



Domains of discontinuity of Lorentzian affine group actions

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Abstract

We prove nonemptiness of domains of proper discontinuity of Anosov groups of affine Lorentzian transformations of \mathbb{R}^n .

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1 Introduction

There is a substantial body of literature, going back to the pioneering work of Margulis [24], on properly discontinuous non-amenable groups of affine transformations, see e.g. [1–3, 9, 10, 14, 25], and numerous other papers, in particular, the recent survey [11]. In this paper we address a somewhat related question of nonemptiness of domains of proper discontinuity of discrete groups acting on affine spaces:

Question 1 *Which discrete subgroups $\Gamma < \text{Aff}(\mathbb{R}^n)$ have nonempty discontinuity domain in the affine space \mathbb{R}^n ?*

In this paper we limit ourselves to the following setting: Suppose that $\Gamma < \mathbb{R}^n \rtimes O(n-1, 1) < \text{Aff}(\mathbb{R}^n)$ is a discrete subgroup such that the linear projection $\ell : \Gamma \rightarrow O(n-1, 1)$ is a *faithful representation with convex-cocompact image*, see e.g. [6] for the precise definition. Given a representation $\ell : \Gamma \rightarrow O(n-1, 1)$, the affine action of Γ is determined by a cocycle $c \in Z^1(\Gamma, \mathbb{R}_\ell^{n-1,1})$. Even in the case $n = 3$ and $\ell(\Gamma)$ a Schottky subgroup of $O(2, 1)$ (which is the setting of Margulis' original examples), while some actions are properly discontinuous on the entire \mathbb{R}^3 (as proven by Margulis, see also [14] for a general description of such

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actions), nonemptiness of domains of discontinuity for *arbitrary* c does not appear to be obvious.¹

The main result of this note is:

Theorem 2 *Every subgroup $\Gamma < \mathbb{R}^n \rtimes O(n-1, 1)$ with faithful convex-cocompact linear representation $\ell : \Gamma \rightarrow O(n-1, 1)$, acts properly discontinuously on a nonempty open subset of the Lorentzian space $\mathbb{R}^{n-1,1}$.*

We will prove this theorem by applying results on domains of discontinuity for discrete group actions on flag-manifolds proven in [21]. More precisely, we will check that Γ is a τ_{mod} -Anosov subgroup of the Lie group $G = O(n, 2)$ for a suitable model simplex (actually, a vertex) $\tau_{mod} \subset \sigma_{mod}$. In section 4 we will equivariantly identify the Lorentzian space $\mathbb{R}^{n-1,1}$ and an open Schubert cell in a partial flag-manifold $F_1 = G/P_{\tau_{mod}}$ of the group $G = O(n, 2)$. In [21] we proved that for each τ_{mod} -Anosov subgroup Γ of a semisimple Lie group G and each *fat thickening* $\text{Th}(\Lambda_{\tau_{mod}}(\Gamma)) \subset F_1$ of the τ_{mod} -limit set $\Lambda_{\tau_{mod}}(\Gamma) \subset F_1$, the group Γ acts properly discontinuously on the open subset $\Omega_{\text{Th}}(\Gamma) = F_1 \setminus \text{Th}(\Lambda_{\tau_{mod}}(\Gamma))$. In Sect. 5 of this paper we verify that $\Omega_{\text{Th}}(\Gamma) \neq \emptyset$ in the context of τ_{mod} -Anosov subgroups $\Gamma < \mathbb{R}^n \rtimes O(n-1, 1) < O(n, 2)$ and the *maximal thickening* Th . This, in turn, will establish nonemptiness of the domain of discontinuity of Γ in $\mathbb{R}^{n-1,1}$.

2 Geometric preliminaries

Symmetric spaces of noncompact type and their visual boundaries For basics of symmetric spaces and their visual boundaries we refer the reader to [4, 12].

Consider a symmetric space of noncompact type $X = G/K$, where G is a semisimple Lie group (with finite center) and K is its maximal compact subgroup. Fix also a base-point $o \in X$ (the choice is ultimately irrelevant), fixed by K . We let d denote the Riemannian distance function on X and $\angle_x(y, z)$ the Riemannian angle between nondegenerate geodesic segments xy, xz emanating from x . The *visual boundary* $\partial_\infty X$ of X , as a set, is identified with the set of equivalence classes $[\rho]$ of geodesic rays $\rho : \mathbb{R}_+ \rightarrow X$ in X , where two rays are equivalent if and only if their images are at a finite Hausdorff distance from each other. One says that every ray ρ representing $\xi = [\rho]$ is *asymptotic to* ρ . The *Tits angle* $\angle_{Tits}(\xi_1, \xi_2)$ between points $\xi_1 = [\rho_1], \xi_2 = [\rho_2]$ is defined as

$$\sup_{x \in X} \angle_x(\rho_1(t), \rho_2(t)),$$

where the supremum is taken over all pairs of rays ρ_1, ρ_2 representing ξ_1, ξ_2 such that $\rho_1(0) = \rho_2(0) = x$. Since X is a symmetric space, there exists a flat $F \subset X$ such that ξ_1, ξ_2 are represented by rays whose images are contained in F . The supremum in the definition of $\angle_{Tits}(\xi_1, \xi_2)$ is realized by pairs of such rays. The Tits angle defines the *Tits metric* on $\partial_\infty X$. This metric is invariant under the natural G -action on $\partial_\infty X$.

The visual boundary of X has two natural topologies. The first one is the *visual topology*: Every $\xi \in \partial_\infty X$ is represented by a unique unit speed geodesic ray emanating from o . Thus, there is a natural bijection between $\partial_\infty X$ and the unit sphere in the tangent space $T_o X$. The visual topology on $\partial_\infty X$ is the one making this bijection a homeomorphism. The natural G -action on $\partial_\infty X$ is continuous with respect to this topology. This topology extends to a *visual compactification* $\bar{X} = X \cup \partial_\infty X$: A sequence (x_n) in X converges to $\xi = [\rho] \in \partial_\infty X$

¹ The reaction to the question that we observed included: “clearly true”, “clearly false”, “unclear”.

if

$$\lim_{n \rightarrow \infty} \angle_o(x_n, \rho(1)) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(o, x_n) = \infty,$$

where $\rho(0) = o$. For a subset $A \subset X$, the *visual boundary* of A is the intersection of $\partial_\infty X$ with the closure of A in \overline{X} with respect to the visual topology.

The second, *Tits topology*, is the one defined by the Tits metric. With respect to this topology, $\partial_\infty X$ has the structure of a certain simplicial complex, the *spherical (Tits) building* $\partial_{Tits} X$, invariant under the action of G . We will fix a *model chamber* of $\partial_\infty X$, i.e. a facet σ_{mod} of this spherical building, a *model maximal flat* $F_{mod} \subset X$, it is the unique maximal flat in X whose visual boundary a_{mod} (the *model apartment* in $\partial_{Tits} X$) is a subcomplex containing σ_{mod} and such that $o \in F_{mod}$. The *Euclidean Weyl chamber* Δ of X is the cone in F_{mod} with the tip o over σ_{mod} (the union of geodesic rays emanating from o and asymptotic to the points of σ_{mod}). The *Weyl group* W of X is the image of $K \cap \text{Stab}_G(F_{mod})$ in the isometry group of the flat F_{mod} . Then Δ is a fundamental domain of the W -action on F_{mod} . The Weyl group W has a standard word-metric; we let w_0 denote the unique longest element of W with respect to this metric. Identifying F_{mod} with \mathbb{R}^r (where r is the rank of X), we get the *opposition involution* $\iota = -w_0$ preserving σ_{mod} . In the case of symmetric spaces of type B , as in this paper, $w_0 = -\text{id}$ and, accordingly, $\iota = \text{id}$.

