Domains of discontinuity of Lorentzian affine group actions

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Abstract

We prove nonemptyness of domains of proper discontinuity of Anosov groups of affine Lorentzian transformations of $\mathbb{R}^n$.

There is a substantial body of literature, going back to the pioneering work of Margulis [Ma], on properly discontinuous non-amenable groups of affine transformations, see e.g. [A, AMS02, AMS11, Dr, DGK, GLM, Me], and numerous other papers. In this paper we address a somewhat related question of nonemptyness of domains of proper discontinuity of discrete groups acting on affine spaces:

Question 1. *Which discrete subgroups $\Gamma < Aff(\mathbb{R}^n)$ have nonempty discontinuity domain in the affine space $\mathbb{R}^n$?*

In this paper we limit ourselves to the following setting: Suppose that $\Gamma < \mathbb{R}^n \rtimes O(n-1,1) < Aff(\mathbb{R}^n)$ is a discrete subgroup such that the linear projection $\ell : \Gamma \to O(n-1,1)$ is a *faithful representation with convex-cocompact image*, see e.g. [Bo] for the precise definition. Given a representation $\ell : \Gamma \to O(n-1,1)$, the affine action of Γ is determined by a cocycle $c \in Z^1(\Gamma, \mathbb{R}^{n-1,1}_\ell)$. Even in the case $n = 3$ and $\ell(\Gamma)$ a Schottky subgroup of $O(2,1)$ (which is the setting of Margulis' original examples), while some actions are properly discontinuous on the entire $\mathbb{R}^3$ (as proven by Margulis, see also [GLM] for a general description of such actions), nonemptyness of domains of discontinuity for *arbitrary* c does not appear to be obvious[1].

The main result of this note is:

Theorem 2. *Every subgroup $\Gamma < \mathbb{R}^n \rtimes O(n-1,1)$ with faithful convex-cocompact linear representation $\ell : \Gamma \to O(n-1,1)$, acts properly discontinuously on a nonempty open subset of the Lorentzian space $\mathbb{R}^{n-1,1}$.*

We will prove this theorem by applying results on domains of discontinuity for discrete group actions on flag manifolds proven in [KLP3]. To this end, we will begin by identifying

[1]The reaction to the question that we observed included: "clearly true", "clearly false", "unclear".

the Lorentzian space $\mathbb{R}^{n-1,1}$ with an open Schubert cell in a partial flag manifold of the group $G = O(n, 2)$.

Consider the group $G = O(n, 2)$ and its symmetric space $X = G/K$, $K = O(n) \times O(2)$. The group G has two partial flag manifolds: the Grassmannian F_1 of isotropic lines and another partial flag manifold F_2 of isotropic planes in $V = \mathbb{R}^{n,2}$, where the quadratic form on V is

$$q = x_1 y_1 + x_2 y_2 + z_1^2 + ... + z_n^2.$$

We will use the notation $\langle \cdot, \cdot \rangle$ for the associated bilinear form on V.

In the paper we will be using the *Tits boundary* $\partial_{Tits} X$ of the symmetric space X and the incidence geometry interpretation of $\partial_{Tits} X$. The Tits boundary $\partial_{Tits} X$ is a metric bipartite graph whose vertices are labelled *lines* and *planes*, these are the elements of F_1 and F_2 respectively. Two vertices $L \in \mathrm{F}_1$ and $p \in \mathrm{F}_2$ are connected by an edge iff the line L is contained in the plane p. The edges of this bipartite graph have length $\pi/4$. We refer the reader to [Br], [G] and [T].

The group G acts transitively on the set of edges of $\partial_{Tits} X$ and we can identify the quotient $\partial_{Tits} X/G$ with σ_{mod}, the *model spherical chamber* of $\partial_{Tits} X$. Thus σ_{mod} is a circular segment of the length $\pi/4$. This segment has two vertices, one of which we denote τ_{mod}, this is the one which is the projection of F_1. The flag manifold F_1 is the quotient G/P_L, where P_L is the stabilizer of an isotropic line L in G; this flag manifold is n-dimensional.

