \mathbb{R}^n$ that is not the restriction of an analytic function on a neighborhood U of Y in $\mathbb{R}^n$. Here Y is given the canonical analytic structure. The following example was provided by Henry King.

Let U_+ and U_- be the open subsets of $\mathbb{R}^3$ defined by

$$U_+ := \{(x, y, z) \in \mathbb{R}^3 : z > -3/4\}, \ U_- := \{(x, y, z) \in \mathbb{R}^3 : z < -1/2\}.$$

Define $f_\pm : U_\pm \to \mathbb{R}$ by $f_+ = (x, y, z) = y \log(1 + z)$ and $f_-(x, y, z) = 0$. Then the pairs $(f_+|X, U_+ \cap X)$ and $(f_-|X, U_- \cap X)$ define an analytic function f on X (with its canonical analytic structure). We have

Lemma 2.8. *(H. King) The function f does not extend to an analytic function in a neighborhood U of X in $\mathbb{R}^3$.*

Proof: Suppose an extension $\tilde{f}$ exists. We claim there is an analytic function g defined on a neighborhood V of $(0, 0, 1)$ in $\mathbb{R}^3$ such that

$$\tilde{f}(x, y, z) = y \log(1 + z) + (x^2 - zy^2)g(x, y, z) \tag{1}$$

in V. Indeed, near $(0, 0, 1)$, X is the union of two smooth analytic hypersurfaces

$$X_+ = \{(x, y, z) : x = \sqrt{z}y\} \quad \text{and} \quad X_- = \{(x, y, z) : x = -\sqrt{z}y\}.$$

Now by assumption $\tilde{f}(x, y, z) - y \log(1 + z)$ vanishes on X_+. Hence there exists $g_1(x, y, z)$ (defined near $(0, 0, 1)$) such that

$$\tilde{f}(x, y, z) - y \log(1 + z) = (x - \sqrt{z}y)g_1(x, y, z).$$

Then we restrict to X_- and find that $g_1(x, y, z)$ is divisible by $x + \sqrt{z}y$ near $(0, 0, 1)$ and the claim follows.

We now take the partial derivative of the both sides of the equation (1) with respect to y and restrict to $X \cap V$ to obtain

$$\frac{\partial f}{\partial y}\Big|_{X \cap V} = \log(1 + z) - 2yzg(x, y, z)\Big|_{X \cap V}.$$

Next restrict this identity to the intersection $X \cap \{y = 0\}$. This intersection near $(0, 0, 1)$ is an interval in the z-axis. After this restriction we get:

$$\frac{\partial f}{\partial y}\Big|_{X \cap V \cap \{y=0\}} = \log(1 + z).$$

By the unique analytic continuation, this identity must hold on $X \cap U \cap \{(x, y, z) : z > -1\} \cap \{y = 0\}$ and on $X \cap U \cap \{(x, y, z) : 0 > z > -1\}$. However $\frac{\partial f}{\partial y}$ extends to all of X and $\log(1 + z)$ blows up as $z \to -1$. Contradiction. $\square$

We obtain the corresponding notions of analytic morphism. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be algebraic subsets. Then an *analytic morphism* $f : X \to Y$ is a map such that for each $x \in X$ there is an open neighborhood (in the classical topology) U of x in $\mathbb{R}^n$ such that $f|U \cap X$ is the restriction of a real analytic map $\tilde{f} : U \to \mathbb{R}^m$. An *analytic isomorphism* is a morphism which has an analytic inverse.

More generally, assume we are given affine subschemes $\mathfrak{X} \subset \mathbb{R}^n$, $\mathfrak{Y} \subset \mathbb{R}^m$ with $\Phi(\mathfrak{X}) = X, \Phi(\mathfrak{Y}) = Y$. We define a morphism

$$\mathfrak{f} : \mathfrak{X}^{an} \to \mathfrak{Y}^{an}$$

to be a morphism of locally-ringed spaces [H, page 72]. In what follows it is important to note

Lemma 2.9. *A morphism*

$$\mathfrak{f} : \mathfrak{X}^{an} \to \mathfrak{Y}^{an}$$

induces an analytic morphism $f : X \to Y$.

Proof: Indeed, f induces a map of point sets $f : X \to Y$. Also, given an open subset $U \subset Y$, f comes with a local homomorphism

$$\tilde{f}_U : \Gamma(U, \mathcal{O}^{an}_{\mathbb{R}^m} \big| U) \to \Gamma(f^{-1}(U), \mathcal{O}^{an}_{\mathbb{R}^n} \big| f^{-1}(U))$$

which commutes with point evaluation since $\tilde{f}_U$ is local. Hence $\tilde{f}_U$ carries $\Gamma(U, \mathcal{J}_{can,Y})$ into $\Gamma(f^{-1}(U), \mathcal{J}_{can,X})$ and we obtain an induced map of quotients. $\square$

Remark 2.10. *The previous lemma also follows from the local Nullstellensatz [ABR, Proposition 2.8 (h), page 216] which implies*

$$\mathcal{J}_{can} = \sqrt[r]{\mathcal{J}}.$$

The above equation means that the maximal ideals $\mathcal{J}_{can,x}$ *and* $\mathcal{J}_x$ *in* $\mathcal{O}^{an}_{\mathbb{R}^n,x}$ *are related by*

$$\mathcal{J}_{can,x} = \sqrt[r]{\mathcal{J}_x}.$$

Suppose $\mathfrak{X}, \mathfrak{Y}, \mathfrak{X}^{an}, \mathfrak{Y}^{an}$ are as above and suppose that $f : X \to Y$ is an analytic morphism. We say that f is a *scheme-theoretic analytic morphism* if it is induced by a morphism $\mathfrak{f} : X^{an} \to \mathfrak{Y}^{an}$. We define *scheme-theoretic analytic isomorphism* to be a scheme-theoretic analytic morphism which has a scheme-theoretic analytic inverse.

We will need the fiber product of analytic germs. Suppose that we are given the following diagram of analytic germs:

$$\begin{array}{ccc} & & (\mathfrak{Y}, y) \\ & & \mathfrak{g} \downarrow \\ (\mathfrak{X}, x) & \xrightarrow{\mathfrak{f}} & (\mathfrak{Z}, z) \end{array}$$

Define the fiber product of germs $(\mathfrak{W}, w) = (\mathfrak{X}, x) \times_{(\mathfrak{Z}, z)} (\mathfrak{Y}, y)$ (where $w = (x, y)$) by the universal property (in the category of analytic germs) analogous to the previous one for affine schemes. The uniqueness of the fiber product is obvious. We prove existence by constructing a presentation of $O^{an}_{(\mathfrak{W},w)}$ from presentations of $\mathcal{O}^{an}_{(\mathfrak{X},x)}$, $\mathcal{O}^{an}_{(\mathfrak{Y},y)}$ and $\mathcal{O}^{an}_{(\mathfrak{Z},z)}$. Choose presentations

$$\mathcal{O}^{an}_{(\mathfrak{X},x)} = \frac{\mathbb{R}\langle X_1, ..., X_n \rangle}{(f_1, ..., f_r)}$$

$$\mathcal{O}^{an}_{(\mathfrak{Y},y)} = \frac{\mathbb{R}\langle Y_1, ..., Y_m \rangle}{(g_1, ..., g_s)}$$

$$\mathcal{O}^{an}_{(\mathfrak{Z},z)} = \frac{\mathbb{R}\langle Z_1, ..., Z_\ell\rangle}{(h_1, ..., h_t)}$$

Then define $\mathcal{O}^{an}_{(\mathfrak{W},w)}$ by

$$\mathcal{O}^{an}_{(\mathfrak{W},w)} = \frac{\mathbb{R}\langle X_1, ..., X_n, Y_1, ..., Y_m\rangle}{I}$$

where

$$I = (f_1, ..., f_r, g_1, ..., g_s, \mathfrak{f}^* Z_1 - \mathfrak{g}^* Z_1, ..., \mathfrak{f}^* Z_\ell - \mathfrak{g}^* Z_\ell).$$

In the above $\mathbb{R}\langle \ldots \rangle$ denotes the $\mathbb{R}$-algebra of convergent power series. The $\mathbb{R}$-algebra $\mathcal{O}^{an}_{(\mathfrak{W},w)}$ is called the *analytic tensor product* of $\mathcal{O}^{an}_{(\mathfrak{X},x)}$ and $\mathcal{O}^{an}_{(\mathfrak{Y},y)}$ over $\mathcal{O}^{an}_{(\mathfrak{Z},z)}$ and denoted

$$\mathcal{O}^{an}_{(\mathfrak{X},x)} \widehat{\otimes}_{\mathcal{O}^{an}_{(\mathfrak{Z},z)}} \mathcal{O}^{an}_{(\mathfrak{Y},y)}.$$

Our definition is an extension of [GR, page 181].

The fiber product operation is again a functor– we leave this to the reader.

We will later need to know that the operations of taking fiber products and passing to germs commute. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ and $\mathfrak{f}, \mathfrak{g}$ be as in the definition of fiber products of affine schemes and let $x \in \mathfrak{X}, y \in \mathfrak{Y}, z \in \mathfrak{Z}$ with $\mathfrak{f}(x) = z$ and $\mathfrak{g}(y) = z$. Let $p_1 : \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \to \mathfrak{X}$ and $p_2 : \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \to \mathfrak{Y}$ be the projections. We have

Lemma 2.11. *The induced map of analytic germs*

$$p_1 \times_{\mathfrak{Z}} p_2 : (\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}, (x, y)) \to (\mathfrak{X}, x) \times_{(\mathfrak{Z},z)} (\mathfrak{Y}, y)$$

is an isomorphism.

Proof: Choose as the local models for the germs $(\mathfrak{X}, x)$, $(\mathfrak{Y}, y)$ and $(\mathfrak{Z}, z)$ the ones determined by the coordinates $(X_1, ..., X_n), (Y_1, ..., Y_m), (Z_1, ..., Z_\ell)$ for the ambient affine spaces containing $\mathfrak{X}, \mathfrak{Y}$ and $\mathfrak{Z}$. The reader will then check, using the above definition of analytic tensor product, that the induced map $(p_1 \times_{\mathfrak{Z}} p_2)^*$ of analytic algebras is the identity map. □

Remark 2.12. *The strange feature that a map between two different objects is represented by the identity comes about because*

$$\mathbb{R}\langle X_1, ..., X_n\rangle \widehat{\otimes} \mathbb{R}\langle Y_1, ..., Y_m\rangle$$

is identified to $\mathbb{R}\langle X_1, ..., X_n, Y_1, ..., Y_m\rangle$ by a canonical isomorphism, see [GR, page 181].

Definition 2.13. *Let $f : X \to Y$ be a scheme-theoretic analytic morphism where $X = \Phi(\mathfrak{X})$, $Y = \Phi(\mathfrak{Y})$. An* **irregular point** *of f is a point where f is not a local analytic isomorphism (in the scheme-theoretic sense).*

Definition 2.14. *Suppose that X, Y are real algebraic sets. Then a finite* **analytically trivial** *covering $f : X \to Y$ is an analytic map such that the restriction of f to each connected component of X is an analytic isomorphism.*

We say that $f : X \to Y$ is an analytically trivial **polynomial** *covering if it is an polynomial morphism which is an analytically trivial regular covering whose group G of deck transformations consists of polynomial automorphisms. We retain the name* **analytically trivial polynomial covering** *for restriction of such f to a G-invariant open subset[5] of X.*

[5] With respect to the classical topology.

Note, that we do not claim here that X splits into disjoint union of Zariski components each of which is polynomially isomorphic to Y. It might happen that the real-algebraic set X is irreducible, but f is not 1-1.

We will now give a version of Definition 2.14 in terms of *Nash functions* (see Lemma 2.18). Let X be a real semi-algebraic set and $U \subset X$ be an open subset (in the classical topology).

Definition 2.15. *A function $f : U \to \mathbb{R}$ is* **Nash** *if it is real-analytic and there exist polynomial functions $p_0, p_1, ..., p_d$ not all equal to zero such that the equation*

$$p_0 + p_1 f + ... + p_d f^d = 0$$

holds identically on U.

We note that an entire rational function $f = p/q$ is Nash– it satisfies the equation $qf - p = 0$.

We define the sheaf $\mathcal{N}_X$ on X by defining $\mathcal{N}_X(U)$ to be the $\mathbb{R}$-algebra of Nash functions on U. Now let X and Y be real algebraic sets and $f : X \to Y$ be a continuous map. Define f to be a *Nash morphism* if $f^*\mathcal{N}_Y \subset f_*\mathcal{N}_X$. An equivalent definition is the following. Choose an embedding $Y \subset \mathbb{R}^n$. Then f is Nash if and only if the components of f are Nash functions on X. We have the following useful criterion for an analytic function to be Nash:

Lemma 2.16. *(See [BCR, Proposition 8.1.7].) Suppose X is a real semi-algebraic set and $f : X \to \mathbb{R}$ is an analytic function on X. Then f is Nash if and only if the graph $Gr(f)$ of f is a semi-algebraic subset of $X \times \mathbb{R}$.*

We will also need

Lemma 2.17. *(See [BCR, Theorem 2.4.5].) Suppose X is a real semi-algebraic set. Then the topological components of X are semi-algebraic sets.*

Now we can prove the result we need:

Lemma 2.18. *Suppose X and Y are real algebraic sets and $X = \cup_{i=1}^k X_i$ is the decomposition of X into connected components. Suppose $f : X \to Y$ is a polynomial map such that each $f_i := f\big|X_i$ is an analytic isomorphism. Then each f_i is a Nash isomorphism.*

Proof: Choose embeddings $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$. Then we have:

$$Gr(f) = \cup_{i=1}^k Gr(f_i)$$

where $Gr(f_i)$ is a semi-algebraic subset of $X \times Y$. Let $g_i := f_i^{-1}$. Then $Gr(g_i) \subset Y \times X$ is the image of $Gr(f_i)$ under the map which exchanges X and Y. Hence $Gr(g_i)$ is semi-algebraic. Let $\pi_j : \mathbb{R}^m \to \mathbb{R}$ be the j-th coordinate projection and $\Pi_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}$ be the projection defined by $\Pi_j(y, x) = (y, \pi_j(x))$. Then $Gr(\pi_j \circ g_i) = \Pi_j(Gr(g_i))$ is the image of a semi-algebraic set under a projection. Hence $Gr(\pi_j \circ g_i)$ is semi-algebraic by [BCR, Theorem 2.2.1]. Therefore $\pi_j \circ g_i$ is Nash by Lemma 2.16 which implies that g_i is a Nash morphism and hence isomorphism. $\square$

Thus, if $f : X \to Y$ is an analytically trivial polynomial covering then X is Nash isomorphic to a disjoint union of a finite number of copies of Y.

We will identify $\mathbb{R}^n$ with the affine part of $\mathbb{RP}^n$. Suppose that $X \subset \mathbb{R}^n$ is an affine real algebraic set. Then X is said to be *projectively closed* it its Zariski closure in $\mathbb{RP}^n$ equals X. Clearly each projectively closed subset must be compact (in the classical topology). It turns out that the converse is "almost true" as well:

Theorem 2.19. *(Corollary 2.5.14 of [AK]) Suppose that $X \subset \mathbb{R}^n$ is a compact affine algebraic set. Then X admits an entire birational isomorphism to a projectively closed affine algebraic subset X' of $\mathbb{R}^n$. Moreover, if X is defined over $\mathbb{Z}$ then X' is defined over $\mathbb{Z}$ as well.*

We will need the following theorem which is a modification of [AK, Corollary 2.8.6] or [T]:

Theorem 2.20. *(Seifert-Nash-Palais-Tognoli) Suppose that M is a smooth compact manifold* [6]*. Then M is diffeomorphic to a projectively closed real affine algebraic set S.*

Remark 2.21. *This theorem is a combination of [AK, Corollary 2.8.6] and Theorem 2.19 (for the assertion that S is projectively closed).*

Notation 2.22. *If $f : X \to X$ then $Fix(f)$ will denote the fixed point set of f.*

Throughout the paper $\mathbf{k}$ will denote either $\mathbb{R}$ or $\mathbb{C}$; $\mathbf{k}^n$ is given the Euclidean metric.

Notation 2.23. *Let $z \in \mathbf{k}^n$, $r > 0$. Then $B_r(z)$ will denote the disk in $\mathbf{k}^n$ of the radius r and center at z.*

A *domain* in $\mathbf{k}^N$ ($k = \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$) is a subset V with nonempty interior, where we use the classical topology.

3. Linkages

3.1. Abstract linkages and their realizations

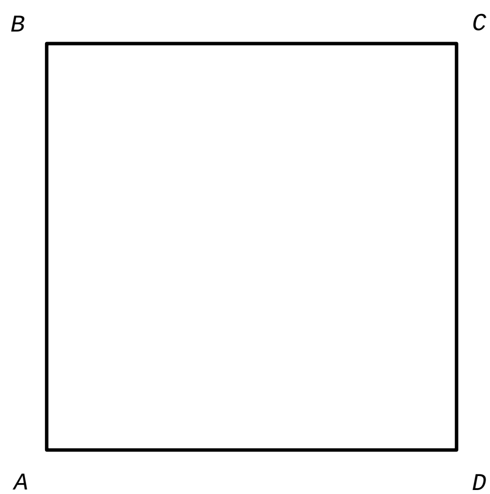

Figure 1: *The abstract square (à la Malevich).*

Throughout this paper we shall use the notation $\mathcal{V}(L)$ for the set of vertices of a graph L and $\mathcal{E}(L)$ for the set of edges of L.

Definition 3.1. *An* **abstract marked linkage** *$\mathcal{L}$ is a triple (L, ℓ, W) consisting of a graph L, an ordered subset $W \subset \mathcal{V}(L)$ and a positive function $\ell : \mathcal{E}(L) \to \mathbb{R}_+$ (a metric on L). The elements of W are called the* **fixed vertices** *of $\mathcal{L}$ and the choice of W is called* **marking**. *If W is empty then we call $\mathcal{L}$ an* **abstract linkage**. *A special case of abstract marked linkage is an abstract* **based** *linkage where W consists of two vertices v_1, v_2 connected by an edge e^*.*

[6] Not necessarily connected.

We do not require ℓ to define a *metric* on L: for instance the triangle with the edge-lengths $1, 1, 3$ satisfies our axioms. We shall also allow a vertex of L to be connected by an edge to itself or two distinct vertices to be connected by more than one edge. However in what follows we will refer to the pair (L, ℓ) as a *metric graph*. In the paper we will sometimes omit the words "abstract" and "marked" when it is clear what kind of linkage we consider.

Definition 3.2. *A morphism between two abstract marked linkages $\phi : \mathcal{L} \to \mathcal{N}$ is a map between corresponding metric graphs which maps vertices to vertices, edges to edges, fixed vertices to fixed vertices and $\ell(\phi(e)) = \ell(e)$ for each edge e of $\mathcal{L}$.*

Definition 3.3. *Let $\mathcal{L} = (L, \ell, W)$ be an abstract linkage. A* **planar realization** *of $\mathcal{L}$ is a map $\phi : \mathcal{V}(L) \to \mathbb{R}^2$ such that: if v and w are joined by an edge $[vw]$ then*

$$|\phi(v) - \phi(w)|^2 = (\ell[vw])^2.$$

We let $C(\mathcal{L}) = C(\mathcal{L}, \emptyset)$ be the set of all planar realizations of $\mathcal{L}$, it is called the **configuration space** *of $\mathcal{L}$.*

Note that if $\mathcal{L}$ contains an edge connecting a vertex to itself of two edges of different lengths connecting two vertices then $C(\mathcal{L}) = \emptyset$.

Definition 3.4. *Let $\mathcal{L} = (L, \ell, W)$ be an abstract linkage, $W = (v_1, ..., v_n)$ be the marking. Let $Z = (z_1, ..., z_n) \in \mathbb{C}^n$, called* **the image of marking**. *A* **relative planar realization** *of $\mathcal{L}$ is a realization $\phi \in C(\mathcal{L})$ such that $\phi(v_j) = z_j$ for all j. We let $C(\mathcal{L}, Z)$ be the set of all relative planar realizations of $\mathcal{L}$, it is called the* **relative configuration space** *of $\mathcal{L}$.*

We note that $C(\mathcal{L})$ and $C(\mathcal{L}, Z)$ are algebraic sets associated to real affine schemes $\mathfrak{C}(\mathcal{L}), \mathfrak{C}(\mathcal{L}, Z)$ defined by the above quadratic equations. The assignments $\mathcal{L} \mapsto C(\mathcal{L}), \mathcal{L} \mapsto \mathfrak{C}(\mathcal{L})$ are functors to be called the *realization functors*, the assignments $(\mathcal{L}, Z) \mapsto C(\mathcal{L}, Z), \mathcal{L} \mapsto \mathfrak{C}(\mathcal{L}, Z)$ are also functors to be called *relative realization functors.*

A special case of the relative configuration space is when $\mathcal{L}$ is a based linkage:

Definition 3.5. *Let $\mathcal{L} = (L, \ell, e)$ be a based linkage. We define the* **moduli space** $\mathcal{M}(\mathcal{L})$ *by*

$$\mathcal{M}(\mathcal{L}) = \{\phi \in C(\mathcal{L}) : \phi(v_1) = (0, 0), \ \phi(v_2) = (\ell(e), 0)\}.$$

Then $\mathcal{M}(\mathcal{L})$ is the algebraic set underlying $\mathfrak{M}(\mathcal{L})$, the real affine scheme defined by the above equations.