Antipodality. Two points ξ, η in $\partial_\infty X$ are called *opposite* if $\angle_{Tits}(\xi, \eta) = \pi$, equivalently, if there exists a geodesic c in X whose opposite subrays are asymptotic to ξ and η respectively. Equivalently, there exists a Cartan involution of X swapping ξ and η . Two simplices $\tau, \hat{\tau}$ in $\partial_{Tits} X$ are *opposite* (or, *antipodal*) if and only if they contain opposite *generic* points in $\partial_\infty X$. (A point in a simplex τ is *generic* if it does not belong to any proper face of τ .) Two simplices in $\partial_{Tits} X$ are opposite if and only if they are swapped by a Cartan involution of X .

Horoballs. For every point $\xi = [\rho]$ in $\partial_\infty X$ one defines the *Busemann function* b_ξ on X (or, more precisely, a family of Busemann functions which differ by additive constants):

$$b_\xi(x) = \lim_{t \rightarrow \infty} (d(\rho(0), x) - t).$$

Busemann functions satisfy the following equivariance condition with respect to the action of isometries g of X :

$$b_{g\xi} = b_\xi \circ g + \text{Const.}$$

Sublevel sets of Busemann functions b_ξ are called *horoballs centered at ξ* and denoted *Hbo*. Busemann functions and, hence, horoballs, are convex. We will need the following lemma that can be found in [4, Lemma 4.10] and [12, Proposition 3.4.3]:

Lemma 3 *For each horoball Hbo in X centered at ξ , the visual boundary of Hbo equals the closed $\frac{\pi}{2}$ -ball $\bar{B}(\xi, \frac{\pi}{2})$ in $\partial_\infty X$ centered at ξ , where the distance is computed in the Tits metric on $\partial_\infty X$.*

Parallel sets Fix two opposite points $\xi, \hat{\xi} \in \partial_\infty X$. The *parallel set* $P(\xi, \hat{\xi})$ is a certain symmetric subspace in X , which is the union of all geodesics l in X that are forward-asymptotic to $\xi \in \partial_\infty X$ and backward-asymptotic to $\hat{\xi} \in \partial_\infty X$. Suppose that $\xi, \hat{\xi}$ are generic points of two opposite simplices $\tau, \hat{\tau}$ in $\partial_{Tits} X$. Then $P(\xi, \hat{\xi})$ splits isometrically as a direct product $F_{\tau, \hat{\tau}} \times Y$, where $F_{\tau, \hat{\tau}}$ is a flat in X of dimension $\dim(\tau) + 1$ and Y is a (totally-geodesic) symmetric subspace of noncompact type in X , called a *cross-section* of $P(\xi, \hat{\xi})$. In the case of interest to us, $\tau, \hat{\tau}$ are vertices in $\partial_{Tits} X$, $F_{\tau, \hat{\tau}}$ is 1-dimensional and Y is a symmetric space of rank 1 (actually, the hyperbolic space). The pointwise stabilizer $G_{\tau, \hat{\tau}}$ of $\{\tau, \hat{\tau}\}$ is a reductive subgroup of P_τ ; it splits off as a product $G_Y \times \mathbb{R}^r$, where \mathbb{R}^r is the group of transvections in G preserving the flat $F_{\tau, \hat{\tau}}$ and G_Y is a semisimple Lie group,

it is the stabilizer of Y in $G_{\tau, \hat{\tau}}$. The action of G_Y on Y (and $P(\xi, \hat{\xi})$) may have a nontrivial (but compact) kernel and the image of G_Y in the isometry group of Y is a subgroup of finite index. The *unipotent radical* $U_\tau \triangleleft P_\tau$ is a normal subgroup such that $P_\tau = U_\tau \rtimes G_{\tau, \hat{\tau}}$. A more refined form of this decomposition is

$$P_\tau = (U_\tau \rtimes G_Y) \rtimes \mathbb{R}^f.$$

The subgroup $U_\tau \rtimes G_Y$ preserves each horoball centered at ξ . See [22, §2.8, 2.10] for more details.

For the material below we refer the reader to [18, 21].

For each point $x \in X$ one defines the Δ -valued distance $d_\Delta(o, x)$ as the unique point of intersection $Kx \cap \Delta$. (This definition extends to general pairs of points in X by G -invariance.) Consider a face τ_{mod} of the spherical Weyl chamber σ_{mod} of X . These faces parameterize *standard parabolic subgroups* $P_{\tau_{mod}}$ of G , their G -stabilizers. The τ_{mod} -*boundary* $\partial_{\tau_{mod}} \sigma_{mod}$ of σ_{mod} is the union of the faces of σ_{mod} which do not contain τ_{mod} . The *open star* $\text{ost}(\tau_{mod})$ of τ_{mod} in σ_{mod} is the complement $\sigma_{mod} \setminus \partial_{\tau_{mod}} \sigma_{mod}$. In the example relevant to us, when $\sigma_{mod} = [u, v]$ is a simplex with the vertices u, v and τ_{mod} is one of the vertices of σ_{mod} , say, u , $\partial_{\tau_{mod}} \sigma_{mod} = \{v\}$ and $\text{ost}(\tau_{mod}) = [u, v) = \sigma_{mod} \setminus \{v\}$. In general, one defines $V(\partial_{\tau_{mod}} \sigma_{mod}) \subset \Delta$ as the cone over $\partial_{\tau_{mod}} \sigma_{mod}$.

Stars at infinity The group G acts transitively on the set of facets of $\partial_\infty X$; thus, a face τ of $\partial_\infty X$ is said to have the type τ_{mod} if they lie in the same G -orbit. One defines *open stars* $\text{ost}(\tau)$ of faces τ of $\partial_\infty X$: One first takes its *star*, $\text{st}(\tau)$, the subcomplex in $\partial_{Tits} X$ which is the union of faces containing τ , and then removes from $\text{st}(\tau)$ those faces which do not contain τ . In the case of interest to us, $\partial_\infty X$ is 1-dimensional (a connected graph of valence continuum at each vertex), τ is a vertex of $\partial_\infty X$, $\text{st}(\tau)$ is the union of edges (including their respective vertices!) containing τ as an end-point and $\text{ost}(\tau)$ is the interior of $\text{st}(\tau)$ with respect to the Tits topology, i.e. the topology of the graph $\partial_{Tits} X$. A point $\xi \in \partial_\infty X$ is said to be τ_{mod} -*regular* if it belongs to $\text{ost}(\tau)$ for some $\tau \in \text{Flag}_{\tau_{mod}}$. One quantifies this notion of regularity by taking a compact subset $\Theta \subset \text{ost}(\tau_{mod})$; a $\xi \in \partial_\infty X$ is said to be Θ -regular if its projection to σ_{mod} belongs to Θ .

Flag-manifolds Fix a model simplex τ_{mod} . The G -orbit $G\tau_{mod}$ is naturally identified with the quotient $G/P_{\tau_{mod}}$. From the viewpoint of the Tits topology, this quotient is discrete, but, it also has a natural manifold topology (the quotient topology of the Lie group G), making it a *partial flag-manifold* $\text{Flag}_{\tau_{mod}}$. Another way to describe this topology is to note that there is a G -equivariant bijection between $G/P_{\tau_{mod}}$ and the orbit $G\xi$ for a generic point $\xi \in \tau_{mod}$. This bijection is a homeomorphism from $G/P_{\tau_{mod}}$ to $G\xi$, where the latter is equipped with the subspace topology inherited from the visual topology on $\partial_\infty X$.