Recall that two vertices of $\partial_{Tits} X$ are opposite iff they are within Tits distance π from each other. In terms of the incidence geometry of the vector space (V, q), two lines $L, \hat{L} \in \mathrm{F}_1$ are opposite iff that they span the plane $\mathrm{span}(L, \hat{L})$ in V such that the restriction of q to $\mathrm{span}(L, \hat{L})$ is nondegenerate, necessarily of the type $(1, 1)$. Two lines $L, L' \in \mathrm{F}_1$ are within Tits distance $\pi/2$ iff they span an isotropic plane in V.

Consider a subgroup $P_L < G$; it is a maximal parabolic subgroup of G; let $U < P_L$ be the unipotent radical of P_L. Choosing a line $\hat{L}$ opposite to L, defines a semidirect product decomposition $P_L = U \rtimes G_{L,\hat{L}}$, where $G_{L,\hat{L}}$ is the stabilizer in P_L of the line $\hat{L}$; equivalently, it is the stabilizer of the *parallel set*[2] $P(L, \hat{L})$. This subgroup is the intersection

$$G_{L,\hat{L}} = P_L \cap P_{\hat{L}}.$$

The orthogonal complement $V_{L,\hat{L}} \subset V$ of the anisotropic plane $\mathrm{span}(L, \hat{L})$ is invariant under $G_{L,\hat{L}}$, hence,

$$G_{L,\hat{L}} \cong \mathbb{R}_+ \times O(V_{L,\hat{L}}, q|_{V_{L,\hat{L}}}) \cong \mathbb{R}_+ \times O(n-1, 1).$$

Here the group $\mathbb{R}_+$ acts via transvections along geodesics in the symmetric space X connecting L and $\hat{L}$. The group $G_{L,\hat{L}}$ acts on both $(V', q') = (V_{L,\hat{L}}, q|_{V_{L,\hat{L}}})$ and on U, where the action of $\mathbb{R}_+$ on $V' = V_{L,\hat{L}}$ is trivial. In order to simplify the notation, we set

$$O(q') = O(V', q').$$

[2]The parallel set $P(L, \hat{L})$ is a certain symmetric subspace in X, which is the union of all geodesics l in X which are forward-asymptotic to $L \in \partial_{Tits} X$ and backward-asymptotic to $\hat{L} \in \partial_{Tits} X$. The parallel set splits isometrically as the product $l \times \mathbb{H}^{n-1}$, where $\mathbb{H}^{n-1}$ is the *cross-section* of $P(L, \hat{L})$.

In terms of linear algebra, $\mathbb{R}_+$ is the identity component of the orthogonal group

$$O(\mathrm{span}(L, \hat{L}), q|_{\mathrm{span}(L,\hat{L})}) \cong O(1,1).$$

We will use the notation

$$G'_L := U \rtimes O(q') < P_L.$$

This subgroup is the stabilizer in P_L of horoballs in X centered at L.

Our next goal is to describe Schubert cells in the Grassmannian F_1. We fix $L \in \mathrm{F}_1$ and define the subvariety $Q_L \subset \mathrm{F}_1$ consisting of all (isotropic) lines $L' \subset V$ such that $\mathrm{span}(L, L')$ is isotropic (the line L or an isotropic plane). In terms of the Tits' distance, $Q_L - \{L\}$ consists of lines $L' \in \mathrm{F}_1$ within distance $\frac{\pi}{2}$ from L. The complement

$$L^{opp} = \mathrm{F}_1 - Q_L$$

consists of lines opposite to L. The group P_L acts transitively on $\{L\}$, $Q_L - \{L\}$ and L^{opp} and each of these subsets is an open Schubert cell of F_1 with respect to P_L and we obtain the P_L-invariant Schubert cell decomposition