If L is connected then $\mathcal{M}(\mathcal{L})$ is a compact real-algebraic subset of $\mathbb{R}^{2r-4}$, where r is the number of vertices in L. The algebraic set $C(\mathcal{L})$ canonically splits as the product $\mathcal{M}(\mathcal{L}) \times E(2)$ (the group $E(2)$ of orientation-preserving isometries of $\mathbb{R}^2$ has obvious real-algebraic structure), thus we shall identify the quotient $C(\mathcal{L})/E(2)$ and $\mathcal{M}(\mathcal{L})$. Note that $\mathcal{M}(\mathcal{L})$ admits an algebraic automorphism induced by the complex conjugation in $\mathbb{C} = \mathbb{R}^2$.

As we shall see in Section 5 for each $(\mathcal{L}, Z)$ there is a based abstract linkage $\mathcal{A}$ so that $\mathcal{M}(\mathcal{A})$ is a 2-fold or 1-fold analytically trivial covering of $C(\mathcal{L}, Z)$.

Suppose that $\mathcal{L}' \subset \mathcal{L}$ is a sublinkage, i.e. $(L', \ell') \subset (L, \ell)$ is a subgraph such that ℓ' is the restriction of ℓ and the marking W' of $\mathcal{L}'$ is the intersection $W \cap L'$. If Z is the image of W, then we have naturally defined Z', the image of W'. Thus we have natural scheme-theoretic *restriction* morphism

$$Res : C(\mathcal{L}, Z) \to C(\mathcal{L}', Z'), \quad \phi \mapsto \phi\big|L'.$$

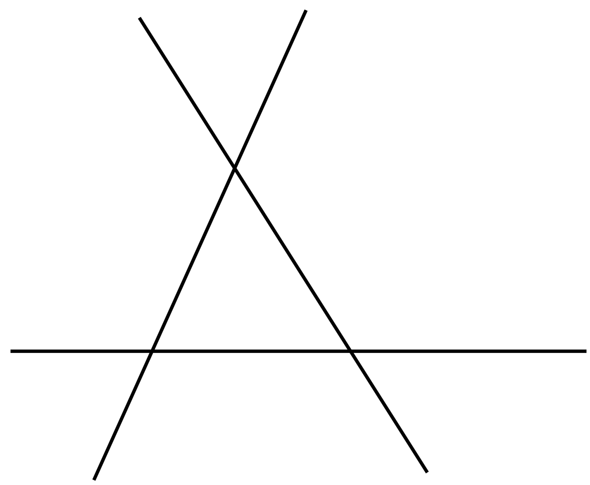

Figure 2: *The moduli space of the square.*

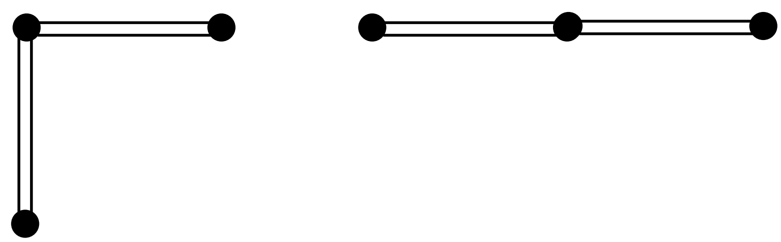

Figure 3: *Degenerate realizations of the abstract square. Black disks denote images of the vertices.*

In particular, if $\mathcal{L}'$ consists of a single vertex $v \in \mathcal{L}$ then we let $eval_v : C(\mathcal{L}, Z) \to \mathbb{R}^2$ be the evaluation map $\phi \mapsto \phi(v)$. We will identify $eval_v$ with the restriction map

$$Res : \phi \mapsto \phi\Big|\{v\} \in C(\{v\}).$$

Many of the problems with the 19-th century work on linkages can be traced to neglecting degenerate realizations of a square.

Definition 3.6. *A* **square** *is the abstract polygonal linkage with four edges all of which have equal length (see Figure 1). We let $e^* := [AB]$. More generally, an* **abstract parallelogram** *S is the polygonal linkage with four edges where the alternating edges have equal lengths.*

We have

Lemma 3.7. *The moduli space of the square is isomorphic to a union of* **three** *smooth curves of degree 2 (each one necessarily rational) in $\mathbb{RP}^6$ such that each pair intersects in a point and at each point of intersection the tangent spaces have 2-dimensional span (see Figure 2).*

Proof: See [GN, Case III, page 120].

Two of the components of the moduli space of the square consist of "degenerate" realizations of the square (Figure 3). We can eliminate the components consisting of degenerate squares by "rigidifying" the square as on Figure 4, see Lemma 3.9 for the proof. We rigidify parallelogram linkages in an analogous way. Henceforth all parallelogram sublinkages that

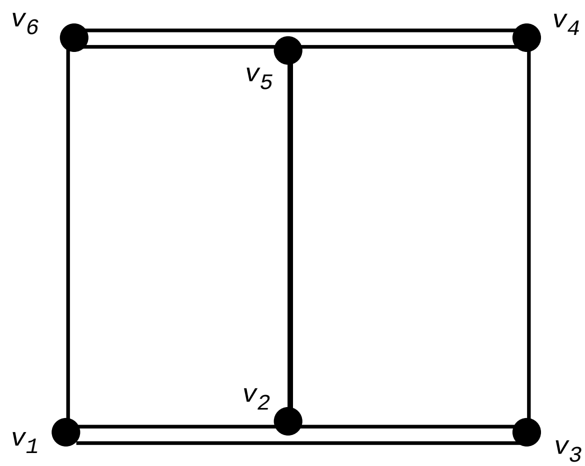

Figure 4: *The rigidified parallelogram* Σ. *We choose:* $\ell[v_1v_2] = \ell[v_2v_3] = \ell[v_1v_3]/2 = \ell[v_6v_5] = \ell[v_5v_4] = \ell[v_6v_4]/2$.

appear in *elementary linkages* (Section 6) will be rigidified– but we will not draw the extra edges. Notice that each rigidified parallelogram $\mathcal{L}$ contains two "degenerate" triangular linkages: one with the vertices v_1, v_2, v_3 and second with the vertices v_4, v_5, v_6. Thus the ring of the configuration scheme of the rigidified parallelogram has nilpotent elements. Since we are interested mostly in the reduced schemes we make the following

Convention 3.8. *Suppose that* $\mathcal{L}$ *is one of the* **elementary linkages** *(translator, pantograph and inversor) in* §*6 and* $\Delta \subset \mathcal{L}$ *is a* **degenerate triangle**, *i.e. a triangle with the vertices* A, B, C *so that*

$$\ell[AC] = \ell[AB] + \ell[BC].$$

Let $r := \ell[AB]/\ell[AC], s := \ell[BC]/\ell[AC]$. *We will add to the defining equations for* $\mathfrak{C}(\mathcal{L}, Z)$ *the linear equation*

$$\phi(B) = r\phi(A) + s\phi(C)$$

for each degenerate triangle. This choice of the scheme is determined by the "mechanical" reasons: to make an actual mechanical model of the abstract linkage $\mathcal{L}$ *drill the hole* B *in the bar* $[AC]$ *within the distances* $\ell[AB]$ *and* $\ell[BC]$ *from the holes* A *and* C *respectively. See Figure 5 for mechanical model of the rigidified parallelogram. The linkages* $\mathcal{L}$ *which will be used in this paper will be built by gluing together the* **elementary** *linkages of* §*6.*

The corresponding realization schemes $\mathfrak{C}(\mathcal{L}, Z)$ will be fiber products of the realization schemes of the elementary linkages (see Theorem 4.1). We will extend the above convention as follows. Add one (vector valued) linear equation as above for each degenerate triangle contained in one of the elementary linkages from which $\mathcal{L}$ is built. Thus the scheme structure of $\mathfrak{C}(\mathcal{L}, Z)$ will depend on a choice $\{\mathcal{L}_i\}_{i\in I}$ of distinguished elementary sublinbkages of $\mathcal{L}$. However the underlying algebraic set $C(\mathcal{L}, Z)$ does not depend on this choice.

We retain the names **realization functor** and **relative realization functor** for the functors $\mathcal{L} \mapsto \mathfrak{C}(\mathcal{L}), (\mathcal{L}, Z) \mapsto \mathfrak{C}(\mathcal{L}, Z)$.

We will use the notation $\mathfrak{M}'(\Sigma)$ for the moduli scheme of the rigidified parallelogram Σ with the partially reduced structure as above.

Recall that S is the (unrigidified) parallelogram linkage with $e^* = [AB]$. Then we have an embedding of affine *schemes* $i : \mathfrak{M}'(\Sigma) \to \mathfrak{M}(S)$.

We let ϕ_1 and ϕ_2 be the degenerate realizations of the rigidified parallelogram Σ (which could be the rigidified square). The following lemma will be very important in what follows.

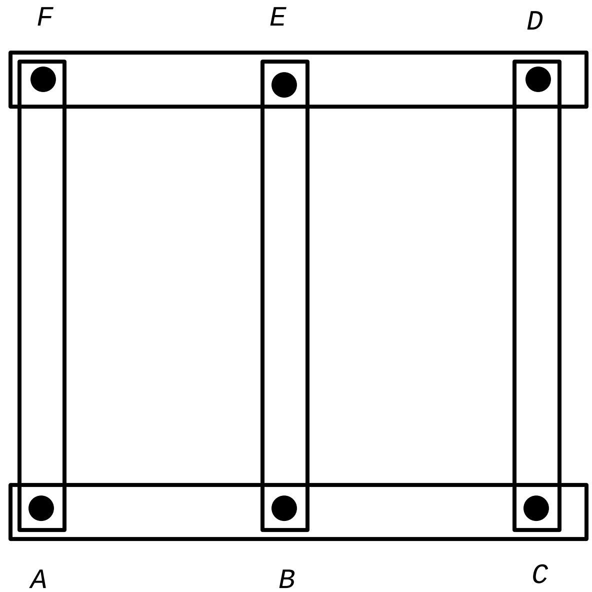

Figure 5: *A mechanical model for the rigidified parallelogram* Σ.

Lemma 3.9. *The real reduced structure on* $\mathfrak{M}'(\Sigma)$ *is a projective line where real points correspond to (convex) parallelograms.* $\mathfrak{M}'(\Sigma)$ *has exactly two singular real points, the degenerate parallelograms* ϕ_1, ϕ_2.

Proof: Recall that the distance function between two oriented straight lines in the Euclidean plane is convex and it is strictly convex unless these lines are parallel. Thus, if $A \neq B$ and $C \neq D$ are points in $\mathbb{C}$ such that

$$\|A - D\| = \|B - C\| = \|(A + B)/2 - (D + C)/2\|$$

then the lines through A, B and D, C are parallel and

$$A - B = D - C.$$

Therefore real points ϕ of $\mathfrak{M}'(\Sigma)$ correspond to parallelograms:

$$\phi(v_1) - \phi(v_3) = \phi(v_6) - \phi(v_4).$$

We will prove the lemma by first determining the algebraic set $\mathcal{M}'(\Sigma)$ underlying $\mathfrak{M}'(\Sigma)$. Then we will prove that all points of $\mathfrak{M}'(\Sigma)$ with the exception of the degenerate parallelograms ϕ_1 and ϕ_2 are smooth points. Lastly we will prove that ϕ_1 and ϕ_2 are singular points.

We first consider the case of a parallelogram S which is not a square. The moduli space $\mathfrak{M}(S)$ for such S is described in [GN, Case II, page 120]. The authors of [GN] describe the (projectivized) moduli space as a real projective subvariety of $\mathbb{P}^6$. They find that the moduli space is the union of a smooth curve of degree two (necessarily isomorphic to $\mathbb{P}^1$) and a smooth curve of degree four which is also isomorphic to $\mathbb{P}^1$ (see [GN, page 119] where it is proved that if the degree is four then the genus is zero). Moreover they prove that the real points of the quartic correspond to the set of "antiparallelogram" (see Figure 6) realizations of the linkage. The authors also prove that the components of $\mathcal{M}(S)$ intersect in two points, the two degenerate parallelograms. It now follows from the paragraph above that $i(\mathcal{M}'(S))$ is the quadratic curve C. We note that the linear equations added by Convention 3.8 merely express the coordinates of E as linear functions of those of D and F and the coordinates of B as linear functions of those of A and C, thereby eliminating the new variables provided by the contribution of the new vertices B and E (see Figure 5). Now let ϕ be a point of $\mathcal{M}'(\Sigma)$ which is different from ϕ_1 and ϕ_2. Then $\mathcal{M}(\Sigma)$ is smooth at ϕ so the ideal $\mathcal{I}_\phi^{an}$ in the analytic local ring $\mathcal{O}_\phi^{an}$ is real reduced. But the functions we have added to pass from $\mathfrak{M}(\Sigma)$ to $\mathfrak{M}'(\Sigma)$ all vanish identically on $\mathcal{M}'(S)$. Hence they already belong to $\mathcal{I}_\phi^{an}$ and the map of analytic germs $(\mathfrak{M}'(\Sigma), \phi) \to (\mathfrak{M}(S), \phi)$ is an isomorphism.

It remains to prove that ϕ_1 and ϕ_2 are in fact singular points of $\mathfrak{M}'(\Sigma)$. We will assume that the images of Σ under ϕ_1 and ϕ_2 are horizontal segments. It suffices to prove that

$$dim T_{\phi_1}(\mathfrak{M}'(\Sigma)) = dim T_{\phi_2}(\mathfrak{M}'(\Sigma)) = 2.$$

We claim that $\mathrm{d}i_{\phi_1}$ and $\mathrm{d}i_{\phi_2}$ are isomorphisms of Zariski tangent spaces. Then in an infinitesimal deformation of either ϕ_1 or ϕ_2 in $\mathfrak{M}(S)$ the vertices A and C stay fixed and F and D move vertically (if F and D move in the same direction we are tangent to the space of parallelograms and if they move in opposite directions we are tangent to the space of antiparallelograms). The equations of Convention 3.8 then tell us how to move the new vertices B and E. But in any case they will move vertically so the distance between B and D is preserved to first order.

The lemma follows in the parallelogram case.

We leave the proof of the lemma in the case of a square to the reader, it is a consequence of [GN, case III, page 120]. In this case $\mathcal{M}(S)$ is the union of three quadratic curves, two of the components correspond to degenerate realizations, see Figures 2, 3. The image of i is the component consisting of rhombi. $\square$

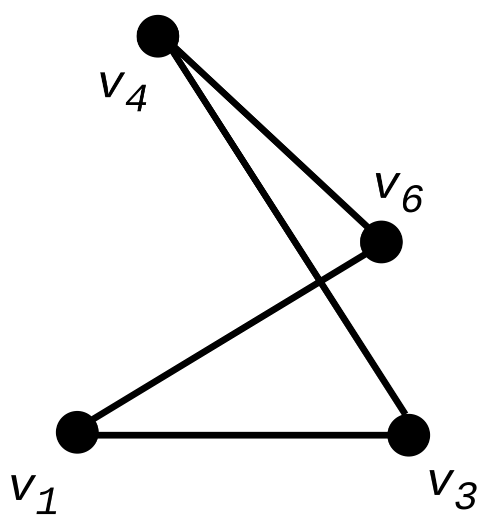

Figure 6: *An antiparallelogram.*

We will not need the next result in what follows but we add it for completeness.

Remark 3.10. *It can be shown that the analytic local rings $\mathcal{O}_{\phi_1}^{an}$ and $\mathcal{O}_{\phi_2}^{an}$ are isomorphic to*

$$\frac{\mathbb{R}\langle x, y\rangle}{(xy, x^2)} \cong \frac{\mathbb{R}\langle x, y\rangle}{(x) \cap (x^2, y)}.$$

Thus the germs $(\mathcal{O}_{\phi_i}^{an}, \phi_i)$ are analytically isomorphic to the union of the y-axis and the embedded nonreduced point given by the equations $x^2 = 0$, $y = 0$. In particular the scheme $\mathfrak{M}'(\Sigma)$ is **nonreduced**.

Let $\mathcal{L}$ be a based triangular linkage with the vertices v_1, v_2, v_3, $e^* := [v_1 v_2]$ so that $\ell[v_1v_2] < \ell[v_2v_3] + \ell[v_1v_3]$, $\ell[v_1v_3] < \ell[v_2v_3] + \ell[v_1v_2]$ and $\ell[v_2v_3] < \ell[v_1v_2] + \ell[v_1v_3]$ (i.e $\mathcal{L}$ is **nondegenerate** and the function ℓ defines a metric on L). Let $\mathcal{A}$ be the (not based) linkage obtained from $\mathcal{L}$ by removing the edge $[v_1v_3]$ and $\mathcal{B}$ be the linkage obtained from $\mathcal{A}$ by removing the vertex v_2 and the edges $[v_1v_2], [v_2v_3]$. A realization $\phi \in C(\mathcal{A})$ is called **nondegenerate** if its image is not contained in a single geodesic in $\mathbb{R}^2$.

Lemma 3.11. *Under the above conditions the moduli space $\mathcal{M}(\mathcal{L})$ consists of two points. If $\phi \in C(\mathcal{A})$ is a* **nondegenerate realization** *then the restriction map of germs*

$$Res : (\mathfrak{C}(\mathcal{A}), \phi) \to (\mathfrak{C}(\mathcal{A}), \phi\big|\mathcal{B})$$

(given by restriction of realizations from $\mathcal{A}$ to $\mathcal{B}$) is an analytic isomorphism.

Proof: The first assertion is obvious. We will only sketch the proof of the second assertion, the details are left to the reader. Both spaces $C(\mathcal{A})$, $C(\mathcal{B})$ are smooth 4-dimensional manifolds. Thus it is enough to verify that the derivative

$$D_\phi Res : T_\phi \mathfrak{C}(\mathcal{A}) \to T_{res(\phi)} \mathfrak{C}(\mathcal{B})$$

is injective, which follows from the fact that the circles centered at $\phi(v_1), \phi(v_2)$ with radii $\ell[v_1v_2], \ell[v_2v_3]$ intersect transversally. $\square$

Notation 3.12. *Throughout the paper we shall use the following notations:*

- *If A, B are distinct points in the plane then (AB) will denote the straight line through A, B.*
- *If A, B, C are points on the plane then $\Delta(A, B, C)$ will denote the triangle with the vertices A, B, C.*
- *If (A, B, C, D) is a quadruple of points on the plane then $\square(ABCD)$ (sometimes we will also put commas between the letters) will denote the quadrilateral with the given vertices (they are connected by edges according to the cyclic order).*

3.2. Fiber sums of linkages

The operation of fiber sum of linkages is analogous to generalized free products of groups (i.e. amalgamated free product or HNN-extension). Let $\mathcal{L}' = (L', \ell', W')$, $\mathcal{L}'' = (L'', \ell'', W'')$ be abstract marked linkages. Suppose that we have a map β between (nonempty) subsets of vertices

$$\beta : S' \subset \mathcal{V}(L') \to S'' \subset \mathcal{V}(L'').$$

If the images Z', Z'' of W', W'' are given we require

$$\phi'(w_j) = \phi''(\beta(w_j))$$

for each $w_j \in W'$ and $\phi' \in C(\mathcal{L}', Z'), \phi'' \in C(\mathcal{L}'', Z'')$.

Then the *fiber sum* $\mathcal{L}$ of linkages $\mathcal{L}', \mathcal{L}''$ associated with β (the fiber sum is denoted $\mathcal{L}' *_\beta \mathcal{L}''$) is constructed as follows:

Step 1. Take the disjoint union of metric graphs $(L', \ell') \sqcup (L'', \ell'')$ and identify v and $\beta(v)$ for all $v \in S'$. The result is the metric graph (L, ℓ).

Step 2. Let W be the image in L of $W' \sqcup W''$, we let W be the marking of the resulting fiber sum $\mathcal{L} := (L, \ell, W)$. If the images Z', Z'' of W', W'' are given, we define the vector

Z (the image of W) as the vector with the coordinates $\phi(w_j)$, where $w_j \in W$ and ϕ is in $C(\mathcal{L}', Z')$ or in $C(\mathcal{L}'', Z'')$.