Thickenings We fix a model face τ_{mod} of σ_{mod} . The W -orbit of τ_{mod} in the model apartment a_{mod} is naturally identified with the quotient $W/W_{\tau_{mod}}$, where $W_{\tau_{mod}}$ is the stabilizer of τ_{mod} in W . The group W acts on $W/W_{\tau_{mod}}$ via the left multiplication. The *strong Bruhat order* \leq on W descends to the *folding (partial) order* on $W/W_{\tau_{mod}}$:

$[w] \leq [w']$ if and only if representatives w, w' or $[w], [w'] \in W/W_{\tau_{mod}}$ can be chosen so that $w \leq w'$.

An *ideal* in the poset $(W/W_{\tau_{mod}}, \leq)$ is a proper subset (i.e., a nonempty subset with nonempty complement) I satisfying the property that with every $[w] \in I$, the ideal contains all smaller elements of $W/W_{\tau_{mod}}$. The poset $(W/W_{\tau_{mod}}, \leq)$ has a unique maximal element $[w_0]$ where w_0 is the longest element of W . Accordingly, $(W/W_{\tau_{mod}}, \leq)$ has a unique maximal ideal J equal to the complement of $\{[w_0]\}$. An ideal I is called *fat* if

$$I \cup w_0 I = W/W_{\tau_{mod}}.$$

For instance, the unique maximal ideal is fat.

For every pair of simplices $\tau, \tau' \in \text{Flag}_{\tau_{mod}}$, there exist $g \in G$ and $w \in W$ such that $g(\tau) = \tau_{mod}$ and $g(\tau') = \nu = w\tau_{mod}$, a simplex in a_{mod} . The simplex ν is not uniquely determined by this, but its W_τ -orbit is uniquely determined. Hence, we define the *relative position* of τ' with respect to τ , $\text{pos}(\tau', \tau)$, as the W_τ -orbit of ν , equivalently, the corresponding $W_{\tau_{mod}}$ -orbit in $W/W_{\tau_{mod}}$ (or, equivalently, the double coset of w in $W_{\tau_{mod}} \backslash W/W_{\tau_{mod}}$). Let $I \subset W/W_{\tau_{mod}}$ be an ideal invariant under the left $W_{\tau_{mod}}$ -action. (For instance, the unique maximal ideal satisfies this condition.) For a simplex $\tau \in \text{Flag}_{\tau_{mod}}$, we define the *thickening* $\text{Th}(\tau) = \text{Th}_I(\tau) \subset \text{Flag}_{\tau_{mod}}$ as the subset consisting of simplices τ' such that $\text{pos}(\tau', \tau) \subset I$. In other words, $\tau' \in \text{Th}(\tau)$ if and only if there exists $g \in G$ such that $g(\tau) = \tau_{mod}$ and $g(\tau') \in I$. The thickening $\text{Th}(\tau)$ is a certain closed subcomplex (a union of Schubert cycles) in a cellular decomposition of $\text{Flag}_{\tau_{mod}}$ relative to τ . The thickenings $\text{Th}(\tau)$ satisfy

$$\text{Th}(g\tau) = g \text{Th}(\tau), g \in G.$$

Given a subset $A \subset \text{Flag}_{\tau_{mod}}$ and a $W_{\tau_{mod}}$ -invariant ideal I in $(W/W_{\tau_{mod}}, \leq)$, we define the corresponding thickening of A as

$$\text{Th}(A) = \bigcup_{\tau \in A} \text{Th}(\tau).$$

It is observed in [21] (see also [17, Lemma 8.18] and Lemma 18 of this paper) that for every closed subset $A \subset \text{Flag}_{\tau_{mod}}$ and an ideal I , the corresponding thickening $\text{Th}(A)$ is a closed subset of $\text{Flag}_{\tau_{mod}}$. A thickening is called *fat* if the corresponding ideal in $W/W_{\tau_{mod}}$ is fat. A thickening is *maximal* if the corresponding ideal is the maximal ideal.

Regularity and flag-convergence A nondegenerate geodesic segment xy in X is said to be τ_{mod} -regular if $d_\Delta(x, y) \in \text{ost}(\tau_{mod})$.

A sequence (x_n) in X is said to be τ_{mod} -regular if the sequence of vectors $d_\Delta(o, x_n) \in \Delta$ diverges away from $V(\partial_{\tau_{mod}} \sigma_{mod})$ as $n \rightarrow \infty$. In the example relevant to us, when G has rank two and, accordingly, Δ is two-dimensional, and τ_{mod} is a vertex of an edge σ_{mod} , $V(\partial_{\tau_{mod}} \sigma_{mod})$ is the null-set of a certain linear functional on Δ , a simple root α . Then τ_{mod} -regularity of (x_n) means that

$$\lim_{n \rightarrow \infty} \alpha(d_\Delta(o, x_n)) = \infty.$$

A sequence (x_n) is said to be *uniformly τ_{mod} -regular* if the sequence of vectors $d_\Delta(o, x_n) \in \Delta$ diverges away from $V(\partial_{\tau_{mod}} \sigma_{mod})$ at a linear speed with respect to $d(o, x_n)$. In a more quantitative way, one describes uniformly regular sequences as follows. Fix a compact subset $\Theta \subset \text{ost}(\tau_{mod})$. A sequence (x_n) is said to be Θ -regular if $d(o, x_n) \rightarrow \infty$ and for all but finitely many members of the sequence, the geodesic rays ρ_n from o through $d_\Delta(o, x_n)$ are asymptotic to points of Θ . Then a sequence (x_n) is uniformly τ_{mod} -regular if and only if it is Θ -regular for some compact $\Theta \subset \text{ost}(\tau_{mod})$.

The same definitions apply to sequences (g_n) in G : A sequence (g_n) is (uniformly) τ_{mod} -regular if for some (equivalently, every) $x \in X$, the sequence $x_n = g_n(x)$ is (uniformly) τ_{mod} -regular.

In [23] we defined a partial compactification of X , $\overline{X}^{\tau_{mod}} = X \cup \text{Flag}_{\tau_{mod}}$. Below we will only describe the notion of *flag-convergence* for τ_{mod} -regular sequences in X to points of $\text{Flag}_{\tau_{mod}}$ with respect to the topology of $\overline{X}^{\tau_{mod}}$. If X has rank 1, then σ_{mod} is a singleton, $\tau_{mod} = \sigma_{mod}$ and $\text{Flag}_{\tau_{mod}} = \partial_\infty X$ (with the visual topology). Accordingly, a sequence (x_n) converges to $\tau \in \text{Flag}_{\tau_{mod}}$ if and only if it converges to $\tau \in \partial_\infty X$ in the visual topology.

In higher rank, a ray geodesic $o\xi_n$ through x_n need not even terminate in a face τ_n of $\partial_{Tits} X$ of type τ_{mod} . But, if it does, then $x_n \rightarrow \tau \in \text{Flag}_{\tau_{mod}}$ if and only if $\tau_n \rightarrow \tau$ in $\text{Flag}_{\tau_{mod}}$.

In general, one defines flag-convergence $x_n \rightarrow \tau \in \text{Flag}_{\tau_{mod}}$ for τ_{mod} -regular sequences (x_n) in X as follows. Due to the τ_{mod} -regularity assumption on (x_n) , one finds (for all sufficiently large n) a unique face τ_n of type τ_{mod} in $\text{Flag}_{\tau_{mod}}$, such that ξ_n belongs to the open star $\text{ost}(\tau_n)$ of τ_n . By the definition, $x_n \rightarrow \tau$ (the sequence (x_n) flag-converges to τ) if and only if $\tau_n \rightarrow \tau$ in $\text{Flag}_{\tau_{mod}}$.