$$\mathrm{F}_1 = \{L\} \sqcup (Q_L - \{L\}) \sqcup L^{opp}.$$

We next describe Q_L more geometrically. A vector $v \in V$ spans an isotropic subspace with L iff $v \in L^{\perp}$ and satisfies the quadratic equation $q(v) = 0$. Since we are only interested in nonzero vectors $v \neq 0$ and their spans $\mathrm{span}(v)$, we obtain the natural identification

$$Q_L \cong \mathbb{P}(q^{-1}(0) \cap L^{\perp}),$$

the right hand-side is the projectivization a conic in $L^{\perp}$. Thus, Q_L is a (projective) conic and $L \in Q_L$ is the unique singular point of the Q_L.

Lemma 3. *Given two opposite isotropic lines $L, \hat{L}$, the intersection of the conics*

$$E = E_{L,\hat{L}} := Q_L \cap Q_{\hat{L}}$$

is an ellipsoid in Q_L.

Proof. As before, let $V' \subset V$ denote the codimension two subspace orthogonal to both $L, \hat{L}$. Then each $L' \in E$ is spanned by a vector $v \in V'$ satisfying the condition $q(v) = 0$. In other words, E is the projectivization of the conic

$$\{v \in V' : q(v) = 0\},$$

i.e. is an ellipsoid. □

Our next goal is to (equivariantly) identify the open cell L^{opp} with the n-dimensional Lorentzian affine space $\mathbb{R}^{n-1,1}$ (where a chosen $\hat{L} \in L^{opp}$ will serve as the origin), so that the group P_L is identified with the group of Lorentzian similarities, where the simply-transitive action $U \curvearrowright L^{opp}$ is identified with the action of the full group of translations of $\mathbb{R}^{n-1,1}$.

We fix nonzero vectors $e \in L$, $f \in \hat{L}$ such that $\langle e, f\rangle = 1$. Then

$$V = \text{span}(e) \oplus \text{span}(f) \oplus V'.$$

We obtain an epimorphism $\eta : P_L \to O(q')$ by sending $g \in P_L$ first to the restriction $g|L^\perp$ and then to the projection of the latter to the quotient space $V' \cong L^\perp/L$ (the quotient of $L^\perp$ by the null-subspace of $q|L^\perp$). Hence, the kernel of this epimorphism is precisely the solvable radical $U \rtimes \mathbb{R}_+$ of P_L.

For each $v' \in V'$ we define the linear transformation (a shear) $s = s_{v'} \in GL(V)$ by its action on e, f and V':

1. $s(e) = e$.
2. $s(f) = -\frac{1}{2}q(v')e + f + v'$.
3. For $w \in V'$, $s(w) = w - \langle v', w\rangle e$.

The next two lemmata are proven by straightforward calculations which we omit:

Lemma 4. *For each $s = s_{v'}$ the following hold:*

1. $s \in P_L$.

2. s lies in the kernel of the homomorphism $\eta : P_L \to GL(V')$ and is unipotent. In particular, $s \in U$ for each $v' \in V$.

Lemma 5. *The map $\phi : v' \mapsto s_{v'}$ is a continuous monomorphism $V' \to U$, where we equip the vector space V' with the additive group structure.*

Since U acts simply transitively on L^{opp}, it is connected and has dimension n. Therefore, the monomorphism ϕ is surjective and, hence, a continuous isomorphism. Thus, ϕ determines a homeomorphism $h : V' \to L^{opp}$

$$h : v' \mapsto s_{v'}(\hat{L}) = span\left(-\frac{1}{2}q(v')e + f + v'\right),$$

$$h(0) = \hat{L}.$$

The group $G_{L,\hat{L}} \cong \mathbb{R}_+ \times O(V', q')$ acts on both L^{opp} and on U (via conjugation). The center of $G_{L,\hat{L}}$ acts on V' trivially while its action on U is via a nontrivial character.