In the case $\mathcal{L}' = \mathcal{L}''$ the above construction has an analogue which we call the *self-fiber sum*. The only difference is that on the first step instead of $L' \sqcup L''$ we take the same graph L' as before. The self-fiber sum will be denoted $\mathcal{L}' *_\beta$. Note that the operations of fiber sum and self-fiber sum can create edges which are loops.

Remark 3.13. *Notice that if $\mathcal{L}' \cong \mathcal{L}''$ then $\mathcal{L}' *_\beta$ is different from $\mathcal{L}' *_\beta \mathcal{L}''$.*

If S', S'' are singletons $\{u\}, \{v\}$ then we will denote $\mathcal{L}' *_\beta$ by $\mathcal{L}' *_{u=v}$. In what follows we will consider $\mathcal{L}', \mathcal{L}''$ being canonically mapped to $\mathcal{L}$.

3.3. Functional linkages

Definition 3.14. *Let X, Y be topological spaces, $f : X \to Y$ be a continuous map and G a group of homeomorphisms acting properly discontinuously and effectively on X. We say that f is a* **regular topological branched covering** *with the* **group of deck-transformations** *G, if f is surjective and*

$$f(x) = f(x') \iff \text{there exists } g \in G \text{ such that } g(x) = x'.$$

We define the **singular set** *$\Sigma(f)$ of f as the collection of points $x \in X$ fixed by a nontrivial element of G.*

Let $\mathbf{k}$ denote either $\mathbb{C}$ or $\mathbb{R}$. We will identify $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{R}$ with the real axis in $\mathbb{C}$. Recall that $C(\mathcal{L}, Z)$ is the real algebraic set underlying the real affine scheme $\mathfrak{C}(\mathcal{L}, Z)$. We now give the main definition.

Definition 3.15. *Let $\mathcal{O} \in \mathbf{k}^m$ and $F : \mathbf{k}^m \to \mathbf{k}^n$ be a map. We define a $\mathbf{k}$-functional linkage $\mathcal{L}$ for the germ $(F, \mathcal{O})$ as follows:*

It is an abstract marked linkage $\mathcal{L} = (L, \ell, W)$ with m distinguished vertices $P_1, .., P_m$ (called the **input** *vertices) and n additional distinguished vertices $Q_1, ..., Q_n$ (called the* **output** *vertices) and a particular choice of a vector $Z \in \mathbb{C}^s$, the image of marking. We require this data to satisfy the axioms:*

(1) The forgetful map $p : C(\mathcal{L}, Z) \to (\mathbb{R}^2)^m$ given by

$$p(\phi) = (\phi(P_1), ..., \phi(P_m)), \quad \phi \in C(\mathcal{L}, Z)$$

is a regular topological branched covering of a domain $Dom(\mathcal{L}, Z)$ in $\mathbf{k}^m$, so that the group $Sym(\mathcal{L}, Z)$ of deck-transformations of p consists of scheme-theoretic automorphisms. We let $Crit(\mathcal{L}, Z)$ denote the union of the set of irregular points of the scheme-theoretic morphism $\mathfrak{p} : \mathfrak{C}(\mathcal{L}, Z) \to \mathbf{k}^m$ with $\Sigma(p)$ (the set of singular points of the topological branched covering p). Let $C^*(\mathcal{L}, Z) := C(\mathcal{L}, Z) - Crit(\mathcal{L}, Z)$. It is clear that $C^*(\mathcal{L}, Z)$ is invariant under $Sym(\mathcal{L}, Z)$. We let $Dom^*(\mathcal{L}, Z) := p(C^*(\mathcal{L}, Z))$. Thus

$$p : C^*(\mathcal{L}, Z) \to Dom^*(\mathcal{L}, Z)$$

is a locally analytically trivial[7] covering. We require $\mathcal{O} \in Dom^*(\mathcal{L})$.

(2) The forgetful map $q : C(\mathcal{L}, Z) \to \mathbb{R}^{2n}$ given by

$$q(\phi) = (\phi(Q_1), ..., \phi(Q_n)), \quad \phi \in C(\mathcal{L}, Z)$$

factors through p and induces the map $F\big|Dom(\mathcal{L}, Z) : Dom(\mathcal{L}, Z) \to \mathbf{k}^n$. We will say that the germ $(F, \mathcal{O})$ is **defined** *by the linkage $\mathcal{L}$ and the vector Z.*

---

[7]It $\phi \in C^*(\mathcal{L}, Z)$ then ϕ is a smooth point of $\mathfrak{C}(\mathcal{L}, Z)$ (since $Dom^*(\mathcal{L}, Z)$ is smooth.) Hence $C^*(\mathcal{L}, Z)$ is real-reduced and p is a regular analytic covering in the scheme-theoretic sense (i.e. $\mathcal{O}^{an}_{Dom^*(L,Z)}$ is the subsheaf of $Sym(\mathcal{L}, Z)$-invariants in $\mathcal{O}^{an}_{C^*(L,Z)}$.

Notice that in the definition of functional linkage for a germ $(F, \mathcal{O})$ the metric ball around $\mathcal{O}$ which is contained in $Dom^*(\mathcal{L}, Z)$ is not specified. We will also need the following modification of the above definition:

Definition 3.16. *Suppose that the pair $(\mathcal{L}, Z)$ as above defines the germ $(F, \mathcal{O})$ and, moreover, U is a neighborhood of $\mathcal{O}$ such that $U \subset Dom^*(\mathcal{L}, Z)$. Then we say that the pair $(\mathcal{L}, Z)$* **defines** *(F, U).*

The group $Sym(\mathcal{L}, Z)$ will be called the *symmetry group* of $\mathcal{L}$ and $Dom(\mathcal{L}, Z)$ will be called the *domain* of $\mathcal{L}$ (of course they both depend on Z). The set of *input vertices* is denoted by $In(\mathcal{L})$ and the set of output vertices by $Out(\mathcal{L})$. We will refer to $\mathbb{R}$-functional linkages as *real functional linkages* and $\mathbb{C}$-functional linkages as *complex functional linkages.* If the choice of $\mathbf{k}, Z, In(\mathcal{L}), Out(\mathcal{L}), \mathcal{O}, U$ and/or F is suppressed then $\mathcal{L}$ is also referred as a functional linkage.

Lemma 3.17. *Suppose that $\mathcal{L}$ is a functional linkage. A point $\phi \in C(\mathcal{L}, Z)$ belongs to $Crit(\mathcal{L}, Z)$ iff either $Dp_\phi : T_\phi\mathfrak{C}(\mathcal{L}) \to T_{p(\phi)}Dom(\mathcal{L}, Z)$ has nonzero kernel or $\phi \in \Sigma(p)$ or ϕ is not a smooth point of $\mathfrak{C}(\mathcal{L}, Z)$.*

Proof: If Dp_ϕ has nonzero kernel then clearly $\phi \in Crit(\mathcal{L}, Z)$. If $\phi \notin Crit(\mathcal{L}, Z)$ and $\phi \notin \Sigma(p)$ then $p(\phi)$ is in the interior of $Dom(\mathcal{L}, Z)$. If $p : (\mathfrak{C}(\mathcal{L}, Z), \phi) \to (Dom(\mathcal{L}, Z), p(\phi))$ is an isomorphism of analytical germs then $(\mathfrak{C}(\mathcal{L}, Z), \phi)$ is necessarily smooth.

Now suppose that $\phi \in C(\mathcal{L}, Z)$, $\phi \notin \Sigma(p)$ and Dp_ϕ has zero kernel. If ϕ is a smooth point then the dimension of $C(\mathcal{L}, Z)$ near ϕ is the same as the dimension of the interior of $Dom(\mathcal{L}, Z)$, thus we can apply the inverse function theorem to conclude that $\phi \in C^*(\mathcal{L}, Z)$. $\square$

Corollary 3.18. *$Dom^*(\mathcal{L}, Z)$ and $C^*(\mathcal{L}, Z)$ are open subsets in $\mathbf{k}^m$ and $C(\mathcal{L}, Z)$ respectively.*

Proof: The set of singular points of $\mathfrak{C}(\mathcal{L}, Z)$ is closed as well as $\Sigma(p)$ and the set of points where Dp_ϕ has nonzero kernel. The restriction of p to $C^*(\mathcal{L}, Z)$ is open. $\square$

Besides functional linkages we will also need *closed functional linkages* defined as follows. Let $\mathcal{L}'$ be a functional linkage and β be a map from a subset of $Out(\mathcal{L}')$ to W (the marking of $\mathcal{L}'$). Then the linkage $\mathcal{L} := \mathcal{L}' *_\beta$ is called a *closed functional linkage.* Such linkage still has the *input map* p (the restriction of the input map p' of $\mathcal{L}'$) and the *domain* $Dom(\mathcal{L}, Z)$ which is the image of p. The group of symmetries $Sym(\mathcal{L}', Z)$ acts naturally on $C(\mathcal{L}, Z)$ and we let $Sym(\mathcal{L}, Z)$ denote the image of $Sym(\mathcal{L}', Z)$ in the group of automorphisms of $C(\mathcal{L}, Z)$.

Let $\mathcal{L}$ be a (possibly closed) $\mathbf{k}$-functional linkage. We have the scheme-theoretic input morphism p. As above we let $Crit(\mathcal{L}, Z) \subset C(\mathcal{L}, Z)$ denote the union of the set of irregular points for p with $\Sigma(p)$, it is invariant under the action of the symmetry group and we let $C^*(\mathcal{L}, Z) := C(\mathcal{L}, Z) - Crit(\mathcal{L}, Z)$, $Dom^*(\mathcal{L}, Z) := p(C^*(\mathcal{L}, Z))$.

If $\mathcal{L}$ is a (possibly closed) functional linkage we let $Wall(\mathcal{L}, Z) \subset C(\mathcal{L}, Z)$ denote the collection of fixed points of nontrivial elements of $Sym(\mathcal{L}, Z)$, i.e. $Wall(\mathcal{L}, Z) = \Sigma(p)$. Components of $Wall(\mathcal{L}, Z)$ will be called *walls* and components of $Crit(\mathcal{L}, Z) - Wall(\mathcal{L}, Z)$ will be called *quasiwalls.* We retain the names *walls* and *quasiwalls* for projections of walls and quasiwall to $Dom(\mathcal{L}, Z)$ via the input map p.

If $\mathcal{L}$ is a (possibly closed) functional linkage and v is an input or output vertex of $\mathcal{L}$ then we let $\pi_v : Dom(\mathcal{L}, Z) \to \mathbf{k}$ to denote the composition $eval_v \circ p^{-1}$. Clearly this composition is a well-defined $\mathbf{k}$-linear mapping.

4. Functionality theorems

In this section we prove three theorems establishing that functionality of linkages is preserved (under appropriate circumstances) by the (self-) fiber sum. To simplify the notations in this section we will suppress the choices of fixed vertices W and their images Z under relative realizations (until we get to the self-fiber sums).

We first establish how configuration spaces of linkages behave under fiber sums of linkages. Let $\mathcal{A}, \mathcal{B}$ be linkages, with N and M vertices respectively and β be a bijection from a subset T of $\mathcal{V}(\mathcal{B})$ to a subset S of $\mathcal{V}(\mathcal{A})$ each consisting of t vertices. Let $\mathcal{G}$ be the graph consisting of t vertices $(G_1, ..., G_t)$ and no edges. Then we have a commutative diagram with all arrows injections

$$\begin{array}{ccc} & \mathcal{A} & \\ \epsilon \nearrow & & \searrow \gamma \\ \mathcal{G} & \longrightarrow & \mathcal{A} *_\beta \mathcal{B} \\ \eta \searrow & & \nearrow \delta \\ & \mathcal{B} & \end{array}$$

where $\beta = \epsilon \circ \eta^{-1}$.

Theorem 4.1. *The configuration space of the fiber sum $\mathfrak{C}(\mathcal{A} *_\beta \mathcal{B})$ is naturally isomorphic to the fiber product of the configuration spaces $\mathfrak{C}(\mathcal{A}) \times_{\mathfrak{C}(\mathcal{G})} \mathfrak{C}(\mathcal{B})$.*

Proof: We will prove this theorem assuming that there are no fixed vertices in $\mathcal{A}$ and $\mathcal{B}$, the general case is similar and is left to the reader. We will also temporarily ignore the issue of degenerate triangles in elementary linkages (see Remark 4.2) since the case of linear equations given by the Convention 3.8 is completely analogous to the quadratic equations considered below.

Since the realization functor is contravariant we obtain a diagram

$$\begin{array}{ccc} & \mathfrak{C}(\mathcal{A} *_\beta \mathcal{B}) & \\ \swarrow & & \searrow \\ \mathfrak{C}(\mathcal{A}) & & \mathfrak{C}(\mathcal{B}) \\ eval_T \searrow & & \swarrow eval_S \\ & \mathbf{k}^t & \end{array}$$

We hence get a map of affine schemes

$$\mathfrak{C}(\mathcal{A} *_\beta \mathcal{B}) \to \mathfrak{C}(\mathcal{A}) \times_{\mathfrak{C}(\mathcal{G})} \mathfrak{C}(\mathcal{B})$$

and a map of coordinate rings

$$\varphi : \mathbb{R}[\mathfrak{C}(\mathcal{A})] \otimes_{\mathbb{R}[\mathfrak{C}(\mathcal{G})]} \mathbb{R}[\mathfrak{C}(\mathcal{B})] \to \mathbb{R}[\mathfrak{C}(\mathcal{A} *_\beta \mathcal{B})].$$

First assume that $\mathcal{A}$ and $\mathcal{B}$ have no edges. Hence $\mathcal{A} \times_\beta \mathcal{B}$ has no edges and all of the above rings are polynomial rings with generators X_v, Y_v corresponding to the vertices v in $\mathcal{V}(\mathcal{A}), \mathcal{V}(\beta)$. ($X_\bullet$ correspond to the first coordinate in $\mathbb{R}^2$ and $Y_\bullet$ to the second coordinate.) The map φ carries $X_v, X_w, Y_v, Y_w, v \in \mathcal{V}(\mathcal{A}), w \in \mathcal{V}(\mathcal{B})$ to $X_{\gamma(v)}, X_{\delta(w)}, Y_{\gamma(v)}, Y_{\delta(w)}$. To check that φ is an isomorphism we divide the vertices of $\mathcal{A} *_\beta \mathcal{B}$ into three classes:

- those in $\gamma\epsilon(\mathcal{G})$,
- those in $\gamma(\mathcal{A}) - \gamma\epsilon(\mathcal{G})$,
- those in $\delta(\mathcal{B}) - \delta\eta(\gamma)$.

We let $U_i', V_i', 1 \leq i \leq t$, $U_j'', V_j'', t+1 \leq j \leq N$ and $U_k''', V_k''', t+1 \leq k \leq M$ be the corresponding indeterminates (here $U_\bullet$ correspond to the first coordinate in $\mathbb{R}^2$ and $V_\bullet$ correspond to the second coordinate). Thus

$$\mathbb{R}[\mathfrak{C}(\mathcal{A} *_\beta \mathcal{B})] = \mathbb{R}[U_i', V_i', U_j'', V_j'', U_k''', V_k''' : 1 \leq i \leq t, t+1 \leq j \leq N, t+1 \leq k \leq M].$$

Similarly we divide the vertices of $\mathcal{A}$ into those in $\epsilon(\mathcal{G})$ and those in $\mathcal{A} - \epsilon(\mathcal{G})$. We let $X_i', Y_i', 1 \leq i \leq t$, and $X_j'', Y_j'', t+1 \leq j \leq N$ be the corresponding indeterminates. Finally we divide up the vertices in $\mathcal{B}$ into those in $\eta(\mathcal{G})$ and those in $\mathcal{B} - \eta(\mathcal{G})$ and let $W_i', Z_i', 1 \leq i \leq t$ and W_j'', Z_j'', $t+1 \leq j \leq M$, be the corresponding indeterminates (here $W_\bullet$ correspond to the first coordinate in $\mathbb{R}^2$ and $Z_\bullet$ correspond to the second coordinate). We find (according to the definition in Section 2) that

$$\mathbb{R}[\mathfrak{C}(\mathcal{A})] \otimes_{\mathbb{R}[\mathfrak{C}(\mathcal{G})]} \mathbb{R}[\mathfrak{C}(\mathcal{B})] = \frac{\mathbb{R}[X_i', Y_i', X_j'', Y_j'', W_i', Z_i', W_k'', Z_k'']}{(X_i' = W_i', Y_i' = Z_i')}.$$

In the above formula and in what follows we use the convention: the i's run from 1 to t, the j's run from $t+1$ to N and the k's run from $t+1$ to M.

The map φ is determined by

$$\varphi(X_i') = U_i', \quad \varphi(Y_i') = V_i', \quad \varphi(X_j'') = U_j'', \quad \varphi(Y_j'') = V_j''$$

$$\varphi(W_i') = U_i', \quad \varphi(Z_i') = V_i', \quad \varphi(W_k'') = U_k''', \quad \varphi(Z_k'') = V_k'''.$$

We obtain an inverse ψ to φ by inverting the above map on the indeterminates (note that $X_i' = W_i'$ and $Y_i' = Z_i'$). We now add edges to $\mathcal{A}$ and $\mathcal{B}$ consequently we add relations to the above $\mathbb{R}$-algebras. It suffices to prove that if e is an edge of $\mathcal{A} *_\beta \mathcal{B}$ and R_e is the corresponding relator then $\psi(R_e)$ is a relator in the tensor product. If e is an edge of $\mathcal{A} *_\beta \mathcal{B}$ then $e = \gamma(e')$ for $e' \in \mathcal{E}(\mathcal{A})$ or $e = \delta(e''), e'' \in \mathcal{E}(\mathcal{B})$. We will treat the first case, the second case is analogous. There are now three cases corresponding to how many of the vertices of e'' are in $\epsilon(\mathcal{G})$. Suppose first that $e' = [v_{j_1}, v_{j_2}], v_{j_1}, v_{j_2} \in \mathcal{A} - \epsilon(\mathcal{G})$. Then

$$R_e = (U_{j_1}'' - U_{j_2}'')^2 + (V_{j_1}'' - V_{j_2}'')^2 - \ell(e)^2$$

and hence

$$\psi(R_e) = (X_{j_1}'' - X_{j_2}'')^2 + (Y_{j_1}'' - Y_{j_2}'')^2 - \ell(e')^2$$

is a relator in $\mathbb{R}[\mathfrak{C}(\mathcal{A})]$, thus in the tensor product. The other two cases are analogous. This concludes the proof of Theorem 4.1. $\square$

Remark 4.2. *In the proof of Theorem 4.1 we did not take into account the linear equations that we added in Convention 3.8. The linear equations added to the defining equations for $\mathfrak{C}(\mathcal{A})$ are determined by the set $\{\mathcal{A}_i\}_{i \in I}$ of distinguished elementary sublinkages of $\mathcal{A}$ and similarly for $\mathfrak{C}(\mathcal{B})$. The reader will verify that in all applications of Theorem 4.1 that follow the degenerate triangles of distinguished elementary sublinkages of $\mathcal{A}$ and $\mathcal{B}$ embed in $\mathcal{A} *_\beta \mathcal{B}$, we define the distinguished elementary sublinkages of $\mathcal{A} *_\beta \mathcal{B}$ to be the images of $\mathcal{A}_i, \mathcal{B}_j$. Thus, if R_Δ is a linear relator for $\mathfrak{C}(\mathcal{A} *_\beta \mathcal{B})$ then $\psi(R_\Delta)$ is a linear relator for either $\mathfrak{C}(\mathcal{A})$ or $\mathfrak{C}(\mathcal{B})$.*

We now restrict to functional linkages. Let $\mathbf{k}$ be either $\mathbb{R}$ of $\mathbb{C}$ and $\overline{\mathbf{k}}$ be its algebraic closure. Consider $\mathcal{A}, \mathcal{B}$ which are $\overline{\mathbf{k}}$- and $\mathbf{k}$-functional linkages respectively for the functions $f : \overline{\mathbf{k}}^n \to \overline{\mathbf{k}}^s$, $g : \mathbf{k}^m \to \mathbf{k}^t$. Let $T = \{P_1, ..., P_t\} \subset In(\mathcal{A})$ be a collection of the input vertices and $Out(\mathcal{B}) = \{Q_1, ..., Q_t\}$. We will assume that $t = n$ in the case $\mathbf{k} \neq \overline{\mathbf{k}}$. Suppose that we are given a bijection $\beta : T \to Out(\mathcal{B}), \beta(P_i) = Q_i$. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$ denote the coordinates in $\overline{\mathbf{k}}^n, \mathbf{k}^m$ respectively.