If (x_n) is uniformly τ_{mod} -regular (i.e., Θ -regular for a compact $\Theta \subset \text{ost}(\tau_{mod})$) one can also describe flag-convergence $x_n \rightarrow \tau$ as follows. First, note that a diverging sequence $x_n \in X$ converges to $\xi \in \partial_\infty X$ with respect to the visual topology on \bar{X} if and only if the sequence (ξ_n) defined above converges to ξ in the visual topology on $\partial_\infty X$. Of course, the sequence (ξ_n) need not converge, but (by compactness of $\partial_\infty X$) it has convergent subsequences. In view of the Θ -regularity of (x_n) , all subsequential limits of (ξ_n) in $\partial_\infty X$ (equivalently, of (x_n) in \bar{X}) are Θ -regular points in $\partial_\infty X$. Then (x_n) flag-converges to $\tau \in \text{Flag}_{\tau_{mod}}$ if and only if the accumulation set of (x_n) in $\partial_\infty X$ is contained in $\text{ost}(\tau)$.

3 Regular and Anosov subgroups

Regular subgroups In what follows, we fix an ι -invariant face τ_{mod} of σ_{mod} . (For the symmetric spaces appearing in this paper, the ι -invariance condition is automatically satisfied since $\iota = \text{id}$.) Importance of this invariance assumption comes from the fact that we will be interested in accumulation points in $\bar{X}^{\tau_{mod}}$ of Γ -orbits of τ_{mod} -regular subgroups $\Gamma < G$. For a typical element $\gamma \in \Gamma$, if a sequence $(\gamma^n)_{n \in \mathbb{N}}$ is τ_{mod} -regular, then the inverse sequence $(\gamma^{-n})_{n \in \mathbb{N}}$ is $\iota\tau_{mod}$ -regular. Hence, to have a satisfactory theory, it makes sense to assume that $\tau_{mod} = \iota\tau_{mod}$.

Remark 4 We must also note that the notion equivalent to τ_{mod} -regularity of subgroups $\Gamma < G$ and the τ_{mod} -limit set was first introduced by Benoist in his highly influential work [5, section 3.6]. For the benefit of an interested reader, his notation for the limit set was Λ_Γ .

We refer the reader to [18, 21] for the detailed discussion of τ_{mod} -regular discrete subgroups $\Gamma < G$ and their τ_{mod} -limit sets (denoted $\Lambda_{\tau_{mod}}(\Gamma)$ in our papers), which are certain closed Γ -invariant subsets of $\text{Flag}_{\tau_{mod}}$.

Below we review the notions of regularity and limit sets. A (necessarily discrete) subgroup $\Gamma < G$ is said to be τ_{mod} -regular if every sequence of distinct elements $\gamma_n \in \Gamma$ is τ_{mod} -regular. Similarly, one defines *uniformly τ_{mod} -regular* subgroups of G . For instance, if X has rank 1, then Δ is 1-dimensional, hence, uniform regularity of a subgroup is equivalent to discreteness.

We next turn to the discussion of limit sets. Following [5], for a discrete (not necessarily regular) subgroup $\Gamma < G$ we define the *visual limit set* $\Lambda(\Gamma) \subset \partial_\infty X$ as the accumulation set of one (equivalently, every) Γ -orbit $\Gamma x \subset X$ with respect to the visual compactification of X . The next lemma is an immediate consequence of Lemma 3:

Lemma 5 *Let $\Gamma < G$ be a discrete subgroup preserving a horoball $Hbo \subset X$ centered at a point $\xi \in \partial_\infty X$. Then*

$$\Lambda(\Gamma) \subset \bar{B}(\xi, \frac{\pi}{2}),$$

the closed ball in $\partial_{Tits} X$, centered at ξ , of radius $\frac{\pi}{2}$ with respect to the Tits metric.

The τ_{mod} -limit set $\Lambda_{\tau_{mod}}(\Gamma)$ of a τ_{mod} -regular subgroup $\Gamma < G$ is the accumulation set in $\text{Flag}_{\tau_{mod}} \subset \bar{X}^{\tau_{mod}}$ of some (equivalently, every) orbit $\Gamma x \subset X$. In other words, $\tau \in \Lambda_{\tau_{mod}}(\Gamma)$ if and only if there exists a sequence (γ_n) in Γ such that the sequence $(\gamma_n(x))$ flag-converges to τ . Since flag-convergence is independent of the base-point, $\Lambda_{\tau_{mod}}(\Gamma)$ is a closed Γ -invariant subset of $\text{Flag}_{\tau_{mod}}$.

By the construction, since $\Lambda_{\tau_{mod}} = \Lambda_{\tau_{mod}}(\Gamma)$ is Γ -invariant, so is $\text{Th}(\Lambda_{\tau_{mod}}) \subset \text{Flag}_{\tau_{mod}}$ for every τ_{mod} -invariant thickening $\text{Th} = \text{Th}_I$. Since $\Lambda_{\tau_{mod}}$ is closed in $\text{Flag}_{\tau_{mod}}$, so is $\text{Th}(\Lambda_{\tau_{mod}})$. If Γ is uniformly τ_{mod} -regular then $\Lambda_{\tau_{mod}}(\Gamma)$ has an alternative description:

$$\Lambda_{\tau_{mod}}(\Gamma) = \{\tau \in \text{Flag}_{\tau_{mod}} : \text{ost}(\tau) \cap \Lambda(\Gamma) \neq \emptyset\}, \tag{6}$$

cf. the alternative description of flag-convergence in the end of the previous section.

Corollary 7 *Under the hypotheses of Lemma 5, assume also that G is a simple Lie group of type B_2 (hence, $\partial_{Tits} X$ is a graph with edges of length $\pi/4$), τ_{mod} is one of the two vertices of σ_{mod} , ξ is a vertex of type τ_{mod} , and $\Gamma < G$ is a uniformly τ_{mod} -regular subgroup. Then*

$$\Lambda_{\tau_{mod}}(\Gamma) \subset \bar{B}(\xi, \frac{\pi}{2}) \cap \text{Flag}_{\tau_{mod}} \subset \partial_{Tits} X.$$

Proof Note that if $\eta \in \partial_{Tits} X$ is a τ_{mod} -regular point, $\tau \in \text{Flag}_{\tau_{mod}} \subset \partial_{Tits} X$, then $\eta \in \text{ost}(\tau)$ if and only if $\angle_{Tits}(\eta, \tau) < \frac{\pi}{4}$. By Lemma 5, $\Lambda(\Gamma) \subset \bar{B}(\xi, \frac{\pi}{2})$. By combining these facts with (6), we obtain

$$\Lambda_{\tau_{mod}}(\Gamma) \subset \bigcup_{\eta \in \Lambda(\Gamma)} B(\eta, \frac{\pi}{4}) \cap \text{Flag}_{\tau_{mod}} \subset B(\xi, \frac{3\pi}{4}) \cap \text{Flag}_{\tau_{mod}} \subset \bar{B}(\xi, \frac{\pi}{2}) \cap \text{Flag}_{\tau_{mod}}.$$

□

A key result used in this paper is Theorem 6.13 from [21]:

Theorem 8 *Let Th be a fat thickening. Then for every τ_{mod} -regular subgroup $\Gamma < G$, the Γ -action on*

$$\Omega_{\text{Th}}(\Gamma) := \text{Flag}_{\tau_{mod}} \setminus \text{Th}(\Lambda_{\tau_{mod}}(\Gamma))$$

is properly discontinuous.

Anosov subgroups An important class of τ_{mod} -regular discrete subgroups $\Gamma < G$ consists of τ_{mod} -Anosov subgroups. Anosov representations $\Gamma \rightarrow G$, whose images are Anosov subgroups, were first introduced in [26] for fundamental groups of closed manifolds of negative curvature, then in [15] for arbitrary hyperbolic groups; we refer the reader to our papers [17, 22, 23], for a simplification of the original definition as well as for alternative definitions and to [18, 20] for surveys of the results.