Proposition 6. *The map h is equivariant with respect to these two actions of $O(V', q')$.*

Proof. Consider a linear transformation $A \in O(V', q')$; as before, we identify $O(V', q')$ with a subgroup of $O(V, q)$ fixing e and f. For an arbitrary $v' \in V'$ we will verify that

$$s_{Av'} = As_{v'}A^{-1}.$$

It suffices to verify this identity on the vectors e, f and arbitrary $w \in V'$. We have:

1. For each $u \in V'$, $s_u(e) = e$, while $A(e) = A^{-1}(e) = e$. It follows that

$$e = s_{Av'}(e) = As_{v'}A^{-1}(e) = e.$$

2. $$s_{Av'}(f) = -\frac{1}{2}q(Av')e + f + Av' = -\frac{1}{2}q(v')e + f + Av'$$

while (since $Ae = e$, $Af = f$)

$$As_{v'}A^{-1}(f) = As_{v'}(f) = A(-\frac{1}{2}q(v')e + f + v') = -\frac{1}{2}q(v')e + f + Av'.$$

3. For $w \in V'$,

$$s_{Av'}(w) = w - \langle Av', w\rangle e = w - \langle v', A^{-1}w\rangle e,$$

while

$$As_{v'}A^{-1}w = As_{v'}(A^{-1}w) = A(A^{-1}w - \langle v', A^{-1}w\rangle e) = w - \langle v', A^{-1}w\rangle e. \quad \square$$

In view of this proposition we will identify V' with the open Schubert cell L^{opp}, which, in turn, enables us to use Lorentzian geometry to analyze L^{opp} and, conversely, to study discrete subgroups of P_L using results of [KLP3] on domains of discontinuity of discrete group actions on the flag manifold F_1. Under the identification $V' \cong L^{opp}$, for each $\hat{L} \in L^{opp}$, the conic $Q_{\hat{L}} \cap L^{opp}$ becomes a translate of the null-cone of the form q' on V' (see Lemma 7 below) and the flag manifold F_1 becomes a compactification of V' obtained by adding to it the "quadric at infinity" Q_L.

Lemma 7. *For all $v' \in V'$, $q'(v') = 0$ iff q vanishes on $\mathrm{span}(f, h(v'))$, i.e. iff $h(v') \in Q_{\hat{L}}$. In other words, $Q_{\hat{L}} \cap L^{opp}$ is the image under h of the null-cone of q' in the vector space V'.*

Proof. Since f and $s_{v'}(f)$ (spanning the line $h(v')$) are null-vectors of q, the vanishing of q on $\mathrm{span}(f, h(v'))$ is equivalent to the vanishing of

$$\langle f, s_{v'}(f)\rangle = -\frac{1}{2}q(v'). \quad \square$$

Lemma 8. *For each neighborhood N of L in Q_L there exists $\hat{L} \in L^{opp}$ such that $E_{L,\hat{L}} \subset N$.*

Proof. We pick $L_\infty \in \mathrm{F}_1$ opposite to L and, as above, identify L_∞^{opp} with (V', q'). Then for a sequence $\hat{L}_i \in L_\infty^{opp}$ contained in the, say, future light cone of $Q_L \cap L_\infty^{opp}$ and converging radially to L, the intersections of null-cones $E_{L,L_i} = Q_{L_i} \cap Q_L$ converge to L. Since $L_i \notin Q_L$, they are all opposite to L. $\square$

For each subset $C \subset \mathrm{F}_1$, we define the *thickening* of C:

$$\mathrm{Th}(C) = \bigcup_{L \in C} Q_L.$$

This notion of thickening is a special case of the one developed in [KLP3] (see also [KL2]): If we restrict to a single apartment a in the Tits building of G, then for the vertex $L \in a$, $\mathrm{Th}(L) \cap a = Q_L \cap a$ consists of three vertices within Tits distance $\frac{\pi}{2}$ from L. Thus, in the terminology of [KLP3], the thickening Th is *fat*.