The goal of the "1-st functionality theorem" below is to prove that the fiber sum $\mathcal{L} = \mathcal{A} *_\beta \mathcal{B}$ is a functional linkage for the composition of the functions f, g and to describe the configuration space, domain, etc., of the linkage $\mathcal{L}$. We will refer to $\mathcal{L} = \mathcal{A} *_\beta \mathcal{B}$ as the *composition* of the linkages $\mathcal{A}, \mathcal{B}$.

We let $eval_T, \pi_T$ be the vector-functions

$$eval_T := (eval_{P_1}, ..., eval_{P_t}) : C(\mathcal{A}) \to \bar{\mathbf{k}}^t, \quad \pi_T := (\pi_{P_1}, ..., \pi_{P_t}).$$

Here $n \geq t$ and we assume that $\mathbf{k}^t$ is canonically embedded in $\mathbf{k}^n = \mathbf{k}^t \times \mathbf{k}^{n-t}$. We let p', p'' be the input maps of $\mathcal{A}, \mathcal{B}$ and q', q'' be the output maps of $\mathcal{A}, \mathcal{B}$.

Theorem 4.3. *(The 1-st functionality theorem.) Suppose that*

$$intDom(\mathcal{B}) \cap g^{-1} intDom(\mathcal{A}) \neq \emptyset.$$

Then:

1. *The scheme $\mathfrak{C}(\mathcal{L})$ is isomorphic to the fiber product $\mathfrak{C}(\mathcal{A}) \times_{eval_T = q''} \mathfrak{C}(\mathcal{B})$.*

2. *$\mathcal{L}$ is a $\mathbf{k}$-functional linkage for the composition h of the functions f, g:*

$$f(g_1(y), ..., g_{t-1}(y), g_t(y), x_{t+1}, ..., x_n)$$

and $In(\mathcal{L}) := In(\mathcal{A}) \cup In(\mathcal{B}) - T$, $Out(\mathcal{L}) := Out(\mathcal{A})$. $Dom(\mathcal{L})$ is isomorphic to $Dom(\mathcal{A}) \times_{\pi_T = g} Dom(\mathcal{B})$ (as semi-algebraic sets).

3. *$Sym(\mathcal{L}) \cong Sym(\mathcal{A}) \times Sym(\mathcal{B})$.*

4. *$Wall(\mathcal{L}) = Wall(\mathcal{A}) \times_{eval_T = q''} C(\mathcal{B}) \cup C(\mathcal{A}) \times_{eval_T = q''} Wall(\mathcal{B})$.*

5. *$Crit(\mathcal{L}) \subset Crit(\mathcal{A}) \times_{eval_T = q''} C(\mathcal{B}) \cup C(\mathcal{A}) \times_{eval_T = q''} Crit(\mathcal{B})$.*

6. *If $O' \in Dom^*(\mathcal{A}), O'' \in Dom^*(\mathcal{B})$ then $(O', O'') \in Dom^*(\mathcal{L})$. If $t = n$ then $Dom^*(\mathcal{B}) \cap g^{-1}(Dom^*(\mathcal{A})) \subset Dom^*(\mathcal{L})$.*

Proof: (1) The first assertion is a special case of Theorem 4.1.

Now we start proving (2). We let p, q be the input and output maps of $\mathcal{L}$ where $In(\mathcal{L}) := In(\mathcal{A}) \cup In(\mathcal{B}) - T$, $Out(\mathcal{L}) := Out(\mathcal{A})$.

We define the isomorphism

$$\theta : Dom(\mathcal{A}) \times_{\pi_T = g} Dom(\mathcal{B}) \to Dom(\mathcal{L}) = p(C(\mathcal{L}))$$

of semi-algebraic sets by the formula:

$$\theta : (x_1, ..., x_n) \times (y_1, ..., y_m) \mapsto (y_1, ..., y_m, x_{t+1}, ..., x_n).$$

The inverse θ^{-1} is given by the formula:

$$(y_1, ..., y_m, x_{t+1}, ..., x_n) \mapsto (g(y), x_{t+1}, ..., x_n) \times (y_1, ..., y_m).$$

The image of p has nonempty interior by the assumption of theorem, i.e. it is a *domain*. We leave the proof of the equality

$$h = q \circ p^{-1}$$

to the reader.

To conclude the proof of (2) it remains to show that p is a regular ramified covering and its group of automorphisms consists of polynomial automorphisms of $C(\mathcal{L})$, this will follow from the proof of (3) and (4) below.

(3) Notice that the group $Sym(\mathcal{A}) \times Sym(\mathcal{B})$ acts naturally on $C(\mathcal{L})$. Indeed, if $\eta = (\eta', \eta'') \in C(\mathcal{L}) \subset C(\mathcal{A}) \times C(\mathcal{B})$ and $\sigma = (\sigma', \sigma'') \in Sym(\mathcal{A}) \times Sym(\mathcal{B})$ then

$$\sigma(\eta) := (\sigma'(\eta'), \sigma''(\eta'')) \in C(\mathcal{A}) \times C(\mathcal{B}).$$

However $\sigma'(\eta')\big|T = \eta'\big|T$, $\sigma''(\eta'')\big|In(\mathcal{B}) = \eta''\big|In(\mathcal{B})$ since $\sigma', \sigma'' \in Sym(\mathcal{A}), Sym(\mathcal{B})$. Hence

$$(\sigma'(\eta'), \sigma''(\eta'')) \in C(\mathcal{L}) = C(\mathcal{A}) \times_{eval_T = q'} C(\mathcal{B}) \subset C(\mathcal{A}) \times C(\mathcal{B}).$$

This implies that σ acts on $C(\mathcal{L})$. We leave it to the reader to verify that the action is faithful. It is clear that for each $\sigma \in Sym(\mathcal{A}) \times Sym(\mathcal{B})$ we have:

$$p \circ \sigma = p, \quad q \circ \sigma = q.$$

Suppose that $\phi, \psi \in C(\mathcal{L})$ are such that $\phi\big|In(\mathcal{L}) = \psi\big|In(\mathcal{L})$. Then

$$\phi\big|In(\mathcal{B}) = \psi\big|In(\mathcal{B})$$

which implies that $\phi\big|T = \psi\big|T$ since $\mathcal{B}$ is functional. Therefore

$$\phi\big|In(\mathcal{A}) = \psi\big|In(\mathcal{A}).$$

It follows that there are symmetries $\sigma' \in Sym(\mathcal{A}), \sigma'' \in Sym(\mathcal{B})$ such that

$$\sigma'(\phi\big|\mathcal{A}) = \psi\big|\mathcal{A}, \quad \sigma''(\phi\big|\mathcal{B}) = \psi|\mathcal{B}.$$

We conclude that $\psi = (\sigma' \times \sigma'')(\phi)$, in particular

$$\psi\big|Out(\mathcal{L}) = \phi\big|Out(\mathcal{L}).$$

This also shows that $Sym(\mathcal{L}) \subset Sym(\mathcal{A}) \times Sym(\mathcal{B})$. This finishes the proof of (3).

(4) Suppose that $\sigma = \sigma' \times \sigma'' \in Sym(\mathcal{L})$. Then $Fix(\sigma)$ is contained in the union

$$Fix(\sigma' \times 1) \cup Fix(1 \times \sigma'')$$

Note that

$$Fix(\sigma' \times 1) = Fix(\sigma') \times_{eval_T = q'} C(\mathcal{B})$$
$$Fix(1 \times \sigma'') = C(\mathcal{A}) \times_{eval_T = q'} Fix(\sigma'').$$

This proves (4). We leave it to the reader to verify that $p|C(\mathcal{L}) - Wall(\mathcal{L})$ is a local homeomorphism into $Dom(\mathcal{L})$. Since $p(\phi) = \psi$ iff there is $\sigma \in Sym(\mathcal{L})$ such that $\psi = \sigma(\phi)$ we conclude that $p|C(\mathcal{L}) - Wall(\mathcal{L})$ is a covering onto its image. This concludes the proof of (2).

(5) Suppose $\phi' \in C^*(\mathcal{A})$, $\phi'' \in C^*(\mathcal{B})$. Hence the maps of germs

$$p' : (\mathfrak{C}(\mathcal{A}), \phi') \to (Dom(\mathcal{A}), p'(\phi'))$$

$$p'' : (\mathfrak{C}(\mathcal{B}), \phi'') \to (Dom(\mathcal{B}), p''(\phi''))$$

are analytic isomorphisms. We wish to prove that the induced map

$$(\mathfrak{C}(\mathcal{B}) \times_{g \circ p'' = \pi_T \circ p'} \mathfrak{C}(\mathcal{A}), (\phi'', \phi')) \to (Dom(\mathcal{B}) \times_{g = \pi_T} Dom(\mathcal{A}), (p''(\phi''), p'(\phi'))$$

is an isomorphism. But under the canonical isomorphism of Lemma 2.11 this map corresponds to the fiber product of the isomorphisms p'' and p' above. Since the fiber product of isomorphisms is an isomorphism the assertion (5) follows.

This proves that

$$Crit(\mathcal{L}) \subset Crit(\mathcal{A}) \times_{eval_T=q'} C(\mathcal{B}) \cup C(\mathcal{A}) \times_{eval_T=q'} Crit(\mathcal{B}).$$

(6) Follows directly from (5). $\square$

Now we consider the self-fiber sums: $\mathcal{L} = \mathcal{A}*_\beta$, where the linkage $\mathcal{A}$ is $\mathbf{k}$-functional for a vector-function $f(x_1, ..., x_n)$ with the components $(f_1, ..., f_m)$.

There will be two cases to consider. In both cases we will apply the following lemma on fiber products.

Suppose that we have diagrams of maps of analytic germs (where the second diagram is the fiber square completion of the first diagram)

$$\begin{array}{ccc} & & (\mathfrak{X}, x) \\ & & \alpha \downarrow \\ & & (\mathfrak{Y}, y) \\ & & \beta \downarrow \\ (\mathfrak{W}, w) & \xrightarrow{\gamma} & (\mathfrak{Z}, z) \end{array} \quad \text{and} \quad \begin{array}{ccc} (\mathfrak{W} \times_{\mathfrak{Z}} \mathfrak{X}, (w, x)) & \longrightarrow & (\mathfrak{X}, x) \\ id \times_{\mathfrak{Z}} \alpha \downarrow & & \alpha \downarrow \\ (\mathfrak{W} \times_{\mathfrak{Z}} \mathfrak{Y}, (w, y)) & \longrightarrow & (\mathfrak{Y}, y) \\ id \times_{\mathfrak{Z}} \beta \downarrow & & \beta \downarrow \\ (\mathfrak{W}, w) & \xrightarrow{\gamma} & (\mathfrak{Z}, z) \end{array}$$

Lemma 4.4. *Suppose that α an isomorphism. Then the pull-back morphism $id \times_{\mathfrak{Z}} \alpha : (\mathfrak{W}, w) \times_{(\mathfrak{Z}, z)} (\mathfrak{X}, x) \to (\mathfrak{W}, w) \times_{(\mathfrak{Z}, z)} (\mathfrak{Y}, y)$ is an isomorphism.*

Proof: If α^{-1} is the inverse to α then $id \times_Z \alpha^{-1}$ is the inverse to $id \times_{\mathfrak{Z}} \alpha$. $\square$

Case I: $\beta : S' = \{v\} \subset In(\mathcal{A}) \to In(\mathcal{A})$, $\beta(v) = w$. Let u denote the image of v (and w) in $\mathcal{A}*_\beta$. Then $\mathfrak{C}(\mathcal{L})$ is the pull-back corresponding to the diagram

$$\begin{array}{ccc} \mathfrak{C}(\mathcal{L}) & \longrightarrow & \mathfrak{C}(\mathcal{A}) \\ \downarrow & & \zeta \downarrow \\ \mathbf{k} & \xrightarrow{\Delta} & \mathbf{k}^2 \end{array}$$

where the vertical map ζ is the forgetful morphism $p'_v \times p'_w$ that records only the position of the vertices v and w and the horizontal map Δ is the diagonal map. Thus we are identifying the two input vertices v and w.

Theorem 4.5. *(The 2-nd functionality theorem.) Suppose that the set $\{x \in Dom^*(\mathcal{A}) : \pi_v(x) = \pi_w(x)\}$ is nonempty. Then: the mapping $p : C(\mathcal{L}) \to Dom(\mathcal{L})$ is a regular ramified covering with the group $Sym(\mathcal{L})$ of covering transformations equal to the image of $Sym(\mathcal{A})$ in the group of automorphisms of $C(\mathcal{L})$ under the restriction map. This covering is a (scheme-theoretic) locally analytically trivial covering over*

$$Dom^*(\mathcal{L}) \supset Dom(\mathcal{L}) \cap Dom^*(\mathcal{A}).$$

The real semi-algebraic set $Dom(\mathcal{L})$ is isomorphic to $\{x \in Dom(\mathcal{A}) : \pi_v(x) = \pi_w(x)\}$. The linkage $\mathcal{L}$ is functional for the restriction of the function f to the hyperplane $\{\pi_v(x) = \pi_w(x)\}$.

Proof: Let p' denote the input map of $\mathcal{A}$. Consider the input mapping

$$p : C(\mathcal{L}) \to Dom(\mathcal{L}) := p(C(\mathcal{L}))$$

which is the restriction of p'. Then $Dom(\mathcal{L})$ equals

$$\{x \in Dom(\mathcal{A}) : \pi_v(x) = \pi_w(x)\}$$

i.e. the intersection of $Dom(\mathcal{A})$ with a hyperplane. We will assume that the intersection of $Dom^*(\mathcal{A})$ with this hyperplane is nonempty. Since $Dom^*(\mathcal{A})$ is open we conclude that $Dom(\mathcal{L})$ has nonempty interior.

The group of symmetries $Sym(\mathcal{A})$ acts naturally on $C(\mathcal{L})$. Let $\phi, \psi \in C(\mathcal{L})$ be realizations such that $\phi|In(\mathcal{L}) = \psi|In(\mathcal{L})$. Then ϕ, ψ lift to realizations $\phi', \psi' \in C(\mathcal{A})$ such that $\phi'|In(\mathcal{A}) = \psi'|In(\mathcal{A})$. It follows that there is a symmetry $\sigma' \in Sym(\mathcal{A})$ such that $\sigma'\psi' = \phi'$, hence $\sigma\psi = \phi$, where σ is the restriction of σ' to $C(\mathcal{L})$. We conclude that $Sym(\mathcal{L})$ is the image of $Sym(\mathcal{A})$ in the group of automorphisms of $C(\mathcal{L})$. In particular,

$$Wall(\mathcal{L}) \subset Wall(\mathcal{A}).$$

Recall that $\pi_v, \pi_w : Dom(\mathcal{A}) \to \mathbf{k}$ and $\pi_u : Dom(\mathcal{L}) \to \mathbf{k}$ are projections corresponding to the positions of the vertices v, w and u. Put $p'_v = \pi_v \circ p'$, $p'_w = \pi_w \circ p'$, $p_u := \pi_u \circ p$. We have diagrams of affine schemes

$$\begin{array}{ccc} \mathfrak{C}(\mathcal{L}) & \longrightarrow & \mathfrak{C}(\mathcal{A}) \\ p \downarrow & & p' \downarrow \\ \mathbf{k}^{n-1} & \longrightarrow & \mathbf{k}^n \\ \pi_u \downarrow & & \pi_v \times \pi_w \downarrow \\ \mathbf{k} & \xrightarrow{\Delta} & \mathbf{k} \times \mathbf{k} \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{C}(\mathcal{L}) & \longrightarrow & \mathfrak{C}(\mathcal{A}) \\ p_u \downarrow & & \pi_v \times \pi_w \downarrow \\ \mathbf{k} & \xrightarrow{\Delta} & \mathbf{k} \times \mathbf{k} \end{array}$$

where Δ is the diagonal embedding. The reader will verify that the square immediately above is a fiber square. This amounts to saying that the coordinate ring of $\mathfrak{C}(\mathcal{L})$ is obtained from that of $\mathfrak{C}(\mathcal{A})$ by imposing the equations $p'_v = p'_w$.

Now suppose $\phi \notin Crit(\mathcal{A})$. We claim $\phi \notin Crit(\mathcal{L})$. It is enough to show that ϕ is not an irregular point of the input map of $\mathcal{L}$. Since $\phi \notin Crit(\mathcal{A})$, the projection

$$p' : (\mathfrak{C}(\mathcal{A}), \phi) \to (Dom(\mathcal{A}), p'(\phi))$$

is an isomorphism. By Lemma 4.4 the induced map

$$p : (\mathfrak{C}(\mathcal{L}), \phi) \to (Dom(\mathcal{L}), \phi)$$

is an isomorphism. We conclude that

$$Crit(\mathcal{L}) \subset Crit(\mathcal{A}) \cap C(\mathcal{L}). \quad \square$$

Case II: Let $\beta : S' \subset Out(\mathcal{A}) \to W$, where W is the collection of *fixed vertices* of $\mathcal{A}$; let $W' := \beta(S')$. Hence the linkage $\mathcal{L} = \mathcal{A}*_\beta$ is a "closed functional linkage". Let $S' = \{Q_1, ..., Q_t\}$, thus $\beta(Q_j)$ is a *fixed vertex* for each $j = 1, ..., t$. Let $z_j := \phi(\beta(P_j))$ for all relative realizations, $Z' = (z_1, ..., z_t)$ is the ordered set which is the image of W' in Z. We have the pull-back diagram

$$\begin{array}{ccc} \mathfrak{C}(\mathcal{L}) & \longrightarrow & \mathfrak{C}(\mathcal{A}) \\ \downarrow & & q'_{S'} \downarrow \\ \{S'\} & \xrightarrow{\iota} & \mathbf{k}^t \end{array}$$

where the image of the one element set $\{S'\}$ is the vector $(z_1, ..., z_t) \in \mathbf{k}^t$. Here q' is the output map for $\mathfrak{C}(\mathcal{A})$ and $q'_{S'}$ means the restriction of q' to $S' \subset Out(\mathcal{A})$. The reader will verify that the diagram is canonically isomorphic to the fiber square:

$$\begin{array}{ccc} \{S'\} \times_{\mathbf{k}^t} \mathfrak{C}(\mathcal{A}) & \longrightarrow & \mathfrak{C}(\mathcal{A}) \\ \downarrow & & q'_{S'} \downarrow \\ \{S'\} & \xrightarrow{\iota} & \mathbf{k}^t \end{array}$$

This amounts to saying that the coordinate ring of $\mathfrak{C}(\mathcal{L})$ is obtained from that of $\mathfrak{C}(\mathcal{A})$ by imposing the equations $f_j(x) = z_j$, $1 \le j \le t$. We now prove

Theorem 4.6. *(The 3-rd functionality theorem.) Suppose that $Dom(\mathcal{L}) \cap Dom^*(\mathcal{A})$ is nonempty. Then the input mapping $p : C(\mathcal{L}) \to Dom(\mathcal{L})$ is a regular ramified covering with the group $Sym(\mathcal{L})$ of deck-transformations where $Sym(\mathcal{L})$ is the quotient of $Sym(\mathcal{A})$ by the subgroup acting trivially on $C(\mathcal{L})$. This covering is locally analytically trivial over $Dom^*(\mathcal{L}) := Dom(\mathcal{L}) \cap Dom^*(\mathcal{A})$ (in the scheme-theoretic sense). The set $Dom^*(\mathcal{L})$ (which is an open[8] subset in $Dom(\mathcal{L})$) is analytically isomorphic to*

$$\{x \in Dom^*(\mathcal{A}) : f_j(x) = z_j, j = 1, ..., t\} \quad .$$

Proof: Consider the input map for $\mathcal{L}$:

$$p : C(\mathcal{L}) \to Dom(\mathcal{A})$$

which is the restriction of the input map p' of $\mathcal{A}$. The image of this restriction is

$$Dom(\mathcal{L}) = \{x \in Dom(\mathcal{A}) : f_j(x) = z_j, j = 1, ..., t\}.$$

Let $\mathfrak{D}om(\mathcal{L})$ denote the corresponding affine scheme. Next consider the group of symmetries of $\mathcal{L}$. Suppose that $\phi, \psi \in C(\mathcal{L})$ are such that $\phi|In(\mathcal{L}) = \psi|In(\mathcal{L})$. It follows that they are restrictions of realizations $\phi', \psi' \in C(\mathcal{A})$ and there is a symmetry $\sigma \in Sym(\mathcal{A})$ such that $\phi' = \sigma\psi'$. Hence $Sym(\mathcal{L})$ is the image of $Sym(\mathcal{A})$ under the restriction map from $C(\mathcal{A})$ to $C(\mathcal{L})$. In particular, $Wall(\mathcal{L}) \subset Wall(\mathcal{A}) \cap C(\mathcal{L})$.