Instead of a detailed discussion of Anosov subgroups, we limit ourselves here to a brief description of their key properties used in this paper. Firstly, suppose that H is a rank one Lie group and X_H be the corresponding rank one symmetric space (the reader can assume that $H = O(n - 1, 1)$ and X_H is the hyperbolic $n - 1$ -space \mathbb{H}^{n-1}). Then the Tits topology on $\partial_{\infty} X_H$ is discrete. Accordingly, there is only one type of visual boundary simplices $\tau_{mod} = \tau_{mod}^H$ and, as we noted earlier, a subgroup $\Gamma < H$ is discrete if and only if it is τ_{mod} -regular. The τ_{mod} -limit set $\Lambda_{\tau_{mod}}(\Gamma) \subset \partial_{\infty} X_H$ is the visual limit set $\Lambda(\Gamma)$. A subgroup $\Gamma < H$ is Anosov (more precisely, τ_{mod}^H -Anosov) if and only if it is *convex-cocompact*, equivalently, if it is discrete, finitely-generated and one, equivalently, every, orbit

map $\Gamma \rightarrow X_H$ is a quasiisometric embedding of Γ (equipped with a word-metric) to the symmetric space X_H . See for instance, Theorem 1.1 in [22] and also [7].

Now consider the case of discrete subgroups of a semisimple Lie group G without any restriction on rank; $X = G/K$ is the associated symmetric space. Suppose that τ_{mod} is an ι -invariant face of σ_{mod} . Below are two of the many characterizations of τ_{mod} -Anosov subgroups $\Gamma < G$ given in [17, 22, 23]:

Theorem 9 *The following are equivalent for a subgroup $\Gamma < G$:*

1. Γ is Gromov-hyperbolic, τ_{mod} -regular (as a subgroup of G), any two distinct limit points in $\Lambda_{\tau_{mod}}(\Gamma) \subset \text{Flag}_{\tau_{mod}}$ are antipodal and there exists an equivariant homeomorphism $\beta : \partial_{\infty}\Gamma \rightarrow \Lambda_{\tau_{mod}}(\Gamma)$. Here $\partial_{\infty}\Gamma$ is the Gromov-boundary of Γ . The map β is called the boundary map of Γ .
2. Γ is finitely generated, uniformly τ_{mod} -regular (as a subgroup of G) and is undistorted, i.e. one (equivalently, every) orbit map $o_x : \Gamma x \subset X$ is a quasiisometric embedding.
3. $\Gamma < G$ is τ_{mod} -Anosov.

Images of rank 1 Anosov subgroups in higher rank lie groups Suppose that G is a semisimple Lie group (the reader can assume that $G = O(n, 2)$) and $H \rightarrow G$ is an embedding of Lie groups (the reader can think of the natural inclusion $O(n - 1, 1) \rightarrow O(n, 2)$; it is the one given by the composition of the embeddings $O(n - 1, 1) \rightarrow G_{L, \dot{L}} \rightarrow G$ discussed in the next section). For simplicity of the discussion (and because it is true in the main example of interest), we assume that the opposition involution ι of the group G is the identity map. Let $X = G/K$ be the symmetric space of G , X_H is the symmetric space of H and let $X_H \rightarrow X$ be a totally-geodesic embedding equivariant with respect to the embedding $H \rightarrow G$. (In the context of $H = O(n - 1, 1) < G = O(n, 2)$, we will discuss the embedding $X_H \rightarrow X$ in Sect. 5.) The embedding $X_H \rightarrow X$ induces an isometric embedding of Tits boundaries $\partial_{Tits} X_H \rightarrow \partial_{Tits} X$ (this embedding is not in general simplicial, but it will be simplicial in the case of interest in this paper); we will identify $\partial_{\infty} X_H$ with its image in $\partial_{\infty} X$. Accordingly, for every point $\eta \in \partial_{Tits} X_H$, there exists a unique smallest simplex $\tau := \xi(\eta)$ in $\partial_{Tits} X$ containing η . (In other words, η is a generic point of τ .) All the simplices $\tau = \xi(\eta)$ have the same type, which we denote τ_{mod} . (In the case of interest, we will see that $\xi(\eta)$ is always a vertex of the type of an isotropic line, i.e. an element of the flag-manifold F_1 . Hence, in this case ξ is the identity embedding.) The map $\xi : \partial_{\infty} X_H \rightarrow \text{Flag}_{\tau_{mod}}$ is continuous, where $\partial_{\infty} X_H$ is equipped with the visual topology. It follows from the main definition of the τ_{mod} -regularity and τ_{mod} -limit set that for a discrete subgroup $\Gamma < H$, its image in G (also denoted Γ) is uniformly τ_{mod} -regular and that $\Lambda_{\tau_{mod}}(\Gamma) = \xi(\Lambda(\Gamma))$, where $\Lambda(\Gamma)$, as we noted earlier, is the limit set of Γ in the visual boundary of X_H . Furthermore, it follows immediately from every characterization of τ_{mod} -Anosov subgroups of G given in [22, 23] (see for instance Theorem 9 above) that if $\Gamma < H$ is convex-cocompact, then $\Gamma < G$ is τ_{mod} -Anosov. This fact was first observed by Labourie in [26, Proposition 3.1] in the *Fuchsian case* and then in [15, Proposition 4.7] in full generality. We summarize these observations in the following proposition:

Proposition 10 *Let G be a semisimple Lie group, $H < G$ is a rank 1 simple Lie subgroup, let $X_H \rightarrow X$ be a totally-geodesic embedding of the associated symmetric spaces, equivariant with respect to the embedding $H \rightarrow G$. Then there exists a model face τ_{mod} of $\partial_{Tits} X$ such that the following hold for every discrete subgroup $\Gamma < H$:*

1. The image of Γ in G is uniformly τ_{mod} -regular.

2. There exists a Γ -equivariant homeomorphism $\beta : \Lambda(\Gamma) \rightarrow \Lambda_{\tau_{mod}}(\Gamma) \subset \text{Flag}_{\tau_{mod}}$ sending each $\lambda \in \Lambda(\Gamma) \subset \partial_\infty X_H$ to the unique simplex of type τ_{mod} in $\partial_{Tits} X$ containing $\lambda \in \partial_\infty X_H \subset \partial_\infty X$.
3. If $\Gamma < H$ is convex-cocompact, then $\Gamma < G$ is τ_{mod} -Anosov.

Note that the map β here is the restriction of the map ξ to $\Lambda(\Gamma) \subset \partial_\infty X_H$. It can be identified with the boundary map of the Anosov subgroup $\Gamma < G$ as in Theorem 9 (the group Γ acts cocompactly the Gromov-hyperbolic space which is the closed convex hull C of $\Lambda(\Gamma)$ in X_H and, hence, $\partial_\infty \Gamma$ can be identified with $\partial_\infty C = \Lambda(\Gamma)$).

4 Lorentzian space $\mathbb{R}^{n-1,1}$ as an open Schubert cell in a partial flag-manifold of the group $G = O(n, 2)$

In this section we will construct an equivariant identification of the Lorentzian space $\mathbb{R}^{n-1,1}$ with an open Schubert cell in a partial flag-manifold F_1 of the group $G = O(n, 2)$, namely, the space of isotropic lines in $V = \mathbb{R}^{n,2}$.

Consider the group $G = O(n, 2)$ and its symmetric space $X = G/K, K = O(n) \times O(2)$. The group G has two partial flag-manifolds: the Grassmannian F_1 of isotropic lines and another partial flag manifold F_2 of isotropic planes in $V = \mathbb{R}^{n,2}$, where the quadratic form on V is

$$q = x_1y_1 + x_2y_2 + z_1^2 + \dots + z_n^2.$$

We will use the notation $\langle \cdot, \cdot \rangle$ for the associated bilinear form on V .