Lemma 9. *For any two opposite lines $L, \hat{L} \in \mathrm{F}_1$ and each compact subset $C \subset Q_{\hat{L}} \cap L^{opp}$, the intersection $Th(C) \cap L^{opp}$ is a proper subset of L^{opp}.*

Proof. Let $H \subset L^{opp} \cong V'$ be an affine hyperplane in V' intersecting $Q_{\hat{L}}$ only at $\hat{L}$. Then

$$C' := \{L' \in H : Q_{L'} \cap C \neq \emptyset\}$$

is compact in H. Next, observe that for $L_1, L_2 \in \mathrm{F}_1$, $L_1 \in Q_{L_2} \iff L_2 \in Q_{L_1}$. Thus, every $L' \in H - C'$ does not belong to $\mathrm{Th}(C)$. ☐

Lemma 10. *For each compact $C \subset Q_L - \{L\}$ the thickening $Th(C)$ is a proper subset of* F_1.

Proof. Lemma 8 implies that there exists $L_\infty \in L^{opp}$ such that E_{L,L_∞} is disjoint from C. Thus, C is contained in L_∞^{opp}. Now the claim follows from Lemma 9. ☐

We now turn to discrete subgroups $\Gamma < G'_L < P_L < G$. We refer the reader to [KLP3] for the notion of τ_{mod}-regular discrete subgroups $\Gamma < G$ and their τ_{mod}-limit sets, which are certain closed Γ-invariant subsets of F_1.

Remark 11. We must also note that the notion equivalent to τ_{mod}-regularity and the τ_{mod}-lit set was first introduced by Benoist in his highly influential work [Ben].

An important class of τ_{mod}-regular discrete subgroups $\Gamma < G$ consists of τ_{mod}-*Anosov subgroups*. Anosov representations $\Gamma \to G$ whose images are Anosov subgroups were first introduced in [La] for fundamental groups of closed manifolds of negative curvature, then in [GW] for arbitrary hyperbolic groups; we refer the reader to our papers [KLP4, KLP5, KL1], for a simplification of the original definition as well as for alternative definitions and to [KL2, KLP2] for surveys of the results.

Lemma 12. *The τ_{mod}-limit set $\Lambda_{\tau_{mod}}(\Gamma)$ of every τ_{mod}-regular discrete subgroup $\Gamma < P_L$ is contained in Q_L.*

Proof. Recall that G'_L and, hence, Γ, preserves each horoball Hbo in X centered at L, where the latter is regarded as a point of the visual boundary of the symmetric space X. Therefore, for each $x \in Hbo$, the closure of Γx in $\overline{X} = X \cup \partial_\infty X$ is contained in the ideal boundary of Hbo, which is the closed $\frac{\pi}{2}$-ball $\bar{B}(L, \frac{\pi}{2})$ in $\partial_\infty X$ centered at L, where the distance is computed in the Tits metric on $\partial_\infty X$. For each vertex τ of the building $\partial_{Tits} X$ which belongs to $\bar{B}(L, \frac{\pi}{2})$ the star $\mathrm{st}(\tau) \subset \partial_\infty X$ is contained in the closed ball in $\partial_\infty X$ of the radius $\frac{3\pi}{4}$ centered at L. Therefore, the intersection of $\mathrm{st}(\tau)$ with the Grassmannian F_1 is contained in $\bar{B}(L, \frac{\pi}{2})$. It follows from the definition of the τ_{mod}-limit set that $\Lambda_{\tau_{mod}}(\Gamma)$ is contained in $\mathrm{F}_1 \cap \bar{B}(L, \frac{\pi}{2}) = Q_L$. ☐

Proposition 13. *Suppose that $\Gamma < G'_L$ is a τ_{mod}-regular discrete subgroup whose τ_{mod}-limit set does not contain L. Then*

$$Th(\Lambda_{\tau_{mod}}(\Gamma)) \neq \mathrm{F}_1$$

and the action

$$\Gamma \curvearrowright \mathrm{F}_1 - Th(\Lambda_{\tau_{mod}}(\Gamma))$$

is properly discontinuous.