We now have the diagram

$$\begin{array}{ccc} \mathfrak{C}(\mathcal{L}) & \longrightarrow & \mathfrak{C}(\mathcal{A}) \\ p \downarrow & & p' \downarrow \\ \mathfrak{D}om(\mathcal{L}) & \longrightarrow & \mathbf{k}^n \\ \downarrow & & f \downarrow \\ \{S'\} & \xrightarrow{\iota} & \mathbf{k}^t \end{array}$$

The reader will verify that the diagram is canonically isomorphic to the fiber square:

$$\begin{array}{ccc} \{Z'\} \times_{\mathbf{k}^t} \mathfrak{C}(\mathcal{A}) & \longrightarrow & \mathfrak{C}(\mathcal{A}) \\ \downarrow & & p' \downarrow \\ \{Z'\} \times_{\mathbf{k}^t} \mathbf{k}^n & \longrightarrow & \mathbf{k}^n \\ \downarrow & & f \downarrow \\ \{Z'\} & \xrightarrow{\iota} & \mathbf{k}^t \end{array}$$

Now suppose $\phi \notin Crit(\mathcal{A})$ and $\phi \in C(\mathcal{L})$. We claim that $\phi \in C^*(\mathcal{L})$. Indeed, since $Wall(\mathcal{L}) \subset Wall(\mathcal{A}) \cap C(\mathcal{L})$, $\phi \notin Wall(\mathcal{L})$. Then, since $\phi \notin Crit(\mathcal{A})$ it follows that

$$p' : (\mathfrak{C}(\mathcal{A}), \phi) \to (Dom(\mathcal{A}), p'(\phi))$$

[8] In the classical topology.

is an isomorphism. By Lemma 4.4 the induced map

$$p : (\mathfrak{C}(\mathcal{L}), \phi) \to (\mathfrak{D}om(\mathcal{L}), p(\phi))$$

is also an isomorphism. We conclude that $Crit(\mathcal{L}) \subset Crit(\mathcal{A}) \cap C(\mathcal{L})$.

Therefore the mapping p is an analytically trivial covering over $Dom^*(\mathcal{L})$ with the group $Sym(\mathcal{L}) \cong (Sym(\mathcal{A})|C(\mathcal{L}))$ of covering transformations. ☐

5. Fixing fixed vertices

The goal of this section is to relate relative configuration spaces $C(\mathcal{L}, Z)$ of marked linkages and the moduli spaces $\mathcal{M}(\mathcal{L})$ of based linkages. Let $\mathcal{L} = (L, \ell, W)$ be a marked linkage, $Z = (z_1, ..., z_s) \in \mathbb{C}^s$ and $W = (w_1, ..., w_s)$. Pick any relative realization $\phi \in C(\mathcal{L}, Z)$.

We first let $\mathcal{L}'$ be the disjoint union of $\mathcal{L}$ and the metric graph $\mathcal{I}$ which consists of a single edge e^* of the unit length connecting the vertices v_1, v_2. Choose the isometric embedding $\phi = \phi_{\mathcal{I}} : \mathcal{I} \to \mathbb{C}$ which maps v_1 to 0 and v_2 to $1 \in \mathbb{R}$. We get a map $\phi : W \cup \mathcal{V}(\mathcal{I}) \to \mathbb{C}$. Then for each pair of vertices $a, b \in W \cup \mathcal{V}(\mathcal{I})$ we do the following:

(a) If $\phi(a) = \phi(b)$ for $\phi \in C(\mathcal{L}, Z)$, we identify the vertices a, b.

(b) Otherwise add to $\mathcal{L}'$ the edge $[ab]$ of the length $|\phi(a) - \phi(b)|$.

Let $\tilde{\mathcal{L}}$ be the resulting based linkage (with the distinguished edge $e^* = [v_1 v_2] \subset \mathcal{I}$).

There are now two different cases: (i) the vector Z is not real (we shall assume $z_1 \notin \mathbb{R}$), (ii) Z is real. In the second case for each realization $\phi \in \mathcal{M}(\tilde{\mathcal{L}})$ we have:

$$(\phi(w_1), \phi(w_2), ..., \phi(w_s)) = (z_1, ..., z_s) \in \mathbb{R}^s$$

Thus the natural (scheme-theoretic) morphism $\iota : C(\mathcal{L}, Z) \to \mathcal{M}(\tilde{\mathcal{L}})$ is a bijection. However (unless the image of $W \cup \mathcal{V}(\mathcal{I})$ in $\tilde{\mathcal{L}}$ consists of two vertices) we had created new nilpotent elements in the ring of $\mathcal{M}(\tilde{\mathcal{L}})$, thus ι is not a scheme-theoretic analytic isomorphism. On the other hand, since we are interested in the real reduced schemes, we can use the same trick as in the case of rigidified parallelograms: we give $\mathcal{M}(\tilde{\mathcal{L}})$ the scheme-theoretic structure of $\mathfrak{C}(\mathcal{L}, Z)$.

In the case (i) for each realization $\phi \in \mathcal{M}(\tilde{\mathcal{L}})$ we have:

$$(\phi(w_1), \phi(w_2), ..., \phi(w_s)) = (z_1, ..., z_s)$$

or

$$(\phi(w_1), \phi(w_2), ..., \phi(w_s)) = (\bar{z}_1, ..., \bar{z}_s) \neq (z_1, ..., z_s).$$

On the other hand, we did not create new nilpotent element in the ring of $\mathcal{M}(\tilde{\mathcal{L}})$ (since for each w_i (if $i \geq 2, z_i \neq 0$) either the triangle $\Delta(v_1 v_2 w_i)$ or the triangle $\Delta(v_1 w_1 w_i)$ is nondegenerate. Thus in the case (i) we get an analytically trivial covering $\tau : \mathcal{M}(\tilde{\mathcal{L}}) \to C(\mathcal{L}, Z)$ given by:

For $\phi \in \mathcal{M}(\tilde{\mathcal{L}})$ we let $\tau(\phi) := \phi|\mathcal{L}$ if $\phi(w_1) = z_1$ and $\tau(\phi) := \overline{\phi}|\mathcal{L}$ if $\phi(w_1) = \overline{z_1}$.
This covering has a section $\sigma : \psi \in C(\mathcal{L}, Z) \mapsto \mathcal{M}(\tilde{\mathcal{L}})$ such that:

$$\sigma(\psi)\big|\mathcal{L} := \psi, \quad \sigma(\psi)|\mathcal{I} := \phi_{\mathcal{I}}.$$

It is clear that the group of automorphisms of the covering τ is $\mathbb{Z}_2$ and is generated by the complex conjugation. We summarize this in the following

Lemma 5.1. *(i) In the case $Z \notin \mathbb{R}^s$ there is an 2-fold analytically trivial covering $\tau : \mathcal{M}(\tilde{\mathcal{L}}) \to C(\mathcal{L}, Z)$.*

(ii) In the case $Z \in \mathbb{R}^s$ there is an isomorphism $\tau : \mathcal{M}(\tilde{\mathcal{L}}) \to C(\mathcal{L}, Z)$.

6. Elementary linkages

In this section we construct several *elementary* functional linkages: translators (for the translation $z \mapsto z+b$), the adder (for the summation $(z,w) \mapsto z+w$), pantographs (for the functions $z \mapsto \lambda z$ and $z \mapsto -z$), inversors (for the functions $z \mapsto t^2/\bar{z}$), the multiplier[9] (for the function $(z,w) \mapsto zw$) and the linkage for straight line motion. These linkages serve as building blocks for the proof of Theorem A. We make the following convention concerning usage of *elementary linkages*:

All elementary linkages come with parameters which do not affect functions that they define but affect domains of the linkages. Thus if we use several elementary linkages with the same name $\mathcal{N}$ in constructing another linkage $\mathcal{L}$ via fiber sum, we allow different choices of the parameters for different appearances of $\mathcal{N}$ in this fiber sum.

We also omit the image of marking Z in the notation for Dom, Dom^* of the elementary linkages.

All *elementary* linkages in the section (with the exception of the multiplier) are modifications of classical constructions, where appropriate modification was made to ensure functionality. We decided to avoid Kempe's construction of the multiplier [Ke1] since computation of Dom and Dom^* for Kempe's linkage presents some difficulties, we use an algebraic trick instead.

6.1. The translators

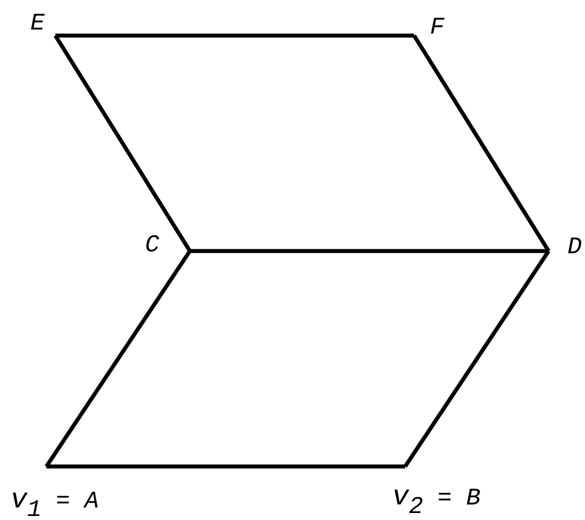

Figure 7: *A translator. The parallelograms* $\square(ACDB)$ *and* $\square(CEFD)$ *are rigidified. The set of intput/output vertices is* $\{E,F\}$ *and* $s = \ell[AC] > t = \ell[CE]$.

Let b be a fixed nonzero complex number. The translation operations $\tau_b : z \mapsto z+b$, $\tau'_b : w \mapsto w-b$ are defined using the translator which is described on Figure 7. Depending on the operation either F or E is the input (resp. output). The point is that if E is the input then by adjusting side-lengths of the corresponding translator $\mathcal{T}_b$ we can get any $z \in \mathbb{C} - \{0\}$ into $Dom^*(\mathcal{T}_b)$. To get $0 \in \mathbb{C}$ into Dom^* we use the point F as the input (and E as the output) of a functional linkage $\mathcal{T}'_b$ for τ'_b. Below we present the details. First of all let $W := (v_1, v_2)$ be the marking of $\mathcal{L}$ (which is either $\mathcal{T}_b$ or $\mathcal{T}'_b$) and $Z := (0,b) \in \mathbb{C}^2$. Below we shall use the relative configuration spaces of $\mathcal{L}$ associated to this data.

[9]Strictly speaking, our multiplier is not so elementary, for instance it would be difficult to draw a picture of the corresponding graph.

The next lemma follows from the triangle inequalities and its proof is left to the reader.

Lemma 6.1. *$Dom(\mathcal{T}_b)$ and $Dom(\mathcal{T}'_b)$ are the annuli given by the inequalities:*

$$Dom(\mathcal{T}_b) = \{\rho := s - t \leq |\phi(E)| \leq R := s + t\},$$

$$Dom(\mathcal{T}'_b) = \{\rho \leq |\phi(F) - b| \leq R\}. \quad \square$$

Notice that the centers of these annuli are at the points $0, b$ respectively.

Lemma 6.2. *Both linkages $\mathcal{T}_b$, $\mathcal{T}'_b$ are functional for the functions τ_b, τ'_b. The walls in the domains of these linkages are boundary circles of the corresponding annuli.*

Proof: We consider the first linkage $\mathcal{L} = \mathcal{T}_b$, the proof for the second linkage is analogous. Notice that for each realization ϕ we have:

$$\phi(F) = \phi(E) + \phi(B), \quad \phi(D) = \phi(C) + \phi(B)$$

and $\phi(A) \neq \phi(E)$ (see Lemma 3.9). Suppose that $\phi \neq \psi$ are relative realizations and $\phi(E) = \psi(E)$. Then $\phi(F) = \psi(F)$. Thus $\psi(C)$ is the reflection of $\phi(C)$ in the line through $\phi(A), \phi(E)$. Therefore $\mathcal{L}$ is functional and the group of symmetries of $\mathcal{L}$ is $\mathbb{Z}_2$. The fixed points of the generator of $Sym(\mathcal{L})$ correspond to realizations for which the triangle

$$\Delta(\phi(A), \phi(C), \phi(E))$$

is degenerate, i.e. one of the inequalities defining $Dom(\mathcal{L})$ is the equality. $\square$

Suppose that $\phi \in C(\mathcal{L}, Z) - Wall(\mathcal{L}, Z)$ is a realization of $\mathcal{L} = \mathcal{T}_b$ or $\mathcal{L} = T'_b$ such that none of the parallelograms $\square(\phi(A)\phi(B)\phi(D)\phi(C))$, $\square(\phi(C)\phi(D)\phi(F)\phi(E))$ is degenerate. We leave it to the reader to verify (using Lemma 3.11) that in this case the morphism of analytic germs $p : (\mathfrak{C}(\mathcal{L}, Z), \phi) \to (\mathfrak{D}om(\mathcal{L}, Z), p(\phi))$ is inevitable and hence such ϕ belongs to $C^*(\mathcal{L}, Z)$. For instance, if F is the input then one can recover $\phi(D)$ as analytic function of $\phi(F)$ and $\phi(B) = b$ (since the triangle $\Delta(\phi(B)\phi(D)\phi(F))$ is nondegenerate). Then one recovers $\phi(C)$ as analytic function of $\phi(A) = 0$ and $\phi(D)$ (since the first of the parallelograms is nondegenerate), etc.

On the other hand, if one of the above parallelograms is degenerate then the derivative

$$Dp_\phi : T_\phi \mathfrak{C}(\mathcal{L}, Z) \to T_{p(\phi)} Dom(\mathcal{L}, Z)$$

has non-zero kernel.

We summarize this in the following lemma

Lemma 6.3. *If $\phi \in C(\mathcal{L}, Z)$ belongs to a quasiwall then one of the parallelogram*

$$\square(\phi(A)\phi(B)\phi(D)\phi(C)), \quad \square(\phi(E)\phi(F)\phi(D)\phi(C))$$

is degenerate. Let $\beta := b/|b|$. The closure of the union of quasiwalls for both $\mathcal{T}_b$, $\mathcal{T}'_b$ is the union of four circles:

$$\{\phi(C) = \pm s\beta, |\phi(E) \mp s\beta| = t\} \cup \{|\phi(E) \pm t\beta| = s\}.$$

Unions of quasiwalls for both $\mathcal{L}$ are disjoint from the collection of ϕ for which $\phi(E)$ belongs to the line through $0, b$.

In particular, if

$$s + t > |b| > s - t > 0, \quad b \in \mathbb{R}$$

then the origin belongs to $Dom^*(\mathcal{T}'_b)$.

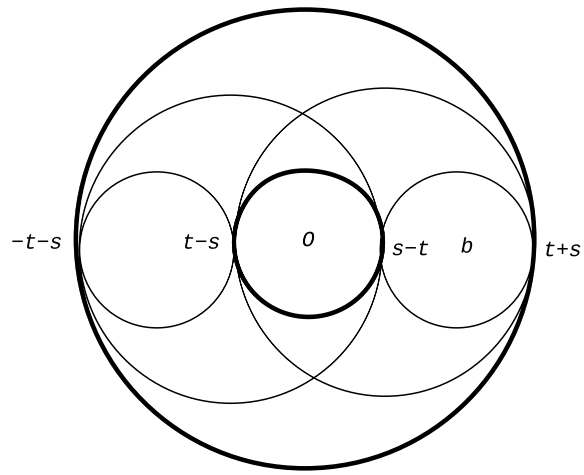

Figure 8: *Quasiwalls in the domain of the translator $\mathcal{T}_b$.*

Remark 6.4. *In the following sections (except in §11) each time when we have to use a translator we shall pick real numbers b.*

The quasiwalls for $\mathcal{T}_b$ are described on Figure 8, the picture for quasiwalls of $\mathcal{T}_b'$ is obtained via translating by b. Notice that

$$int(B_t(\pm s)) \subset Dom^*(\mathcal{T}_b), \quad int(B_t(\pm s + b)) \subset Dom^*(\mathcal{T}_b')$$

(provided that $b \in \mathbb{R}$).

Corollary 6.5. *Let $0 < r < |b|, b \in \mathbb{R}$. Then the parameters t, s can be chosen so that*

$$B_r(b) \subset Dom^*(\mathcal{T}_b), \quad B_r(0) \subset Dom^*(\mathcal{T}_b').$$

Proof: Choose t, s so that $|t - s| \to 0$, $|t + s| \to \infty$. Then $Dom^*(\mathcal{T}_b)$ converges to the union of half-planes $\{Re(z) \neq 0\}$ and $Dom^*(\mathcal{T}_b')$ converges to $\{Re(z) \neq b\}$. □

6.2. The pantograph

The (rigidified) pantograph $\mathcal{P}$ is described on Figure 9, recall that we use the Convention 3.8 for the two degenerate triangles in $\mathcal{P}$ as well as for the rigidified parallelogram. The pantograph is a versatile linkage, its role in engineering[10] was as a functional linkage for the functions $z \mapsto \lambda z, z \mapsto \lambda^{-1} z$, $\lambda > 1$.

Remark 6.6. *In the next section we shall also use the pantograph to construct the adder.*

In the case of the function $z \mapsto \lambda z$ we let $W := \{A\}$ be the fixed vertex, $Z := 0$, take D as input and G as output, let $\mathcal{P}_\lambda$ be the resulting linkage (it will be functional for $z \mapsto \lambda z$). By switching input and output we obtain a functional linkage $\mathcal{P}_{1/\lambda}$ for $z \mapsto z/\lambda$.

By letting $\{D\} = W$ instead of A, the same Z as before, $\lambda = 2$ and taking A as input and G as output we obtain a functional linkage for the function $z \mapsto -z$ in the complex plane. Notice that the condition $s \neq t$ implies that for each realization ϕ the points $\phi(A)$, $\phi(D)$, $\phi(G)$ are pairwise distinct.

Below we describe Dom and Dom^* of the pantograph, the proofs are similar to the previous section and are left to the reader.

[10]That goes back to at least 17-th century, see [Sch].

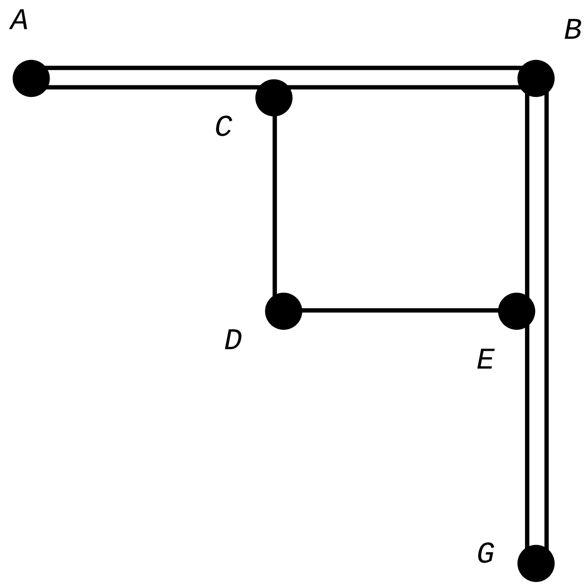

Figure 9: *The rigidified pantograph $\mathcal{P}$: the parallelogram $BCDE$ is rigidified, $\lambda > 1$. This linkage is not marked, we shall use different choices of input/output vertices later on. We take: $s = \ell[AB] = \lambda\ell[AC] \neq t = \ell[BG] = \lambda\ell[BE]$.*

Lemma 6.7. *For each choice of the fixed vertex and input/output vertices described above, the pantograph is a functional linkage. The group of symmetries is isomorphic to $\mathbb{Z}/2$ and is generated by the reflection of $\phi(B)$ in the line ν through the points $\phi(A)$, $\phi(D)$, $\phi(G)$. The walls are described by the condition: $\phi(B) \in \nu$. There are no quasiwalls. $Dom^*(\mathcal{P})$ is the interior of $Dom(\mathcal{P})$. If D or A is the input and G is the output then the domain of the pantograph is the annulus given by the inequalities*

$$|s-t|/\lambda \leq |\phi(D) - \phi(A)| \leq (s+t)/\lambda.$$

If G is the input and D is the output then the domain of the pantograph is the annulus given by the inequalities

$$|s-t| \leq |\phi(G) - \phi(A)| \leq s+t.$$

For given λ as $|s-t| \to 0$ and $|s+t| \to \infty$ the domains $Dom^*(\mathcal{P})$ are convergent to punctured complex planes.