In the paper we will be using the the incidence geometry interpretation of $\partial_{Tits} X$, the Tits boundary of the symmetric space of the group $G = O(n, 2)$. The Tits boundary $\partial_{Tits} X$ (as a spherical building) has the structure of a metric bipartite graph whose vertices are labelled *lines* and *planes*, these are the elements of F_1 and F_2 respectively. Two vertices $L \in F_1$ and $p \in F_2$ are connected by an edge iff the line L is contained in the plane p . The edges of this bipartite graph have length $\pi/4$. We refer the reader to [8, 13, 27].

The group G acts simply transitively on the set of edges of $\partial_{Tits} X$ and we can identify the quotient $\partial_{Tits} X/G$ with σ_{mod} , the model spherical chamber of $\partial_{Tits} X$. Thus σ_{mod} is a circular segment of the length $\pi/4$. This segment has two vertices, one of which we denote τ_{mod} , this is the one which is the projection of F_1 . The flag-manifold F_1 is the quotient G/P_L , where P_L is the stabilizer of an isotropic line L in G ; this flag-manifold is n -dimensional.

Recall that two vertices of $\partial_{Tits} X$ are opposite iff they are within Tits distance π from each other. In terms of the incidence geometry of the vector space (V, q) , two lines $L, \hat{L} \in F_1$ are opposite iff the restriction of q to $\text{span}(L, \hat{L})$ is nondegenerate, necessarily of the type $(1, 1)$. Two lines $L, L' \in F_1$ are within Tits distance $\pi/2$ iff they span an isotropic plane in V .

Consider a subgroup $P_L < G$; it is a maximal parabolic subgroup of G ; let $U < P_L$ be the unipotent radical of P_L . Choosing a line \hat{L} opposite to L , defines a semidirect product decomposition $P_L = U \rtimes G_{L, \hat{L}}$, where $G_{L, \hat{L}}$ is the stabilizer in P_L of the line \hat{L} ; equivalently, it is the stabilizer of the *parallel set* $P(L, \hat{L})$.² This subgroup is the intersection

$$G_{L, \hat{L}} = P_L \cap P_{\hat{L}}.$$

² The parallel set $P(L, \hat{L})$ splits isometrically as the product $l \times \mathbb{H}^{n-1}$, where \mathbb{H}^{n-1} is the *cross-section* of $P(L, \hat{L})$.

The orthogonal complement $V_{L,\hat{L}} \subset V$ of the anisotropic plane $\text{span}(L, \hat{L})$ is invariant under $G_{L,\hat{L}}$, hence,

$$G_{L,\hat{L}} \cong \mathbb{R}^\times \times O(V_{L,\hat{L}}, q|_{V_{L,\hat{L}}}) \cong \mathbb{R}^\times \times O(n - 1, 1). \tag{11}$$

The subgroup $\mathbb{R}_+ < \mathbb{R}^\times$ acts via transvections along geodesics in the symmetric space X connecting L and \hat{L} . The group $G_{L,\hat{L}}$ acts on both $(V', q') = (V_{L,\hat{L}}, q|_{V_{L,\hat{L}}})$ and on U , where the action of \mathbb{R}_+ on $V' = V_{L,\hat{L}}$ is trivial. In order to simplify the notation, we set

$$O(q') = O(V', q').$$

In terms of linear algebra, $\mathbb{R}_+ < \mathbb{R}^\times$ is the identity component of the orthogonal group

$$O(\text{span}(L, \hat{L}), q|_{\text{span}(L,\hat{L})}) \cong O(1, 1).$$

We will use the notation

$$G'_L := U \rtimes O(q') < P_L.$$

This subgroup is the stabilizer in P_L of horoballs in X centered at L .

Our next goal is to describe Schubert cells in the Grassmannian F_1 . We fix $L \in F_1$ and define the subvariety $Q_L \subset F_1$ consisting of all (isotropic) lines $L' \subset V$ such that $\text{span}(L, L')$ is isotropic (the line L or an isotropic plane). In terms of the Tits' distance, $Q_L \setminus \{L\}$ consists of lines $L' \in F_1$ within distance $\frac{\pi}{2}$ from L . The complement

$$L^{opp} = F_1 \setminus Q_L$$

consists of lines opposite to L . The group P_L acts transitively on $\{L\}$, $Q_L \setminus \{L\}$ and L^{opp} and each of these subsets is an open Schubert cell of F_1 with respect to P_L and we obtain the P_L -invariant Schubert cell decomposition

$$F_1 = \{L\} \sqcup (Q_L \setminus \{L\}) \sqcup L^{opp}.$$

We next describe Q_L more geometrically. A vector $v \in V$ spans an isotropic subspace with L iff $v \in L^\perp$ and satisfies the quadratic equation $q(v) = 0$. Since we are only interested in nonzero vectors $v \neq 0$ and their spans $\text{span}(v)$, we obtain the natural identification

$$Q_L \cong \mathbb{P}(q^{-1}(0) \cap L^\perp),$$

the right hand-side is the projectivization a singular quadric hypersurface in L^\perp . Thus, Q_L is a (projective) singular quadric and $L \in Q_L$ is the unique singular point of the Q_L .

In the next lemma, by an *ellipsoid* in a real projective space \mathbb{RP}^{k-1} we mean the projectivization E of a quadric in \mathbb{R}^k given by a quadric form of signature $(k - 1, 1)$. (The reason for the name is that in a suitable affine patch in \mathbb{RP}^{k-1} , E becomes an ellipsoid.)

Lemma 12 *Given two opposite isotropic lines L, \hat{L} , the intersection of the quadrics*

$$E = E_{L,\hat{L}} := Q_L \cap Q_{\hat{L}}$$

is an ellipsoid in $\mathbb{P}(L^\perp \cap \hat{L}^\perp)$. In particular, $E \cap \{L, \hat{L}\} = \emptyset$.

Proof As before, let $V' \subset V$ denote the codimension two subspace orthogonal to both L, \hat{L} . Then each $L' \in E$ is spanned by a vector $v \in V'$ satisfying the condition $q(v) = 0$. In other words, E is the projectivization of the quadric

$$\{v \in V' : q(v) = 0\}.$$

i.e. is an ellipsoid, since q restricted to V' has signature $(n - 1, 1)$. □

Our next goal is to (equivariantly) identify the open cell L^{opp} with the n -dimensional Lorentzian affine space $\mathbb{R}^{n-1,1}$ (where a chosen $\hat{L} \in L^{opp}$ will serve as the origin), so that the group P_L is identified with the group of Lorentzian similarities, where the simply-transitive action $U \curvearrowright L^{opp}$ is identified with the action of the full group of translations of $\mathbb{R}^{n-1,1}$. In particular, for now, and until the end of the proof of Proposition 15, we fix isotropic opposite lines L and \hat{L} . After the end of the proof of Proposition 15 we will allow the line \hat{L} to vary.

We fix nonzero vectors $e \in L, f \in \hat{L}$ such that $\langle e, f \rangle = 1$. Then

$$V = \text{span}(e) \oplus \text{span}(f) \oplus V'$$

We obtain an epimorphism $\eta : P_L \rightarrow O(q')$ by sending $g \in P_L$ first to the restriction $g|_{L^\perp}$ and then to the projection of the latter to the quotient space $V' \cong L^\perp/L$ (the quotient of L^\perp by the null-subspace of $q|_{L^\perp}$). Hence, the kernel of this epimorphism is precisely the solvable radical $U \rtimes \mathbb{R}_+$ of P_L .