Proof. Since $\Lambda_{\tau_{mod}}(\Gamma)$ is a compact subset of Q_L, the first statement of the proposition is a special case of Lemma 10. The proper discontinuity statement is a special case of a general theorem [KLP3, Theorem 6.13] since the thickening Th is fat. $\square$

We now describe certain conditions on τ_{mod}-regular discrete subgroups $\Gamma < G'_L$ which will ensure that $\Lambda_{\tau_{mod}}(\Gamma)$ does not contain the point L. Each subgroup $\Gamma < G'_L$ has the *linear part* Γ_0, i.e. its projection to $O(q') \cong O(n-1,1)$, which is identified with the semisimple factor of the stabilizer in P_L of some $\hat{L} \in L^{opp}$. We now assume that:

- Γ_0 is a convex-cocompact subgroup of $O(n-1,1)$.
- The projection
$$\ell : \Gamma \to \Gamma_0$$
is an isomorphism.

Since $\Gamma_0 < O(q')$ is convex-cocompact and $O(q') < P_L$ is the Levi subgroup of the parabolic group P_L stabilizing a face of type τ_{mod} of $\partial_{Tits}X$, it follows that $\Gamma_0 < G$ is a τ_{mod}-Anosov subgroup of G; the τ_{mod}-limit set of Γ_0 is contained in the visual boundary of the cross-section (isometric to $\mathbb{H}^{n-1}$) of the parallel set $P(L, \hat{L})$; in particular, $\Lambda_{\tau_{mod}}(\Gamma_0)$ does not contain L.

Given a subgroup $\Gamma_0 < O(q')$, the inverse $\rho : \Gamma_0 \to \Gamma$ to $\ell : \Gamma \to \Gamma_0$ is determined by a cocycle $c \in Z^1(\Gamma_0, V')$ which describes the translational parts of the elements of Γ:

$$\rho(\gamma) : v \mapsto \gamma v + c(\gamma), v \in V' \cong \mathbb{R}^{n-1,1}.$$

Pick some $t \in \mathbb{R}_+$; then tc is again a cocycle corresponding to the conjugate representation ρ^t, where we identity $t \in \mathbb{R}_+$ with a central element of $G_{L,\hat{L}}$. Sending $t \to 0$ we obtain:

$$\lim_{t\to 0} \rho^t = id,$$

the identity embedding $\Gamma_0 \to O(n-1,1) < P_L$. In view of stability of Anosov representations (see [GW] and [KLP1]) we conclude that all representations ρ^t are τ_{mod}-Anosov and the τ_{mod}-limit sets of $\Gamma_t = \rho^t(\Gamma_0)$ vary continuously with t; moreover,

$$t\Lambda_{\tau_{mod}}(\Gamma_{t_1}) = \Lambda_{\tau_{mod}}(\Gamma_{t_2})$$

where $t = t_2/t_1$. In particular,

$$\Lambda_{\tau_{mod}}(\Gamma) \subset Q_L - \{L\}$$

is a compact subset. Proposition 13 now implies:

Corollary 14. *For each* Γ *as above,*

$$Th(\Lambda_{\tau_{mod}}(\Gamma)) \neq \mathrm{F}_1$$

and the action

$$\Gamma \curvearrowright \mathrm{F}_1 - Th(\Lambda_{\tau_{mod}}(\Gamma))$$

is properly discontinuous.

Thus, we proved that each discrete subgroup $\Gamma < P_L$ as above has nonempty domain of discontinuity in the vector space V'. Theorem 2 follows. $\square$

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