Note that zero does not belong to Dom^* of the pantograph for any choice of input/output points. To resolve this problem we compose $\mathcal{P}$ with the appropriate translators:

$$-z = -(z+b) + b = \tau_b(-\tau'_{-b}(z))$$

$$\lambda z = \lambda(z+b) - \lambda b = \tau_{-\lambda b}(\lambda \tau'_{-b}(z))$$

$$z/\lambda = (z+b)/\lambda - b/\lambda = \tau_{-b/\lambda}(\tau'_{-b}(z)/\lambda)$$

where $b \in \mathbb{R} - \{0\}$. We call the linkages computing the above functions the *modified pantographs* and denote them $\mathcal{P}'_-, \mathcal{P}'_\lambda, \mathcal{P}'_{1/\lambda}$ respectively. We would like $Dom^*(\mathcal{P}'_\lambda), Dom^*(\mathcal{P}'_{1/\lambda})$ to contain arbitrarily large compacts. This is done as follows:

Lemma 6.8. *Fix $\lambda > 1$ and let $r > 0$. Then we can choose $b \in \mathbb{R}$ and edge-lengths for the translators and for the pantographs $\mathcal{P}_\lambda, \mathcal{P}_{1/\lambda}$ so that*

$$B_r(0) \subset Dom^*(\mathcal{P}'_\lambda), \quad B_r(0) \subset Dom^*(\mathcal{P}'_{1/\lambda}).$$

Proof: We consider the case of the modified pantograph $\mathcal{L} := \mathcal{P}'_\lambda$, the second case is similar. By Theorem 4.3 we have:

$$Dom^*(\mathcal{L}) \supset Dom^*(\mathcal{T}'_{-b}) \cap [Dom^*(\mathcal{P}_\lambda) - b] \cap [\lambda^{-1} Dom^*(\mathcal{T}_{-\lambda b}) - b].$$

Below we analyze the triple intersection:
(1) By choosing appropriately the parameters t, b in $\mathcal{T}'_{-b}$ we can guarantee that $Dom^*(\mathcal{T}'_{-b})$ contains arbitrarily large discs around the origin (see Corollary 6.5). In particular, $B_r(0) \subset Dom^*(\mathcal{T}'_{-b})$.

(2) The domain $[Dom^*(\mathcal{P}_\lambda) - b]$ is obtained by translating $Dom^*(\mathcal{P}_\lambda)$ by $-b$. Recall that $Dom^*(\mathcal{P}_\lambda)$ is an open annulus centered at zero. By adjusting parameters in $\mathcal{P}_\lambda$ (and keeping λ fixed) we can guarantee that $Dom^*(\mathcal{P}_\lambda)$ contains the disk $B_\rho(b)$ for each $\rho < |b|$, see Lemma 6.7. We conclude that if $|b| > r$ (and under appropriate choice of edge-lengths in $\mathcal{P}_\lambda$, $\mathcal{T}'_{-b}$) the domain

$$Dom^*(\mathcal{T}'_{-b}) \cap [Dom^*(\mathcal{P}_\lambda) - b]$$

contains the disk $B_r(0)$.

(3) Lastly we consider the domain $\lambda^{-1} Dom^*(\mathcal{T}_{-\lambda b}) - b$, it contains the disk $B_r(0), r = R/\lambda$, provided that $Dom^*(\mathcal{T}_{-\lambda b})$ contains the disk of radius R centered at the point λb. The domain $Dom^*(\mathcal{T}_{-\lambda b})$ is again an annulus centered at zero. By adjusting parameters of the linkage $\mathcal{T}_{-\lambda b}$ (and keeping λ, b fixed) we can guarantee that $Dom^*(\mathcal{T}_{-\lambda b})$ contains arbitrary annuli centered at zero. Hence for each $R < |\lambda b|$ (i.e. $r < |b|$) and under appropriate choice of edge-lengths in $\mathcal{T}_{-\lambda b}$, the domain $Dom^*(\mathcal{T}_{-\lambda b})$ contains the disk $B_R(\lambda b)$. We conclude that for each $\lambda > 0$ and $r > 0$ if we choose b such that $r < |b|$, then the edge-lengths of the linkage $\mathcal{L}$ can be chosen so that

$$B_r(0) \subset Dom^*(\mathcal{L}). \qquad \square$$

6.3. The adder

We again consider the rigidified pantograph, only now $\lambda = 2$, the vertices A, G are the inputs, D is the output and there is no fixed vertices at all. We will use the notation $\mathcal{Q}$ for the resulting linkage. Similarly to the previous section $\mathcal{Q}$ is $\mathbb{C}$-functional for the function

$$(z, w) \mapsto (z + w)/2$$

(the input A corresponds to z and the input G corresponds to w). As before the domain of $\mathcal{Q}$ is given by

$$\{(z, w) \in \mathbb{C}^2 : t \leq |z - w| \leq 3t\}$$

and $Dom^*(\mathcal{Q})$ is the interior of $Dom(\mathcal{Q})$. Note that the point $(b, -b)$ belongs to $Dom^*(\mathcal{Q})$ provided that $t < 2|b| < 3t$. The point $(0, 0)$ does not belong to $Dom^*(\mathcal{Q})$, similarly to the previous section we use appropriate translators to resolve this problem:

$$(z + w)/2 = [(z + b) + (z - b)]/2$$

where $t < 2|b| < 3t$. Thus we get a modified linkage $\mathcal{Q}'$ for $(z, w) \mapsto (z + w)/2$ such that $(0, 0) \in Dom^*(\mathcal{Q}')$. To get the linkage $\mathcal{L}_A$ for the addition we combine $\mathcal{Q}'$ and the modified pantograph $\mathcal{P}'_2$ for the multiplication by 2:

$$(z, w) \mapsto (z + w)/2 \mapsto z + w.$$

Then $(0, 0) \in Dom^*(\mathcal{L}_A)$.

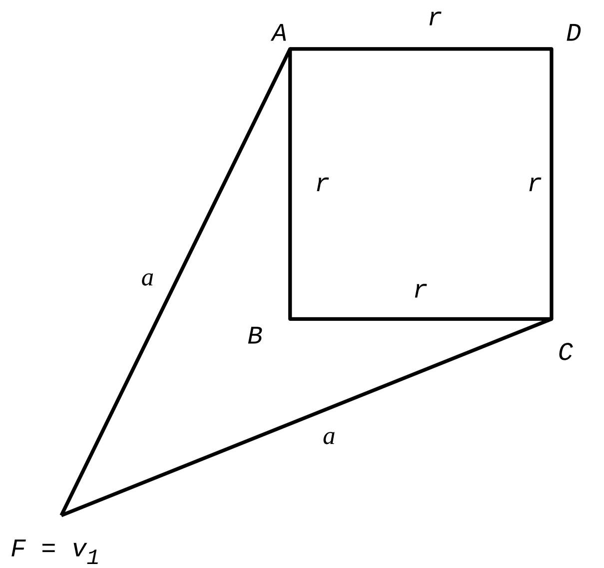

Figure 10: *The Peaucellier inversor.*

6.4. The modified inversor

The most famous functional linkage is the Peaucellier inversor (see [HCV, page 273] and [CR, page 156]) depicted on Figure 10 (with $a^2 - r^2 = t^2$).

The vertex $F = v_1$ is the only *fixed vertex* of the inversor, $Z := (0)$. According to the 19-th century work on linkages, the Peaucellier inversor is supposed to be the functional for the inversion $J_t(z) = t^2/\bar{z}$ with the center at zero and radius t.

Unfortunately this is not true for our definition of functional linkage because of the degenerate realizations ϕ, ψ with $\phi(B) = \phi(D)$ and $\psi(A) = \psi(C)$. Note that there is a 3-torus of degenerate realizations ψ with $\psi(A) = \psi(C)$, so even the dimension of $C(\mathcal{L}, Z)$ is not correct for a functional linkage with $n = m = 1$.

Many of the degenerate realizations can be eliminated by rigidifying the square $ABCD$, but there remains an $\mathbb{S}^1 \times \mathbb{S}^1$ of degenerate realizations with $\psi(A) = \psi(C)$ for which $\psi(B)$ and $\psi(D)$ are not in general related by inversion. We eliminate these by attaching a "hook"[11] to $\{A, C\}$ as on the Figure 11.

Lemma 6.9. *The modified Peaucellier inversor $\mathcal{J}_t$ (with B as input and D as output) is a functional linkage for $J_t(z) = t^2/\bar{z}$. The domain of $\mathcal{J}_t$ is the annulus*

$$\{z \in \mathbb{C} : \rho \leq |z| \leq \rho^{-1}\}$$

where $\rho = \sqrt{a^2 - \epsilon^2} - \sqrt{r^2 - \epsilon^2}$. The only quasiwall of the inversor is the circle of inversion $\{z : |z| = t\}$. $Wall(\mathcal{J}_t)$ is the boundary of the annulus $Dom(\mathcal{J}_t)$.

Proof: If the image of the square $[ABCD]$ under a realization ϕ is nondegenerate then $\phi(D) = J_t(\phi(B))$, see [CR], [HCV]. Notice that $\phi(A) \neq \phi(C)$ for any realization ϕ (the hook!). If $\phi(B) = \phi(D)$ then we still have $\phi(D) = J_t(\phi(B))$ since for such a realization the point $\phi(B)$ bisects the segment $[\phi(A), \phi(C)]$. It is clear from the triangle inequalities that the above annulus equals $Dom(\mathcal{J}_t)$. Suppose that we are given $\phi(B) = z \in \mathbb{C} - \{0\}$. Then the location of $\phi(A), \phi(C)$ is uniquely determined up to reflection σ_1 in the line λ through $0, z$ which interchanges these points. This determines the point $\phi(D) \in \lambda$ as well. The

[11] Notice that by attaching this hook we had created an extra symmetry on the moduli space: the transformation which fixes images of all vertices except $\phi(E)$ and reflects $\phi(E)$ with respect to the line $(\phi(A)\phi(C))$.

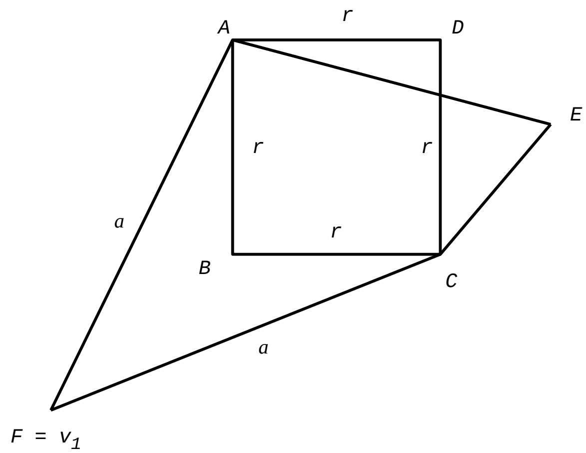

Figure 11: *The modified Peaucellier inversor $\mathcal{J}_t$: the square $ABCD$ is rigidified and $\ell[AE]-\ell[EC]=2\epsilon>0$, $\ell[EC]>r$.*

reflection σ_1 determines an element of order 2 in $Sym(\mathcal{J}_t)$ which we denote by σ_1 as well. However the action of σ_1 on $C(\mathcal{J}_t, Z)$ is *free* since $\phi(A)\neq\phi(C)$ for all realizations ϕ. Once the images of A, B, C, D are determined there remains a single indeterminacy for ϕ: we can reflect $\phi(E)$ via the reflection σ_2 in the line through $\phi(A), \phi(C)$. This reflection determines an element $\sigma_2\in Sym(\mathcal{J}_t)$, the fixed-point set of which consists of realizations for which $|\phi(A)-\phi(C)|$ is minimal, i.e. equal to 2ϵ. The projection of the set of these realizations to $Dom(\mathcal{J}_t)$ forms the boundary of the annulus $Dom(\mathcal{J}_t)$. This identifies the union of walls in $C(\mathcal{J}_t, Z)$. We also conclude that the group of symmetries $Sym(\mathcal{J}_t)$ is $\mathbb{Z}_2\times\mathbb{Z}_2$ and is generated by the above involutions σ_1, σ_2.

It remains to identify the quasiwalls. Suppose that $\xi\in T_\phi C(\mathcal{J}_t, Z)$ is an infinitesimal deformation which vanishes on B, recall that $\phi|v_1=0$ as well and ϕ does not belong to a wall. The above description of the domain of $\mathcal{J}_t$ implies that the triangles

$$\Delta(0,\phi(B),\phi(A)),\quad \Delta(0,\phi(B),\phi(C))$$

are nondegenerate for each realization ϕ as above. Hence $\xi|A=0, \xi|C=0$. If the rhombus

$$\Box(\phi(B),\phi(A),\phi(D),\phi(C))$$

is nondegenerate then Lemma 3.11 implies that $\xi|D=0$. Since ϕ is not on a wall, the triangle

$$\Delta(\phi(E),\phi(C),\phi(A))$$

is nondegenerate and $\xi|E=0$. This proves that the projection of the quasiwall to $Dom(\mathcal{J}_t)$ is the circle

$$\{\phi(D)=\phi(B)\}=\{\phi(B):|\phi(B)|=t\}$$

i.e. the circle of the inversion J_t. $\Box$

Remark 6.10. *Notice that unlike the cases of other linkages, the quasiwall $\{|z|=t\}$ does not move if we alter edge-lengths of the functional linkage $\mathcal{J}_t$ for the given function J_t. On the other hand, by adjusting the parameters $\epsilon, a, r, \ell[AE]$ and keeping t fixed we can get any point $z\in\mathbb{C}^*$ to the interior of $Dom(\mathcal{J}_t)$.*

Notation 6.11. *We shall use the notation $\mathcal{J}$ for $\mathcal{J}_1$.*

6.5. The multiplier

Our construction of the multiplier is quite different from the one that was used by Kempe [Ke1] and other people (see for instance [B]). The idea is to use algebra instead of geometry: if one has addition, subtraction and inversion then one also gets the function $z \mapsto z^2$ via composition as follows. Consider the identity

$$\frac{1}{\bar{z}-0.5}+[-\frac{1}{\bar{z}+0.5}]=\frac{1}{\bar{z}^2-0.25}$$

Hence we can combine the following linkages:

- Three translators for the functions $\tau'_{\pm 0.5}: z \mapsto z \mp 0.5$, $\tau_{0.25}: z \mapsto z+0.25$.
- One pantograph for the germ of the function $w \mapsto -w$ at the point -2.
- Three inversors $\mathcal{J}$ for the function $J_1: z \mapsto 1/\bar{z}$.
- The modified adder $\mathcal{L}_A$ for the germ of addition at $(-2,-2)$.

to get a functional linkage $\mathcal{Q}$ for the function $z \mapsto z^2$.

Lemma 6.12. *Under the following restrictions:*

$$\epsilon < r/2, \quad a+r > \sqrt{3}/2$$

on the parameters a, r, ϵ for the inversor $\mathcal{J}$, the origin belongs to $Dom^*(\mathcal{Q})$.

Proof: First of all we need:

$$\pm 0.5 \in Dom^*(\mathcal{J})$$

None of the points ± 0.5 belong to the circle of inversion, hence if we use a, r, ϵ as above then

$$1 > 0.5 > \rho$$

and $\pm 0.5 \in Dom^*(\mathcal{L})$.

The point $(-2,-2)$ belongs to $Dom^*(\mathcal{L}_A)$ provided that the parameter t is chosen so that $t < 2 < 3t$. Finally, we apply the inversor $\mathcal{J}$ again to compute

$$\frac{1}{\bar{z}^2-0.25} \mapsto z^2-0.25$$

For this operation we need: $-4 \in Dom^*(\mathcal{J})$. Direct computation again shows that $4 < \rho^{-1}$ under the above restrictions on a, r, ϵ, which implies that $-4 \in Dom^*(\mathcal{J})$. □

Thus, we have a functional linkage $\mathcal{Q}$ for the computation of the function $z \mapsto z^2$ so that $0 \in Dom^*(\mathcal{L})$. Then we use the identity

$$zw = [(z+w)^2+(-(z^2+w^2))]/2$$

to construct a functional linkage for the complex multiplication. The linkages which are used for this computation are: the three copies of modified adder, two modified pantographs (for the functions $x \mapsto -x$ and $y \mapsto y/2$) and three copies of the linkage $\mathcal{Q}$ for squaring. On each step of the composition of linkages all we need are functional linkages with the origin in Dom^*, which is true for all the above linkages.

6.6. The straight-line motion linkage

In this section we modify the usual *Peaucellier straight-line motion linkage* (see [CR], [HCV]) to obtain a real functional linkage $\mathcal{S}$ for the inclusion $\mathbb{R} \to \mathbb{C} = \mathbb{R}^2$.

Start with the rigidified inversor $\mathcal{J}_t$ and add the edge $[GD]$ of the length t. The vertices F, G are the *fixed vertices* of the new linkage $\mathcal{S}$. Their images are: $\phi(F) = -\phi(G) = \pm\sqrt{-1}t/2$. Take the vertex B as both the input vertex and the output vertex. See Figure 12.

Remark 6.13. *This choice is somewhat strange from the classical point of view since the linkage $\mathcal{S}$ was invented to transform periodic linear motion of the vertex B to the circular motion of the vertex D (from this point of view B is the input and D is the output). However we do not use the linkage $\mathcal{S}$ to transform linear to circular motion but to restrict motion of the input-vertex B to the real axis.*

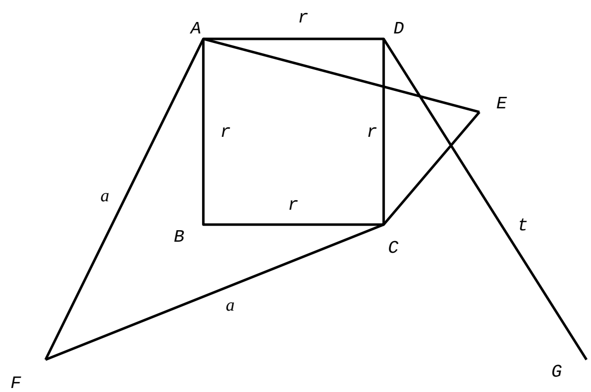

Figure 12: *The Peaucellier straight-line motion linkage $\mathcal{S}$. The vertices F, G are fixed. The vertex B is the input and output.*

The point $\phi(D)$ is now restricted to the circle with the center at $\phi(G)$ and radius t. The input $\phi(B)$ is obtained from $\phi(D)$ by inversion with the center at $\phi(F)$ and radius t. Whence the input $\phi(B)$ moves along a segment in the real axis. Notice that $\mathcal{S}$ has a symmetry that interchanges $\phi(F)$ and $\phi(G)$, and maps $\phi(D)$ to $\overline{\phi(D)}$; this symmetry is induced by the complex conjugation $\mathbb{C} \to \mathbb{C}$.

We use the following restrictions on the side-lengths of the linkage:

$$0 < 2\epsilon = \ell[AE] - \ell[CE] ,$$

$$\ell[CE] > 2r, \quad a > r > \epsilon, \quad 17r > 15a$$

Under these conditions the linkage $\mathcal{S}$ is a real functional linkage for the inclusion map $id : \mathbb{R} \to \mathbb{C} = \mathbb{R}^2$ and the input map $p : C(\mathcal{S}, Z) \to \mathbb{R} \subset \mathbb{R}^2$ has the following property:

$Dom^*(\mathcal{S}, Z)$ contains the open interval $(-\frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2}t)$.

Notice that $\phi(F), \phi(G) \notin \mathbb{R}$. We will need a modification $\tilde{\mathcal{S}}$ of $\mathcal{S}$ where images of all fixed vertices are real numbers. The *based* linkage $\tilde{\mathcal{S}}$ is produced from $\mathcal{S}$ via the construction in Section 5, see Figure 13.

We will also need another modification $\mathcal{S}^m$ of the linkage $\tilde{\mathcal{S}}$. Namely, take m isomorphic copies $\tilde{\mathcal{S}}_j$ of the linkage $\tilde{\mathcal{S}}$. Then take their fiber sum by identifying the vertices with the labels v_1, v_2, F, G for each pair $\tilde{\mathcal{S}}_j$, $\tilde{\mathcal{S}}_i$. We leave it to the reader to verify that $\mathcal{S}^m$ is a real functional linkage for the inclusion map $id : \mathbb{R}^m \to \mathbb{C}^m$ and

$$Dom^*(\mathcal{S}^m) \supset (-\frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2}t)^m.$$

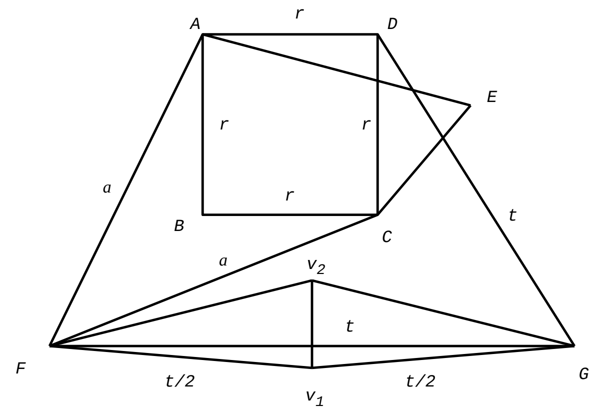

Figure 13: *The based Peaucellier straight-line motion linkage $\tilde{\mathcal{S}}$: $\ell[v_1v_2]^2 + (t/2)^2 = \ell[Fv_2]^2 = \ell[Gv_2]^2$. The image of B under all realizations lies on a segment of the real axis which contains the open interval $(-\frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2}t) \subset Dom^*(\tilde{\mathcal{S}})$.*

The linkage $\mathcal{S}^m$ is used for constructing *real* functional linkages from the *complex* ones.