For each $v' \in V'$ we define the linear transformation (a shear) $s = s_{v'} \in GL(V)$ by its action on e, f and V' :

- (1) $s(e) = e.$
- (2) $s(f) = -\frac{1}{2}q(v')e + f + v'.$
- (3) For $w \in V', s(w) = w - \langle v', w \rangle e.$

The next two lemmata are proven by straightforward calculations which we omit:

Lemma 13 *For each $s = s_{v'}$ the following hold:*

- 1. $s \in P_L.$
- 2. s lies in the kernel of the homomorphism $\eta : P_L \rightarrow GL(V')$ and is unipotent. In particular, $s \in U$ for each $v' \in V'.$

Lemma 14 *The map $\phi : v' \mapsto s_{v'}$ is a continuous monomorphism $V' \rightarrow U$, where we equip the vector space V' with the additive group structure.*

Since U acts simply transitively on L^{opp} , it is connected and has dimension n . Therefore, the monomorphism ϕ is surjective and, hence, a continuous isomorphism. Thus, ϕ determines a homeomorphism $h : V' \rightarrow L^{opp}$

$$h : v' \mapsto s_{v'}(\hat{L}) = \text{span} \left(-\frac{1}{2}q(v')e + f + v' \right),$$

so that in particular

$$h(0) = \hat{L}.$$

The group $G_{L,\hat{L}} \cong \mathbb{R}^\times \times O(V', q')$ acts on both L^{opp} and on U (via conjugation). The center of $G_{L,\hat{L}}$ acts on V' trivially while its action on U is via a nontrivial character.

Proposition 15 *The map h is equivariant with respect to these two actions of $O(V', q')$.*

Proof Consider a linear transformation $A \in O(V', q')$; as before, we identify $O(V', q')$ with a subgroup of $O(V, q)$ fixing e and f . For an arbitrary $v' \in V'$ we will verify that

$$s_{Av'} = A s_{v'} A^{-1}.$$

It suffices to verify this identity on the vectors e, f and arbitrary $w \in V'$. We have:

1. For each $v' \in V'$, $s_{v'}(e) = e$, while $A(e) = A^{-1}(e) = e$. It follows that

$$e = s_{Av'}(e) = As_{v'}A^{-1}(e) = e.$$

2.

$$s_{Av'}(f) = -\frac{1}{2}q(Av')e + f + Av' = -\frac{1}{2}q(v')e + f + Av'$$

while (since $Ae = e, Af = f$)

$$As_{v'}A^{-1}(f) = As_{v'}(f) = A(-\frac{1}{2}q(v')e + f + v') = -\frac{1}{2}q(v')e + f + Av'.$$

3. For $w \in V'$,

$$s_{Av'}(w) = w - \langle Av', w \rangle e = w - \langle v', A^{-1}w \rangle e,$$

while

$$As_{v'}A^{-1}w = As_{v'}(A^{-1}w) = A(A^{-1}w - \langle v', A^{-1}w \rangle e) = w - \langle v', A^{-1}w \rangle e.$$

□

In view of this proposition we will identify V' with the open Schubert cell L^{opp} , which, in turn, enables us to use Lorentzian geometry to analyze L^{opp} and, conversely, to study discrete subgroups of P_L using Theorem 8 on domains of discontinuity of τ_{mod} -regular group actions on the flag-manifold F_1 . Under the identification $V' \cong L^{opp}$, for each $L' \in L^{opp}$ (in particular, for $L' = \hat{L}$), the conic $Q_{L'} \cap L^{opp}$ becomes a translate of the null-cone of the form q' on V' (see Lemma 16 below) and the flag-manifold F_1 becomes a compactification of V' obtained by adding to it the “quadric at infinity” Q_L .

Lemma 16 For all $v' \in V'$, $q'(v') = 0$ iff q vanishes on $\text{span}(f, h(v'))$, i.e. iff $h(v') \in Q_{\hat{L}}$. In other words, $Q_{\hat{L}} \cap L^{opp}$ is the image under h of the null-cone of q' in the vector space V' .

Proof Since f and $s_{v'}(f)$ (spanning the line $h(v')$) are null-vectors of q , the vanishing of q on $\text{span}(f, h(v'))$ is equivalent to the vanishing of

$$\langle f, s_{v'}(f) \rangle = -\frac{1}{2}q(v').$$

□

Lemma 17 For each neighborhood N of L in Q_L there exists $L' \in L^{opp}$ such that $E_{L,L'} \subset N$.

Proof We pick $L_\infty \in F_1$ opposite to L and, as above, identify L_∞^{opp} with (V', q') . Then for a sequence $L_i \in L_\infty^{opp}$ contained in the, say, future light cone of $Q_L \cap L_\infty^{opp}$ and converging radially to L , the intersections of null-cones $E_{L,L_i} = Q_{L_i} \cap Q_L$ converge to L . Since $L_i \notin Q_L$, they are all opposite to L . Taking $L' = L_i$ for a sufficiently large i concludes the proof. □

For each subset $C \subset F_1$, we define the *thickening* of C :

$$\text{Th}(C) = \bigcup_{L \in C} Q_L.$$

This notion of thickening is a special case of the one discussed in Sect. 2: If we restrict to a single apartment a in the Tits building of G , then for the vertex $L \in a$, $\text{Th}(L) \cap a = Q_L \cap a$ consists of three vertices within Tits distance $\frac{\pi}{2}$ from L . Thus, the thickening Th is maximal and, hence, *fat* (see Sect. 2).

Lemma 18 *For every compact subset $C \subset F_1$, the thickening $Th(C) \subset F_1$ is compact.*

Proof Compactness of thickenings (of closed subsets of general flag-manifolds) is a general fact observed in [21, p. 193], a proof can be found in [17, Lemma 8.18], we add a proof here for the sake of completeness (it is the same as in [17]). Compactness of $Th(L) = Q_L$ for each $L \in F_1$ is clear. Observe that for $g \in G$, $g Th(L) = Th(gL)$. Consider a closed subset $C \subset F_1$. Take a sequence $L_k \in C$ converging to $L_0 \in C$. There exists a sequence $g_k \in K$ such that $g_k(L_1) = L_k$ for all $k \in \mathbb{N}$ (since the maximal compact subgroup $K < G$ acts transitively on F_1). In view of compactness of the subgroup $K < G$, without loss of generality, we may assume that the sequence g_k converges to some $g_0 \in K$. Thus, the sequence of subsets $g_k(L_1) \subset Th(C)$ converges to $g_0(L_1)$ with respect to the Hausdorff metric on the set of nonempty closed subsets of F_1 . At the same time, the sequence $g_k(L_1) = L_k$ converges to L_0 , which implies that $L_0 = g_0(L_1)$. Thus, the limit of the sequence of thickenings $g_k(Th(L_1))$ equals the thickening $Th(L_0) \subset Th(C)$. It follows that $Th(C) \subset F_1$ is closed; compactness of F_1 implies compactness of $Th(C)$.

Lemma 19 *For any two opposite isotropic lines $L, \hat{L} \in F_1$ and each compact subset $C \subset Q_{\hat{L}} \cap L^{opp}$, the intersection $Th(C) \cap L^{opp}$ is a proper subset of L^{opp} .*

Proof Let $H \subset L^{opp} \cong V'$ be an affine hyperplane in V' intersecting $Q_{\hat{L}}$ only at \hat{L} . Then

$$C' := \{L' \in H : Q_{L'} \cap C \neq \emptyset\}$$

is compact in H . Next, observe that for $L_1, L_2 \in F_1, L_1 \in Q_{L_2} \iff L_2 \in Q_{L_1}$. Thus, every $L' \in H \setminus C'$ does not belong to $Th(C)$. □

Lemma 20 *For each compact $C \subset Q_L \setminus \{L\}$ the thickening $Th(C)$ is a proper compact subset of F_1 .*

Proof Lemma 17 implies that there exists $L_\infty \in L^{opp}$ such that E_{L,L_∞} is disjoint from C . Thus, C is contained in L_∞^{opp} . In particular, Lemma 19 implies that $Th(C)$ is a proper subset of F_1 . Compactness of $Th(C)$ was proven in Lemma 18. □

5 Proof of the main theorem

We continue with the notation introduced in the previous section. Consider the subgroups $G'_L < P_L < G$ with $G = O(n, 2)$. The subgroup $H = O(n - 1, 1) < G$ stabilizes two opposite points in $\partial_\infty X$, which are isotropic lines L, \hat{L} and, hence, preserves the parallel set $P(L, \hat{L})$ consisting of geodesics in X asymptotic to both L, \hat{L} . This parallel set splits as the product $\mathbb{R} \times Y$, where Y is a totally-geodesic symmetric subspace in X (necessarily of rank one) and for each $y \in Y$ the product $\mathbb{R} \times \{y\}$ is one of the geodesics in X asymptotic to L, \hat{L} . The subgroup H preserves $P(L, \hat{L})$; it also necessarily preserves the product decomposition. The identity component $H_0 < H$ also necessarily preserves each $\{t\} \times Y$ (for otherwise, we obtain a nontrivial isometric action of H on the real line). Moreover, since H preserves both L and \hat{L} , the entire group H preserves each $\{t\} \times Y$. Pick a point $y \in Y$ and take a visual boundary point $\eta \in \partial_\infty Y$. We have two geodesic rays in $P(L, \hat{L})$ emanating from y : One is asymptotic to L , another (contained in Y) asymptotic to η . These rays are obviously contained in a 2-dimensional flat in X and are orthogonal to each other. Hence, the Tits angle between η, L equals $\pi/2$. Since the Tits boundary of X is a bipartite graph with edge-length $\pi/4$, and L is a vertex of this graph, the point η is also a vertex and has the same type as L , i.e. the

type of an isotropic line. Similarly, η has the Tits distance $\pi/2$ from \hat{L} . Since the subgroup $H = O(n - 1, 1) < G = O(n, 2)$ preserves each $\{t\} \times Y$, there exists a totally-geodesic isometric embedding of the symmetric space X_H of H into $\{t\} \times Y$. (Actually, (11) implies that X_H is the entire $\{t\} \times Y$ but we will not need this fact.)

From now on, τ_{mod} is a vertex of the Tits building $\partial_\infty X$ which has the type of an isotropic line. In view of Proposition 10, we conclude:

Lemma 21 *Let $\Gamma < H$ be a discrete subgroup. Then the image of Γ under the embedding $H \rightarrow G$ is τ_{mod} -regular. Every τ_{mod} -limit point η of Γ is a vertex of $\partial_\infty X$ of the type of an isotropic line, which is at the Tits distance $\pi/2$ from both L, \hat{L} . Accordingly, η belongs to the intersection $Q_L \cap Q_{\hat{L}}$.*

Corollary 22 *If $\Gamma < H$ is a convex-cocompact subgroup, then its image in G is τ_{mod} -Anosov and its τ_{mod} -limit set $\Lambda_{\tau_{mod}}(\Gamma)$ is contained in $Q_L \cap Q_{\hat{L}}$.*

We next consider the slightly more general case of uniformly τ_{mod} -regular discrete subgroups $\Gamma < P_L$:

Lemma 23 *The τ_{mod} -limit set $\Lambda_{\tau_{mod}}(\Gamma)$ of every uniformly τ_{mod} -regular subgroup $\Gamma < P_L < G$ is contained in Q_L .*

Proof According to Corollary 7, $\Lambda_{\tau_{mod}}(\Gamma) \subset \bar{B}(L, \frac{\pi}{2}) \cap F_1$. The latter intersection is Q_L since both consist of isotropic lines $L' \subset V$ such that $\text{span}(L, L')$ is an isotropic subspace of V . □

Proposition 24 *Suppose that $\Gamma < G'_L$ is a τ_{mod} -regular discrete subgroup whose τ_{mod} -limit set does not contain L . Then $\text{Th}(\Lambda_{\tau_{mod}}(\Gamma))$ is closed in F_1 , $\text{Th}(\Lambda_{\tau_{mod}}(\Gamma)) \neq F_1$, and the action*

$$\Gamma \curvearrowright \Omega_{\text{Th}}(\Gamma) = F_1 \setminus \text{Th}(\Lambda_{\tau_{mod}}(\Gamma))$$

is properly discontinuous.

Proof Since $\Lambda_{\tau_{mod}}(\Gamma)$ is a compact subset of Q_L , the first statement of the proposition is a special case of Lemma 20. The proper discontinuity statement is a special case of Theorem 8 since the thickening Th is fat. □

We now describe certain conditions on τ_{mod} -regular discrete subgroups $\Gamma < G'_L$ which will ensure that $\Lambda_{\tau_{mod}}(\Gamma)$ does not contain the point L . Each subgroup $\Gamma < G'_L$ has the linear part Γ_0 , i.e. its projection to $O(q') \cong O(n - 1, 1)$, which is identified with the semisimple factor of the stabilizer in P_L of some $\hat{L} \in L^{opp}$. We now assume that:

- Γ_0 is a convex-cocompact subgroup of $O(n - 1, 1)$.
- The projection

$$\ell : \Gamma \rightarrow \Gamma_0$$

is an isomorphism.

As we proved in Corollary 22, $\Gamma_0 < G$ is a τ_{mod} -Anosov subgroup of G and $\Lambda_{\tau_{mod}}(\Gamma_0) \subset Q_L \cap Q_{\hat{L}}$. In particular, $\Lambda_{\tau_{mod}}(\Gamma_0)$ does not contain L by Lemma 12.

Given a subgroup $\Gamma_0 < O(q')$, the inverse $\rho : \Gamma_0 \rightarrow \Gamma$ to $\ell : \Gamma \rightarrow \Gamma_0$ is determined by a cocycle $c \in Z^1(\Gamma_0, V')$ which describes the translational parts of the elements of Γ :

$$\rho(\gamma) : v \mapsto \gamma v + c(\gamma), v \in V' \cong \mathbb{R}^{n-1,1}.$$

Pick some $t \in \mathbb{R}_+$; then $t\mathcal{C}$ is again a cocycle corresponding to the conjugate representation ρ^t , where we identify $t \in \mathbb{R}_+$ with a central element of $G_{L,\hat{L}}$. Sending $t \rightarrow 0$ we obtain:

$$\lim_{t \rightarrow 0} \rho^t = id,$$

the identity embedding $\Gamma_0 \rightarrow O(n - 1, 1) < P_L$. In view of stability of Anosov representations (see [15, Theorem 5.13] and [19, Theorems 1.10, 1.11], [16, Corollary 6.14]) we conclude that all representations ρ^t are τ_{mod} -Anosov and the τ_{mod} -limit sets of $\Gamma_t = \rho^t(\Gamma_0)$ vary continuously with t ; moreover,

$$t \Lambda_{\tau_{mod}}(\Gamma_{t_1}) = \Lambda_{\tau_{mod}}(\Gamma_{t_2})$$

where $t = t_2/t_1$. In particular,

$$\Lambda_{\tau_{mod}}(\Gamma) \subset \mathcal{Q}_L \setminus \{L\}$$

is a compact subset. Proposition 24 now implies:

Corollary 25 *For each Γ as above,*

$$Th(\Lambda_{\tau_{mod}}(\Gamma)) \neq F_1$$

and the action

$$\Gamma \curvearrowright \Omega_{Th}(\Gamma) = F_1 \setminus Th(\Lambda_{\tau_{mod}}(\Gamma))$$

is properly discontinuous.

Thus, we proved that each discrete subgroup $\Gamma < P_L$ as above has nonempty domain of discontinuity in the vector space V' . Theorem 2 follows. \square

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Declaration

Conflict of interest The authors declare that they have no Conflict of interest.

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