7. Expansion of domains of functional linkages

Convention 7.1. *In this section we will suppress choices of fixed vertices and their images for functional linkages.*

We apply the results of Section 6.2 to expand domains of functional linkages:

Lemma 7.2. *Suppose that $g(x)$ is a homogeneous polynomial of degree d, $\mathcal{L}$ is a functional linkage which defines the germ $(g, 0)$. Then for any $r > 0$ we can modify $\mathcal{L}$ so that the new linkage $\tilde{\mathcal{L}}$ is functional for the function g and $Dom^*(\tilde{\mathcal{L}})$ contains the disk $B_r(0)$.*

Proof: We consider the case $d > 0$, the case of constant functions ($d = 0$) is left to the reader. By the assumption $Dom^*(\mathcal{L})$ contains a disk $B_\epsilon(0)$ centered at the origin, we can assume $\epsilon < r$. Choose positive $\lambda < \epsilon/r < 1$. Let $\mu := \lambda^{-d} > 1$. We use the formula

$$g(y) = \lambda^{-d} g(\lambda y) = \mu g(\lambda y)$$

to construct a functional linkage $\tilde{\mathcal{L}}$ for the function g as a composition of the following linkages:

- $\mathcal{P}'_\lambda$ (the modified pantograph for the multiplication by λ),
- the linkage $\mathcal{L}$,
- $\mathcal{P}'_\mu$ (the modified pantograph for the multiplication by μ).

The linkages $\mathcal{P}'_\lambda$, $\mathcal{P}'_\mu$ are chosen so that

$$B_r(0) \subset Dom^*(\mathcal{P}'_\lambda), \quad B_R(0) \subset Dom^*(\mathcal{P}'_\mu)$$

where $R := \max\{g(y) : y \in B_\epsilon(0)\}$. Let's check that this choice guarantees that $B_r(0)$ is contained in $Dom^*(\tilde{\mathcal{L}})$. By Theorem 4.3 we have:

$$Dom^*(\tilde{\mathcal{L}}) \supset [Dom^*(\mathcal{P}'_\lambda)] \cap [\lambda^{-1}(Dom^*(\mathcal{L}))] \cap [\lambda^{-1} g^{-1}(Dom^*(\mathcal{P}'_\mu))].$$

Then, since $\lambda^{-1}\epsilon > r$, it is enough to verify that

$$B_r(0) \subset \lambda^{-1}g^{-1}(Dom^*(\mathcal{P}'_\mu)).$$

However $B_R(0) \subset Dom^*(\mathcal{P}'_\mu)$, hence (by the choice of R)

$$B_{\epsilon/\lambda}(0) \subset \lambda^{-1}g^{-1}(Dom^*(\mathcal{P}'_\mu))$$

which together with the inequality $r \leq \lambda^{-1}\epsilon$ implies the assertion. $\square$

As a corollary we get the following Theorem:

Theorem 7.3. *(Theorem on expansion of domain.) Suppose that $f : \mathbf{k}^m \to \mathbf{k}^n$ be a polynomial morphism, $\mathcal{L}$ is a functional linkage which defines the germ $(f, 0)$. Then for any $r > 0$ we can modify $\mathcal{L}$ so that the new linkage $\tilde{\mathcal{L}}$ is functional for the morphism f and $Dom^*(\tilde{\mathcal{L}})$ contains the disk $B_r(0)$.*

Proof: We will consider the case when $n = 1$, the general case follows from Theorem 4.5. Write $f(x)$ as

$$f(x) = \sum_{j \leq d} f_j(x)$$

where each f_j is a homogeneous polynomial of degree j. Let $g(y) := y_1 + ... + y_d$. Hence we can represent f as a composition of homogeneous polynomials $f_j, j \leq d$, and g. Now the assertion follows from the previous lemma and Theorems 4.3, 4.5. $\square$

8. Realization of complex polynomial maps by functional linkages

In this section we prove Theorem A (the complex case). We first consider the case $f : \mathbb{C}^m \to \mathbb{C}$, i.e. $n = 1$. Let

$$f(x) = a_0 + \sum_j a_j g_j(x)$$

where $g_j = x_1^{\alpha_1} ... x_m^{\alpha_m}$ are monomials of positive degrees and $a_j \in \mathbb{C}$ are constants ($j = 0, 1, ..., N$). Let $y = (y_0, ..., y_N)$. Consider the function

$$\hat{f}(x, y) = y_0 + \sum_j y_j g_j(x).$$

This function is obtained via composition of the multiplication and addition operations. Hence we use the elementary linkages for the addition and multiplication we get a complex functional linkage $\hat{\mathcal{L}}$ for the germ $(\hat{f}, 0)$. Then we use Theorem 7.3 (on expansion of domain): for each given $\rho > 0$ we can modify $\hat{\mathcal{L}}$ to $\tilde{\mathcal{L}}$ so that $\tilde{\mathcal{L}}$ is functional for the pair $(\hat{f}, B_\rho(0))$, $0 \in \mathbb{C}^{m+N}$. We use ρ so large that $B_\rho(0)$ contains the disk

$$\{(x, y) : x \in B_r(\mathcal{O}), y_j = a_j, j = 0, ..., N\}.$$

We represent f as a composition of the function $\hat{f}$ and the constant function

$$a : (y_0, ..., y_N) \mapsto (a_0, ..., a_N).$$

The constant function is defined by a functional linkage as follows:

Let $\mathcal{A}$ be the graph which consists of the set of vertices $[In(\mathcal{A}) = (P_1, ..., P_m)] \cup [Out(\mathcal{A}) = (Q_1, ..., Q_N)]$, no edges, $W = Out(\mathcal{A})$ and $Z = (a_0, ..., a_N)$.

Clearly $Dom^*(\mathcal{A}) = \mathbb{C}^m$. Thus the 1-st and 2-nd functionality theorems 4.3, 4.5 imply that composition of the linkages $\tilde{\mathcal{L}}$ and $\mathcal{A}$ gives us a functional linkage for the pair $(f, B_r(\mathcal{O}))$. This proves Theorem A in the complex case when $n = 1$.

To get functional linkages for polynomial vector-functions we use repeatedly the 2-nd functionality theorem 4.5:

If we have a functional linkage $\mathcal{L}_1$ for the germ $(f_1(x_1, ..., x_n), A_1)$ and a functional linkage $\mathcal{L}_2$ for $(f_2(x_1, ..., x_n), A_2)$ we glue inputs of $\mathcal{L}_1$ and $\mathcal{L}_2$ to construct a functional linkage $\mathcal{L}$ for the germ $((f_1, f_2), (A_1, A_2))$.

Thus we proved

Theorem 8.1. *Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map, $\mathcal{O} \in \mathbb{C}^m$ and $r > 0$. Then there is a marked functional linkage $\mathcal{L} = (L, \ell, W)$ together with a vector $Z \in \mathbb{C}^s$ so that:*

The ball $B_r(\mathcal{O})$ is contained in $Dom^*(\mathcal{L}, Z)$, $q \circ p^{-1} : Dom(\mathcal{L}, Z) \to \mathbb{C}^m$ equals the restriction of the vector-function f, i.e. $(\mathcal{L}, Z)$ defines $(f, B_r(\mathcal{O}))$.

There is a special case when f has real coefficients and $Z \in \mathbb{R}^s$. Recall that we use only real numbers b for the translators in $\mathcal{L}$. We apply the construction described in the Section 5 and modify $\mathcal{L}$ to a based linkage $\tilde{\mathcal{L}}$. According to Lemma 5.1 we get an algebraic isomorphism

$$\tau : \mathcal{M}(\tilde{\mathcal{L}}) \to C(\mathcal{L}, Z).$$

Hence the based linkage $\tilde{\mathcal{L}}$ also defines the pair $(f, B_r(\mathcal{O}))$. This proves Theorem A of the Introduction (in the complex case).

9. Transition from complex to real functional linkages

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial function, $\mathcal{O} \in \mathbb{R}^m$ be a point and $r > 0$. Our goal is to produce a real functional linkage for the pair $(f, B_r(\mathcal{O}))$.

We extend f by complexification to a morphism $f^c : \mathbb{C}^m \to \mathbb{C}^n$ and construct a complex functional linkage $\mathcal{L}'$ for $(f^c, B_r^c(\mathcal{O}))$, where $B_r^c(\mathcal{O})$ is the ball of radius r in $\mathbb{C}^m$ centered at $\mathcal{O}$.

Take the real functional linkage $\mathcal{S}^m$ for the identity map $id : \mathbb{R}^m \to \mathbb{R}^m$, see §6.6. We choose the parameter t in $\mathcal{S}^m$ so large that

$$B_r(\mathcal{O}) \subset (-\frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2}t)^m \subset Dom^*(\mathcal{S}^m).$$

Next we alter $\mathcal{L}'$ via fiber product with the linkage $\mathcal{S}^m$. Namely, take the bijection $\beta : In(\mathcal{L}') \to In(\mathcal{S}^m) = Out(\mathcal{S}^m)$ which maps each input vertex P_j' of $\mathcal{L}'$ to the input vertex P_j'' of $\mathcal{S}^m$. Let $\mathcal{L} := \mathcal{L}' *_\beta \mathcal{S}^m$. Then the 1-st functionality theorem 4.3 implies that $\mathcal{L}$ is a real-functional linkage for the polynomial f and $Dom^*(\mathcal{L})$ contains the disk $B_r(\mathcal{O}) \subset \mathbb{R}^m$.

Notice that for each fixed vertex $v \in W$ of $\mathcal{S}^m$ we have: $z = \phi(v) \in \mathbb{R}$, $\phi \in C(\mathcal{S}^m, Z)$. The same is true for $\mathcal{L}'$ since the polynomial f has only real coefficients and we use only real numbers b for the translators. Then (as in as in the Section 8) Lemma 5.1 implies that we can modify $\mathcal{L}$ to a based linkage $\tilde{\mathcal{L}}$ so that

$$\tau : \mathcal{M}(\tilde{\mathcal{L}}) \to C(\mathcal{L}, Z)$$

is an algebraic isomorphism. This concludes the proof of Theorem A in the real case. $\square$

10. Realization of algebraic sets and smooth manifolds as moduli spaces of planar linkages

In this section we derive Theorem B from Theorem A. Let X be a compact affine algebraic subset of $\mathbb{R}^m$, choose a polynomial $f : \mathbb{R}^m \to \mathbb{R}$ such that $X = f^{-1}(0)$. We may assume $M \subset B_r(0)$. By Theorem A (the real case) we have a *based* functional linkage $\mathcal{L}$ for the pair $(f, B_r(0))$. The output mapping p of $\mathcal{L}$ is an analytically trivial[12] polynomial covering over $B_r(0)$. Now glue the output vertices of $\mathcal{L}$ to the basic vertex v_1 to obtain a linkage $\mathcal{L}_0$. Let p_0 be the output mapping of $\mathcal{L}_0$. The images of the input vertices of $\mathcal{L}_0$ under $\phi \in \mathcal{M}(\mathcal{L}_0)$ are now constrained to $f^{-1}(0)$. Thus the mapping $\mathfrak{p}_0 : \mathfrak{M}(\mathcal{L}_0) \to \mathfrak{X}_{can}$ is a local analytic isomorphism (by the 3-rd functionality theorem, Theorem 4.6). Hence by Lemma 2.9 the mapping $p_0 : \mathcal{M}(\mathcal{L}_0) \to X$ is also a local analytic isomorphism (now $\mathcal{M}(\mathcal{L}_0)$ has the reduced analytic structure). Hence p_0 is an analytically trivial polynomial covering over X. Theorem B follows. $\square$

Proof of Theorem B' is similar and is left to the reader.

To prove Corollary C we use Theorem 2.20 and Theorem B to get a linkage $\mathcal{L}_0$ and an analytically trivial covering $p : \mathcal{M}(\mathcal{L}_0) \to M$ where we identify M with a real algebraic subset of $\mathbb{R}^m$. $\square$

11. How to draw algebraic curves

In this section we prove Theorem 11.2 according which one can "draw" arbitrary algebraic curves in $\mathbb{R}^2$ using planar mechanical linkages. For instance, if Γ is a compact connected algebraic curve in $\mathbb{R}^2$ then there is a closed complex functional linkage $\mathcal{L}_0$ with a single input vertex P so that as realizations ψ of $\mathcal{L}_0$ vary along an arbitrary connected component C of $\mathcal{M}(\mathcal{L}_0)$, the vertex P traces the curve Γ and the projection $C \to \Gamma$ is an analytic isomorphism.

We first need a functional linkage for the complex conjugation. There are several ways to do it.

The 1-st construction. Using Theorem B' construct a closed functional linkage $\mathcal{L}_0$ for the germ of the complex algebraic set $zw = 1$ at the point $(2, 1/2) \in \mathbb{C}^2$. Let P_1, P_2 be the input vertices of $\mathcal{L}_0$. Then let $\mathcal{L}^0$ be the linkage $\mathcal{L}_0$ where we declare P_1 the sole input vertex and $Q_1 := P_2$ the output vertex. Then $\mathcal{L}^0$ is a complex functional linkage for the germ of the function $z \mapsto z^{-1}$ at the point 2. Recall that we have the complex functional linkage $\mathcal{J}$ for the germ of the inversion

$$w \mapsto 1/\bar{w}$$

at the point $1/2$. Hence we compose $z \mapsto z^{-1} \mapsto \bar{z}$ and compose the linkages $\mathcal{J}$, $\mathcal{L}^0$ to get a $\mathbb{C}$-functional linkage $\bar{\mathcal{L}}$ for the germ of the map $z \mapsto \bar{z}$ at the point 2. Finally, we use the formula

$$\bar{z} = \overline{z+2} - 2$$

and composition of $\bar{\mathcal{L}}$ with two translators to get a $\mathbb{C}$-functional linkage $\mathcal{L}'$ for the germ $(z \mapsto \bar{z}, 0)$.

The 2-nd construction. We start with the linkage $\mathcal{B}$ described on Figure 14: an abstract (rigidified) square with a "hook" attached. We let

$$\ell[AP] = \sqrt{2}, \quad |\ell[AC] - \ell[BC]| > 2$$

[12] In the scheme-theoretic sense.

Let $\mathcal{S}^2$ be the linkage from the section 6.6 where $t = 1$, let P_1, P_2 be the input vertices of $\mathcal{S}^2$ (they are the output vertices as well). Take $\beta : A \mapsto P_1, B \mapsto P_2$ and $\mathcal{L} := \mathcal{B} *_\beta \mathcal{S}^2$. We declare $P \in \mathcal{B}$ the input and $Q \in \mathcal{B}$ the output of the linkage $\mathcal{L}$. We leave it to the reader to verify that the linkage $\mathcal{L}$ is a $\mathbb{C}$-functional linkage for the germ of $z \mapsto \bar{z}$ at the point $\sqrt{-1}$.

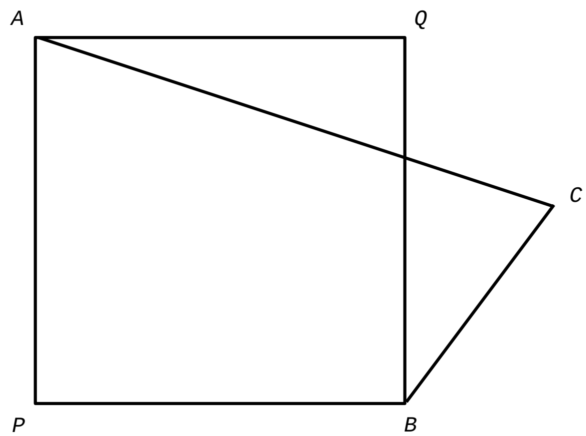

Figure 14: *Construction of a functional linkage for the complex conjugation.*

Remark 11.1. *Under all realizations ϕ of $\mathcal{L}$ we have: $\phi(A), \phi(B) \in \mathbb{R}$, $\phi(A) \neq \phi(B)$ and $\phi(Q) = \overline{\phi(Q)}$.*

Finally, we use the formula

$$\bar{z} = \overline{z + i} - i$$

and composition of $\mathcal{L}$ with two translators to get a $\mathbb{C}$-functional linkage $\mathcal{L}''$ for the germ $(z \mapsto \bar{z}, 0)$.

Theorem 11.2. *Let $f = f(z, \bar{z}), f : \mathbb{C} \to \mathbb{R}$ be a polynomial function of the variables $z, \bar{z}$ and $\Gamma := f^{-1}(0) \subset \mathbb{C}$ be a real-algebraic curve. Pick an open (in the classical topology) bounded subset $U \subset \Gamma$. Then there is a closed $\mathbb{C}$-functional linkage $\mathcal{L}_0$ so that the input map $p_0 : C(\mathcal{L}_0, Z) \to \mathbb{C}$ is an analytically trivial polynomial covering over U.*

Proof: Let $U \subset B_r(\mathcal{O})$. Our argument is exactly the same as in the proof of Theorem B. Namely, as in Theorem A we first construct a functional linkage $\mathcal{L}$ for the pair $(f, B_r(\mathcal{O}))$ (now we use the composition of addition, multiplication and the complex conjugation). Then we attach the output vertex Q of $\mathcal{L}$ to the distinguished vertex v_1 (such that $\phi(v_1) = 0$ for all relative realizations). Let $\mathcal{L}_0$ be the resulting closed functional linkage and p_0 be its input map. Then as in the proof of Theorem B we have: p_0 is an analytically trivial polynomial covering over U. $\square$

12. Universality theorem for arrangements in $\mathbb{P}^2$

In the section we review notions of configuration spaces for arrangements and universality theorems proven in [KM2], which extend earlier results of N. Mnev [Mn].

Let $\mathcal{A}$ be an *abstract arrangement*, i.e. a bipartite graph with the parts $\mathcal{P}$ and $\mathcal{L}$. We say that a "point" $P \in \mathcal{P}$ is incident to a "line" $L \in \mathcal{L}$ if P and L are connected by an edge. A projective realization ϕ of $\mathcal{A}$ is a map

$$\phi : \mathcal{P} \cup \mathcal{L} \to \mathbb{P}^2 \cup (\mathbb{P}^2)^\vee, \;\; \phi(\mathcal{P}) \subset \mathbb{P}^2, \;\; \phi(\mathcal{L}) \subset (\mathbb{P}^2)^\vee$$

such that if P and L are incident then $\phi(P) \in \phi(L)$. This condition defines a projective scheme $R(\mathcal{A})$ over $\mathbb{Z}$. We let $R(\mathcal{A}, \mathbb{CP}^2)$ and $R(\mathcal{A}, \mathbb{RP}^2)$ denote the sets of complex and real points of $R(\mathcal{A})$.

Here and below we use the symbol $\vee$ for polarity between points and lines in the projective plane, this polarity is determined by the standard bilinear form on $\mathbb{R}^3$ given by $\|(x, y, z)\|^2 = x^2 + y^2 + z^2$.

We now want to pass to the quotient of $R(\mathcal{A})$ by PGL_3. We do this by restricting to realizations in a "general position" and then taking a cross-section. To make it precise we first define based arrangements.

Definition 12.1. *The* **standard triangle** *is the abstract arrangement T consisting of 6 point-vertices and 6 line-vertices that corresponds to a triangle with its medians, see Figure 15.*

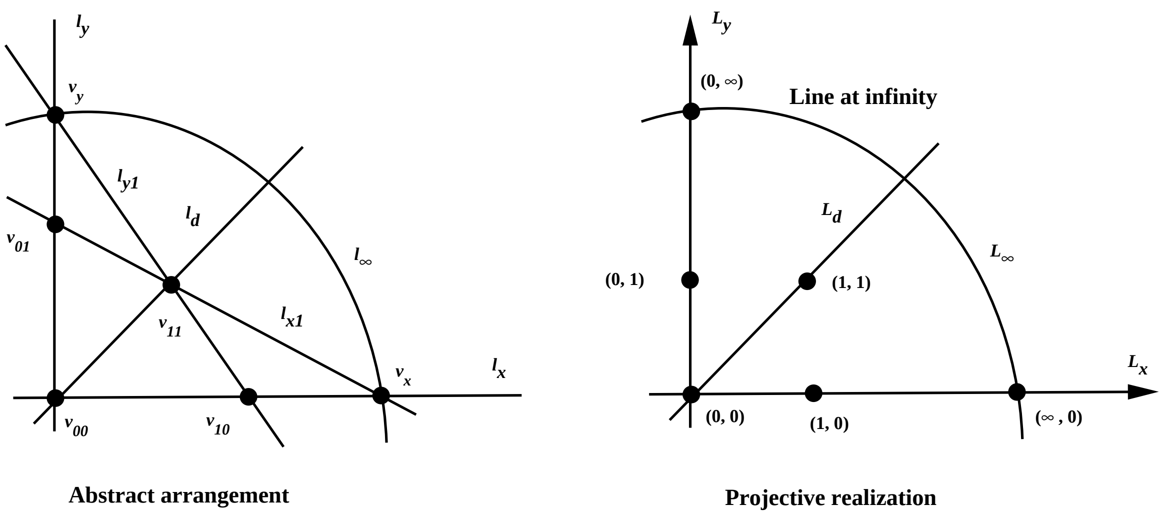

Figure 15: *The standard triangle T and its standard realization.*

Definition 12.2. *The* **standard realization** *ϕ_T of the standard triangle T is determined by:*

$$\phi_T(v_{00}) = (0, 0), \phi_T(v_x) = (\infty, 0), \phi_T(v_y) = (0, \infty), \phi_T(v_{11}) = (1, 1).$$

Here $(0, 0), (\infty, 0), (0, \infty), (1, 1)$ are points in the affine plane $\mathbb{A}^2 \subset \mathbb{P}^2$ which have the homogeneous coordinates: $(0 : 0 : 1), (1 : 0 : 0), (0 : 1 : 0), (1 : 1 : 1)$ respectively.

We say that an abstract arrangement $\mathcal{A}$ is *based* if it comes equipped with an embedding $i : T \to \mathcal{A}$. Let $(\mathcal{A}, i)$ be a based arrangement. We say that a projective realization ϕ of $\mathcal{A}$ is *based* if $\phi \circ i = \phi_T$. Let $BR(\mathcal{A}, \mathbb{P}^2(\mathbf{k}))$ be the subset of $R(\mathcal{A}, \mathbb{P}^2(\mathbf{k}))$ consisting of based realizations, $\mathbf{k} = \mathbb{R}, \mathbb{C}$.

Lemma 12.3. *(See [KM2, Theorem 8.20].) $BR(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ is the set of complex points of a projective scheme over $\mathbb{Z}$ which is a scheme-theoretic quotient of $R(\mathcal{A})$ by the action of PGL_3.*

Definition 12.4. *A* **functional arrangement** *is a based arrangement $(\mathcal{A}, i)$ with two subsets of marked point-vertices $\mu = (P_1, ..., P_m)$ (the* **input-vertices***) and point-vertices $\nu = (Q_1, ..., Q_n)$ (the* **output-vertices***) such that all the marked vertices are incident to the line-vertex $l_x \in i(T)$ (which corresponds to the x-axis) and such that the following two axioms are satisfied:*

Let $BR_0(\mathcal{A}_\mu) \subset BR(\mathcal{A})$ denote the open subset which consists of realizations ϕ such that $\phi(P_j) \in \mathbb{A}^2$ for all j, we define $BR_0(\mathcal{A}_\nu)$ similarly. Then we require:

(1) $BR_0(\mathcal{A}_\mu) \subset BR_0(\mathcal{A}_\nu)$.

(2) The projection $p : BR_0(\mathcal{A}_\mu) \to \mathbb{A}^m$ given by $p(\phi) = (\phi(P_1), ..., \phi(P_m))$ is an isomorphism of schemes over $\mathbb{Z}$.

Each functional arrangement determines a morphism $f : \mathbb{A}^m \to \mathbb{A}^n$ (which is defined over $\mathbb{Z}$) by the formula:

$$f(x) = q \circ p^{-1}(x)$$

where $q(\phi) = (\phi(Q_1), ..., \phi(Q_n))$.

Theorem 12.5. *(See [KM2, Lemma 9.7].) Let $f : \mathbb{A}^m \to \mathbb{A}^n$ be any polynomial mapping with integer coefficients. Then there is a functional arrangement $\mathcal{A}$ which determines f.*

Let $S \subset \mathbb{A}^m$ be a closed subscheme defined over $\mathbb{Z}$, $S = f^{-1}(0)$ for some morphism $f : \mathbb{A}^m \to \mathbb{A}^n$. Let $\mathcal{A}$ be a functional arrangement which determines f as in the above theorem. By gluing the output vertices of $\mathcal{A}$ to v_{00} we obtain an arrangement $\mathcal{A}^0$ containing distinguished vertices $P_1, ..., P_m$. Again define $BR_0(\mathcal{A}^0)$ by requiring $\phi(P_i)$ to be finite. We get an induced morphism (easily seen to be an embedding) $p : BR_0(\mathcal{A}^0) \to \mathbb{A}^m$. We then have

Theorem 12.6. *(Theorem 1.3 of [KM2]) Let S be a closed subscheme of $\mathbb{A}^m$ (again over $\mathbb{Z}$). Then there exists a based marked arrangement $\mathcal{A}$ such that the input mapping $p : BR_0(\mathcal{A}) \to \mathbb{A}^m$ induces an isomorphism of schemes $BR_0(\mathcal{A}) \to S$.*

We will use the following version of the above theorem:

Theorem 12.7. *Let X be a compact real algebraic set defined over $\mathbb{Z}$. Then there exists a based arrangement $\mathcal{A}$ such that X is entire birationally isomorphic to a Zariski open and closed subset C in $BR(\mathcal{A}, \mathbb{P}^2(\mathbb{R}))$.*

Proof: Using Theorem 2.19 we may assume that X is projectively closed. We choose the projective scheme $\mathfrak{X} \subset \mathbb{P}^m$ whose set of real points is X so that the corresponding affine scheme $\mathfrak{X}_a \subset \mathbb{A}^{m+1}$ corresponds to a real reduced ideal. Thus X is Zariski dense in $\mathfrak{X}(\mathbb{C})$.

Define the affine scheme $\mathfrak{X}_A = \mathfrak{X} \cap \mathbb{A}^m$. Now apply Theorem 12.6 to construct a based marked arrangement $\mathcal{A}$ so that $BR_0(\mathcal{A})$ is isomorphic to $\mathfrak{X}_A$ (as a scheme), hence the sets of real points of these schemes are isomorphic as well. Thus X is (polynomially) isomorphic to $BR_0(\mathcal{A}, P^2(\mathbb{R}))$ which is Zariski open. It remains to show that $BR_0(\mathcal{A}, P^2(\mathbb{R}))$ is also Zariski closed.

Recall that $BR(\mathcal{A})$ embeds canonically in a product $(\mathbb{P}^2)^N \times (\mathbb{P}^1)^m$ where the last m factors correspond to the input vertices. The morphism $p : BR_0(\mathcal{A}) \to \mathbb{A}^m$ is the restriction of the projection on the last m factors:

$$(\mathbb{P}^2)^N \times (\mathbb{P}^1)^m \to (\mathbb{P}^1)^m$$

The subset $BR_0(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ is constructible, hence its closure with respect to the classical topology is the same as its closure $\overline{BR_0}(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ with respect to the Zariski topology in $(\mathbb{P}^2(\mathbb{C}))^N \times (\mathbb{P}^1(\mathbb{C}))^m$.
Suppose that there is a real point $z \in \overline{BR_0}(\mathcal{A}, \mathbb{P}^2)$ that does not belong to $BR_0(\mathcal{A}, \mathbb{P}^2(\mathbb{R}))$. Then z is the limit of a sequence $z_j \in BR_0(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$. However $p(z_j) \in \mathbb{C}^m$ are obtained by "forgetting" all but the last m coordinates of z_j, hence $p(z_j)$ will converge to a real point x of $\mathfrak{X}$. It is clear that $x \notin \mathbb{R}^m$ and hence does not belong to X. This contradicts the fact that X is projectively closed. $\square$

Remark 12.8. *In general $BR(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ is different from $\overline{BR_0}(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$. As an example consider the linkage $\mathcal{A}$ corresponding (via the construction in [KM2]) to the system of equations:*

$$x + y = 0, \quad x + y = 1, \quad x = y$$

in $\mathbb{A}^2$. The set solutions of this system of equations is empty (even in the projective compactification of $\mathbb{A}^2$). Thus $BR_0(\mathcal{A}, \mathbb{P}^2(\mathbb{C})) = \emptyset$, on the other hand: $BR(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ is a single point.

Now we construct metric graphs corresponding to based abstract arrangements. Suppose that $\mathcal{A}$ is a based arrangement. We start by identifying the point-vertex v_{00} with the line-vertex l_∞, the point-vertex v_x with the line-vertex l_y and the point-vertex v_y with the line-vertex l_x in the standard triangle T. We also introduce the new edges

$$[v_{10}v_{00}], \quad [v_{01}v_{00}], \quad [v_{10}v_x], \quad [v_{01}v_y]$$

(Here $v_{10}, v_{00}, v_{11}, v_{01}, \ldots$ are the *point*-vertices in the standard triangle T.) We will use the notation L for the resulting graph. We construct a length-function ℓ on the set of edges $e \subset L$ as follows:

1) We assign the length $\pi/4$ to the new edges.
2) We assign the length $\pi/2$ to the rest of the edges.

13. The relation between the two universality theorems

The goal of this section is to establish a relation between the two universality theorems for realizability of real algebraic sets (Theorems B and 12.7). Consider an abstract based arrangement $\mathcal{A}$. We choose $v_{00}, v_x, v_y, v_{01}, v_{10}$ as distinguished vertices of the corresponding metric graph L. Let $\mathcal{L}$ denote the metric graph L with the distinguished set of vertices as above. Let X be either $\mathbb{S}^2$ or $\mathbb{RP}^2$ with the standard metric d (so that the standard projection $\mathbb{S}^2 \to \mathbb{RP}^2$ is a local isometry). Define the *configuration space* $C(\mathcal{L}, X)$ of realizations of $\mathcal{L}$ in X to be the collection of mappings ψ from the vertex-set $\mathcal{V}(\mathcal{L})$ of $\mathcal{L}$ to X such that

$$d(\psi(v), \psi(w))^2 = (\ell[vw])^2$$

for all vertices v, w of $\mathcal{L}$ connected by an edge.

Remark 13.1. *Notice that if $a, b \in \mathbb{RP}^2$ are within the distance $\pi/2$ then there are two minimal geodesics connecting a to b. This is the reason to define $C(\mathcal{L}, X)$ as the set of maps from $\mathcal{V}(\mathcal{L})$ rather than from $\mathcal{L}$ itself.*

One can easily see that $C(\mathcal{L}, X)$ has natural structure of a real algebraic set. The subsets

$$\mathcal{M}(\mathcal{L}, \mathbb{RP}^2) := \{\psi \in C(\mathcal{L}, \mathbb{RP}^2) : \psi(v_{00}) = (0,0), \psi(v_x) = (\infty, 0),$$
$$\psi(v_{10}) = (1,0), \psi(v_{01}) = (0,1)\}$$
$$\mathcal{M}(\mathcal{L}, \mathbb{S}^2) := \{\psi \in C(\mathcal{L}, \mathbb{S}^2) : \psi(v_{00}) = (0,0,1), \psi(v_y) = (0,1,0),$$
$$\psi(v_x) = (1,0,0), \psi(v_{10}) = (1,0,1), \psi(v_{01}) = (0,1,1)\}$$

form cross-sections to the actions of the groups of isometries $PO(3, \mathbb{R}), O(3, \mathbb{R})$ of X on $C(\mathcal{L}, X)$. We call $\mathcal{M}(\mathcal{L}, X)$, the *moduli spaces* of realizations of $\mathcal{L}$ in X (where $X = \mathbb{S}^2, \mathbb{RP}^2$).

Remark 13.2. *Now it is convenient to use the full group of isometries of $\mathbb{S}^2$ instead of the group of orientation-preserving isometries that we used for planar linkages.*

Lemma 13.3. *The moduli space $\mathcal{M}(\mathcal{L}, \mathbb{RP}^2)$ is (polynomially) isomorphic to the real algebraic set $BR(\mathcal{A}, \mathbb{RP}^2)$.*

Proof: The key to the proof is the fact that a point $P \in \mathbb{RP}^2$ is incident to a line $L \in (\mathbb{RP}^2)^\vee$ iff

$$d(P, L^\vee) = \pi/2.$$

Thus we construct a morphism

$$\mu : BR(\mathcal{A}_0, \mathbb{RP}^2) \to \mathcal{M}(\mathcal{L}, \mathbb{RP}^2), \quad \mu : \phi \mapsto \psi$$

so that for each point-vertex $P \in \mathcal{A}$ we have $\psi(P) = \phi(P)$ and for each line-vertex $L \in \mathcal{A}$ we have $\psi(L) = \phi(L)^\vee$. This morphism has algebraic inverse given by the same formula (since $(L^\vee)^\vee = L$). □

Let $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$ be the image of $BR_0(\mathcal{A}_0, \mathbb{RP}^2)$ under the isomorphism μ. Consider the standard 2-fold covering $\mathbb{S}^2 \to \mathbb{RP}^2$. It induces a (locally trivial) analytical covering

$$\alpha : \mathcal{M}(\mathcal{L}, \mathbb{S}^2) \to \mathcal{M}(\mathcal{L}, \mathbb{RP}^2).$$

The group of automorphisms of α is $(\mathbb{Z}_2)^r$, where r is the number of (point) vertices in $[L - \mathcal{P}(T)] \cup \{v_{11}\}$. The generators of this group are indexed by the vertices $v \in [L - \mathcal{P}(T)] \cup \{v_{11}\}$:

$$g_v : \psi(v) \mapsto -\psi(v), g_v : \psi(w) \mapsto \psi(w), w \neq v.$$

Proposition 13.4. *For each arrangement $\mathcal{A}$ as in Theorem 12.7, the covering α is analytically trivial over $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$.*

Proof: The proposition will follow from the following:

For each point-vertex v in $\mathcal{L}$ there is a line λ in $\mathbb{RP}^2$ and for each line-vertex $v \in \mathcal{L}$ there is a line λ' in $(\mathbb{RP}^2)^\vee$ so that:

$\phi(v) \notin \lambda$ for all $\phi \in BR_0(\mathcal{A}, \mathbb{RP}^2)$ (if v is a point-vertex) and $\phi(v) \notin \lambda'$ for all $\phi \in BR_0(\mathcal{A}, \mathbb{RP}^2)$ (if v is a line-vertex).

To prove this property recall (see [KM2]) that $\mathcal{A}$ is obtained from "elementary" arrangements for the addition and multiplication via fiber sums. Thus it is enough to verify the above property for the arrangements C_A, C_M for the addition and multiplication that are described in [KM2]. The verification is straightforward and is left to the reader. □

Now we identify the moduli space of spherical linkages $\mathcal{M}(\mathcal{L}, \mathbb{S}^2)$ with a moduli space of Euclidean linkages in $\mathbb{R}^3$ as follows:

Add an extra vertex v_0 to the graph $\mathcal{L}$ and connect it to each vertex of $\mathcal{L}$ by edge of the unit length. Modify the other side-lengths as follows:

$$\ell'(e) := \sqrt{2 - 2\cos(\ell(e))}, \quad e \in \mathcal{E}(\mathcal{L}).$$

Let $\mathcal{L}'$ be the resulting metric graph with the distinguished set of vertices $[\mathcal{P}(T) - \{v_{11}\}] \cup \{v_0\}$. Define the configuration space

$$C(\mathcal{L}', \mathbb{R}^3) := \{\psi : \mathcal{V}(\mathcal{L}') \to \mathbb{R}^3 : |\psi(v) - \psi(w)|^2 = \ell'[vw]^2\}.$$

Again is is clear that

$$\mathcal{M}(\mathcal{L}', \mathbb{R}^3) := \{\psi \in C(\mathcal{L}', \mathbb{R}^3) : \psi(v_0) = (0, 0, 0),$$

and the same normalization on $\mathcal{P}(T) - \{v_{11}\}$ as we used for $\mathcal{M}(\mathcal{L}, \mathbb{S}^2)\}$

is a real-algebraic set which is a cross-section for the action of $Isom(\mathbb{R}^3)$ on $C(\mathcal{L}', \mathbb{R}^3)$. Obviously we have an isomorphism

$$\mathcal{M}(\mathcal{L}, \mathbb{S}^2) \cong \mathcal{M}(\mathcal{L}', \mathbb{R}^3)$$

of real-algebraic sets. We let $\mathcal{M}_0(\mathcal{L}', \mathbb{R}^3)$ be the subset of $\mathcal{M}(\mathcal{L}', \mathbb{R}^3)$ corresponding to $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$ under the isomorphism

$$\mathcal{M}(\mathcal{L}, \mathbb{RP}^2) \cong \mathcal{M}(\mathcal{L}, \mathbb{S}^2) \cong \mathcal{M}(\mathcal{L}', \mathbb{R}^3).$$

Thus, as a corollary of Theorem 12.7 we obtain the following:

Theorem 13.5. *Let S be a compact real algebraic set defined over $\mathbb{Z}$. Then there exist abstract linkages $\mathcal{L}, \mathcal{L}'$ and Zariski open and closed subsets $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2) \subset \mathcal{M}(\mathcal{L}, \mathbb{RP}^2)$, $\mathcal{M}_0(\mathcal{L}', \mathbb{R}^3) \subset \mathcal{M}(\mathcal{L}', \mathbb{R}^3)$, so that:*

(1) $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$ is entire birationally isomorphic to S.

(2) $\mathcal{M}_0(\mathcal{L}', \mathbb{R}^3)$ is an analytically trivial entire rational covering of S.

14. A brief history of "Kempe's theorem"

This story began with the invention of the steam engine by Newcomen in 1722. One problem that appeared naturally was to transform a periodic linear motion (of the "input" vertex) to a circular motion (of the "output" vertex). The "parallelogram" invented by Watt in the late 18-th century gave an approximate solution to this problem. The "input" motion was not exactly linear, however the input vertex traces a curve with a point of zero curvature, hence the output approximates a straight line up to the 2-nd order. After discovery in the first half of the 19-th century of several "unsolvable" geometric problems (like squaring a circle, etc.), for a while it was a common opinion that the problem of transforming linear to circular motion also has no exact solution. This opinion was shared for instance by Chebyshev who after thinking about this problem introduced *Chebyshev polynomials*, partial motivation for which was the optimal approximate solution of the problem.

This was the situation until 1864 when French navy officer Peaucellier published a letter [Pe1] where he claimed a positive solution, without giving any details[13].

There are several opinions on what happened next (this caused a serious controversy between Russian and French-British mathematical schools in the late 19-th century). In 1871 Lippman Lipkin[14] published the first detailed solution [L]. Two years later (in 1873) Peaucellier published a paper [Pe2] which also contained a detailed solution (the *Peaucellier inversor*) identical to Lipkin's. Immediately after that several other ways to "draw a straight line" were discovered [Ha], [Ke2]. As far as applications are concerned it turned out that all the mechanisms that transform linear motion to circular are too complicated to be used instead of Watt's parallelogram, invention of efficient lubricants had closed the problem. The only practical application of the inversor we are aware of was in air engines which ventilated the British parliament in 1870-1880-s (see [W, Page 182]).

The rest of the story is mostly pure mathematics. In 1875 A. B. Kempe published [Ke1] where (in the present terminology) he outlined a proof of the following theorem analogous to Theorem 11.2:

[13] It seems that in 1860-s Peaucellier explained his solution to some other people, cf. [Ma], so his letter [Pe1] was probably not a hoax. However [Ma] contains only the title so we cannot be sure if Mannheim really knew the construction of the inversor.

[14] That time Lipkin was a graduate student of Chebyshev. Lipkin had died in 1875 at the age of 25 from the smallpox.

Theorem 14.1. *Suppose that $S \subset \mathbb{R}^2$ is an algebraic curve, $\mathcal{O} \in S$. Then there exists an abstract closed $\mathbb{C}$-functional linkage $\mathcal{L}$, a Zariski closed algebraic subset $C \subset \mathcal{M}(\mathcal{L})$ (which is a union of irreducible components) and a closed* [15] *neighborhood U of $\mathcal{O}$ in S so that the restriction of the input map p to C is onto U.*

Remark 14.2. *However, if one follows Kempe's arguments, C is not open in $\mathcal{M}(\mathcal{L})$, $U \neq S$ (even if S is compact) and the mapping $p : C \to U$ is not a trivial covering.*

Versions of Kempe's proof were reproduced in a number of places (see for instance [B]), however (as far as we know) even the assertion was not made precise and details of the proof were not given. Recently several (written) attempts were made to improve Theorem 14.1, i.e. to make the subset C open and $U = S$ (see [HJW]) and the projection $p|C$ injective, however, as far as we can tell, they were unsuccessful. Finally there was a work of W. Thurston on this subject that we have discussed in the Introduction.

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